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ON THE MINIMAL GRAPH WITH A GIVEN NUMBER OF SPANNING TREES

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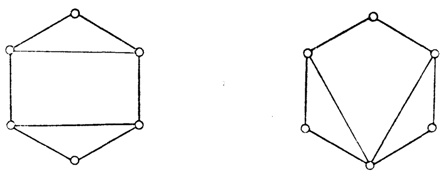
Let G be a finite connected graph without loops or multiple edges. A maximal tree subgraph T of G is called a spanning tree of G. Denote by $k(G)$ the number of all trees spanning the graph G. A. Rosa formulated the following problem (private communication): Let $x(\neq 2)$ be a given positive integer and denote by $\alpha(x)$ the smallest positive integer y having the following property: There exists a graph G on y vertices with x spanning trees. Investigate the behavior of the function $\alpha(x)$.

Obviously, $\alpha(1)=1$, $\alpha(3)=3$, $\alpha(4)=4$, $\alpha(5)=5$, $\alpha(6)=6$, $\alpha(7)=7$, $\alpha(8)=4$, $\alpha(9)=5$, etc. The minimal graph $G_{\min}$ need not be unique. For instance, we have $\alpha(30)=6$ and Figures 1 and 2 show two nonisomorphic graphs $G_{\min}$ on six vertices with $k(G_{\min})=30$.



Figures 1 and 2.

It is easy to see that $\alpha(x)\leq x$ for each positive integer $x\neq 2$. Further, the following estimates are true: If $x\equiv 0 \pmod 3$, $x>6$ then by considering the graph consisting of a circuit of length $x/3$ and circuit of length 3, having one vertex in common,

$$\alpha(x) \leq \frac{x}{3}+2.$$

If $x\equiv 2 \pmod 3$, $x>5$ then by considering the graph consisting of a circuit of length $(x+4)/3$ with a single chord,

$$\alpha(x) \leq \frac{x+4}{3}.$$

If $x=p_1^{a_1}p_2^{a_2}p_3^{a_3}\ldots p_s^{a_s}$ and if $p_1, p_2, p_3, \ldots, p_s$ are odd primes then by considering the graph with a_i circuits of length p_i meeting at a single vertex,

$$\alpha(x) \leq 1+\sum_{i=1}^{s} a_i(p_i-1).$$

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In the following theorem $1g\ x$ means the natural logarithm of x. The symbol $[n]$ means the maximal integer m with $m \le n$.

THEOREM 1.

$$\alpha(x) > [\sqrt{1g\,x}] \quad \textit{for each } x \neq 2.$$

Proof. For $x=1$ this is obvious. Assume $x \ge 3$ and put $R=[\sqrt{1g\,x}]$. If G is a graph on n vertices with $n \le R$ then we have $k(G) \le R^{R-2}$ (Cayley's formula). Therefore

$$1g\,k(G) \le (R-2)\,1g\,R \le R\,1g\,R < R^2 = [\sqrt{1g\,x}]^2 \le 1g\,x$$

hence $k(G) < x$. Thus $\alpha(x) > R$. Hence the proof.

Theorems 2 and 3 are easy corollaries of the preceding Theorem 1.

THEOREM 2.

$$\lim_{x \to \infty} \alpha(x) = +\infty.$$

THEOREM 3. *For every positive integer n the equation $\alpha(x)=n$ has at most a finite number of roots.*

In a former paper [1] we defined the set A_n as a set of all natural numbers m for which there exists a graph G with n vertices and $k(G)=m$. Put $A_n=\{x_1, x_2, \ldots, x_r\}$, where $x_1 < x_2 < \cdots < x_r$. We showed that for $n \ge 8$ we have

$$(1) \qquad \begin{cases} x_r = n^{n-2}, & x_{r-1} = (n-2)n^{n-3}, \\ x_{r-2} = (n-2)^2 n^{n-4}, & x_{r-3} = (n-1)(n-3)n^{n-4}, \\ x_{r-4} = (n-2)^3 n^{n-5}, & x_{r-5} = (n-3)(n-2)(n-1)n^{n-5}, \\ x_{r-6} = (n-2)(n^2-4n+2)n^{n-5}, & x_{r-7} = (n-3)^2 n^{n-4}, \\ x_{r-8} = (n-4)(n-1)^2 n^{n-5}, & x_{r-9} = (n-2)^4 n^{n-6}. \end{cases}$$

Further

$$A_1 = A_2 = \{1\},\ A_3 = \{1, 3\},\ A_4 = \{1, 3, 4, 8, 16\},$$
$$A_5 = \{1, 3, 4, 5, 8, 9, 11, 12, 16, 20, 21, 24, 40, 45, 75, 125\}.$$

Let n be a given positive integer. We shall engage in solving the equation $\alpha(x)=n$. For $\alpha(x)=1$ we have $x=1$. The equation $\alpha(x)=2$ has no solution. For $\alpha(x)=3$ we get $x=3$ and for $\alpha(x)=4$, $x=4$, 8,16. The equation $\alpha(x)=5$ has the roots $x=5$, 9, 11, 12, 20, 21, 24, 40, 45, 75, 125. Denote by $M(n)$ the set of all roots of the equation $\alpha(x)=n$. We have $M(1)=\{1\}$ and for $n>1$ the following relation holds:

$$M(n) = A_n - \bigcup_{i=1}^{n-1} A_i = A_n - A_{n-1}.$$

In [2] we showed that $|A_{n+1}| \ge |A_n| + n - 1$. From this inequality Theorem 4 follows immediately:

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THEOREM 4.

$$\lim_{n\to\infty} |M(n)| = +\infty.$$

Let be given an equation $\alpha(x)=n$, where $n\geq 8$. We are able to determine ten roots by using the formulas (1). Obviously $(n-1)^3<(n-2)^4$ and $(n-1)^{n-6}<n^{n-6}$, hence $(n-1)^{n-3}<(n-2)^4 n^{n-6}$. We conclude $\alpha(x_i)=n$ for $i=r, r-1, r-2, \ldots, r-9$.

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