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BEST TRIGONOMETRIC APPROXIMATION, FRACTIONAL ORDER DERIVATIVES AND LIPSCHITZ CLASSES

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1. Introduction. Let $C_{2\pi}$ denote the space of 2π-periodic continuous functions and Π_n the set of trigonometric polynomials of degree $\leqq n$, where $n \in \mathbf{P} = \{0, 1, \ldots\}$. Given $\theta > 0$, the well-known theorem of Stečkin and its converse state that the best approximation of an $f \in C_{2\pi}$ with respect to the max-norm satisfies

$$(1) \qquad E_n(f; C_{2\pi}) = \inf_{t_n \in \Pi_n} \|f - t_n\|_{C_{2\pi}} = O(n^{-\theta}) \quad (n \to \infty)$$

if and only if, for some $r < \theta$, $r \in \mathbf{N}$ (= naturals),

$$(2) \qquad \|f^{(r)} - (t_n^*)^{(r)}\|_{C_{2\pi}} = O(n^{r-\theta}) \quad (n \to \infty),$$

where $t_n^* = t_n^*(f) \in \Pi_n$ denotes the polynomial of best approximation to f. Moreover, assertion (1) is, by Zamansky's theorem and its converse (due to P. L. Butzer, S. Pawelke and G. Sunouchi), equivalent to

$$(3) \qquad \|(t_n^*(f))^{(s)}\|_{C_{2\pi}} = O(n^{s-\theta}) \quad (n \to \infty),$$

for every $s \in \mathbf{N}$, $s > \theta$.

One purpose of this paper is to extend these results to derivatives of fractional order (see Section 3). Furthermore, the classical Jackson and Bernstein theorems, stating that (1) is equivalent to the smoothness condition

$$(4) \qquad \omega_s(f; \delta; C_{2\pi}) = O(\delta^\theta), \quad (\delta \to 0+),$$

for every $s \in \mathbf{N}$, $s > \theta$, can be extended to moduli of continuity of fractional index s. The resulting characterization has the advantage that the continuous scale of θ now has as its counterpart a continuous scale of s, which can be used to extend to the fractional situation several results on Lipschitz classes, such as identity theorems for different indices and a reduction theorem (Theorem 3 and Remark 7). For example, denoting by $D^\beta f$ the fractional derivative of order $\beta > 0$ of f (see Definition 1), the reduction theorem implies that

$$\omega_\alpha(f; \delta; C_{2\pi}) = O(\delta^\theta), \quad (\delta \to 0+)$$

for $\alpha > \theta$, if and only if

$$D^\beta f \in \begin{cases} \mathrm{Lip}_1(\theta - \beta; C_{2\pi}), & 0 < \theta - \beta < 1 \\ \mathrm{Lip}_2(\theta - \beta; C_{2\pi}), & 0 < \theta - \beta < 2. \end{cases}$$

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Basic properties of moduli of continuity of fractional index, including their relation to the K-functional (Proposition 2) and Marchaud's inequality (Corollary 2), will be studied in Section 4.

For the proofs, we employ the results of Butzer–Scherer [**3**;**4**] on the equivalence of the fundamental theorems (i)–(iv) of best approximation (Lemma 5 in Section 3), extend the results of Westphal [**16**] and Butzer–Westphal [**5**] on the fractional difference approach to fractional derivatives, as well as use recent inequalities of Taberski (see [**14**]) and Nessel–Wilmes [**13**]. Basic in our approach is Proposition 2 connecting the moduli of continuity of fractional index with the K-functional. In our proofs we do not make use of the corresponding results in the classical integral case; in particular, we supply a self-contained proof of the fractional analog of Jackson's inequality (Proposition 1 in Section 3).

2. Preliminaries. Let $X_{2\pi}$ be one of the spaces $C_{2\pi}$, $L_{2\pi}^p$, $1 \leqq p < \infty$, with norms

$$||f||_{C_{2\pi}} = \sup_{x\in[-\pi,\pi]} |f(x)|, \quad ||f||_{L_{2\pi}^p} = \left\{(1/2\pi) \int_{-\pi}^{\pi} |f(x)|^p dx\right\}^{1/p}.$$

The finite Fourier transform of $f \in X_{2\pi}$ is denoted by

$$f^\wedge(k) = (1/2\pi)\int_{-\pi}^{\pi} f(u)e^{-iku}du, \quad k \in \mathbf{Z}$$

and the convolution of $f \in L_{2\pi}^1$, $g \in X_{2\pi}$ by

$$(f * g)(x) = (1/2\pi)\int_{-\pi}^{\pi} f(x-u)g(u)du.$$

As in [**5**], we define the (right) difference of $f \in X_{2\pi}$ of fractional order $\alpha > 0$ with respect to the increment $h \in \mathbf{R}$ by

$$(2.1) \qquad (\Delta_h^\alpha f)(x) = \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} f(x-hj) \quad (x \in \mathbf{R}).$$

For convenience we assemble some basic properties of the fractional difference.

LEMMA 1. *Let* $f, g \in X_{2\pi}$, $\alpha, \beta > 0$, $x, h \in \mathbf{R}$. *Then*

(i) $||\Delta_h^\alpha f||_{X_{2\pi}} \leqq M(\alpha)||f||_{X_{2\pi}}, \quad M(\alpha) = \sum_{j=0}^{\infty} \left|\binom{\alpha}{j}\right| \leqq 2^{\{\alpha\}}$

where $\{\alpha\} = \inf\{k \in \mathbf{P}; k \geqq \alpha\}$,

(ii) $[\Delta_h^\alpha f]^\wedge(k) = (1-e^{-ikh})^\alpha f^\wedge(k) \quad (k \in \mathbf{Z})$,

(iii) $(\Delta_h^\alpha (f*g))(x) = ((\Delta_h^\alpha f)*g)(x) \quad (a.e.)$,

(iv) $(\Delta_h^\alpha(\Delta_h^\beta f))(x) = (\Delta_h^{\alpha+\beta} f)(x) \quad (a.e.)$,

(v) $||\Delta_h^{\alpha+\beta} f|| \leqq 2^{\{\beta\}}||\Delta_h^\alpha f||$,

(vi) $\lim_{h\to 0} ||\Delta_h^\alpha f||_{X_{2\pi}} = 0$.

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The proof is essentially contained in [**5**]: Property (i), which is simple, yields (ii) (see [**5**, p. 126]), which in turn gives (iii) and (iv). Part (v) follows by (iv) and (i). Concerning (vi), since the operator Δ_h^α is uniformly bounded in h by (i), it suffices to establish (vi) for the fundamental set $\{e^{ikx}\}_{k\in\mathbf{Z}}$ which is trivial in view of (ii).

Definition 1. (a) If for $f \in X_{2\pi}$ there exists $g \in X_{2\pi}$ such that

$$(2.2)\qquad \lim_{h\to 0+} \|h^{-\alpha}\Delta_h^\alpha f - g\|_{X_{2\pi}} = 0,$$

then g is called the *Liouville-Grünwald derivative of order* $\alpha > 0$ *of* f *in the* $X_{2\pi}$*-norm*, denoted by $g = D^\alpha f$.

(b) The integral of $f \in X_{2\pi}$ of order $\alpha > 0$ is defined by

$$(2.3)\qquad (I^\alpha f)(x) = (f * \psi_\alpha)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-u)\psi_\alpha(u)du,$$

ψ_α being the $L_{2\pi}{}^1$-function given by

$$(2.4)\qquad [\psi_\alpha]^\wedge(k) = \begin{cases} (ik)^{-\alpha} & , k \in \mathbf{Z}, k \neq 0 \\ 0 & , k = 0. \end{cases}$$

From [**5**, p. 129, 130] we cite:

LEMMA 2. (a) *The following assertions are equivalent for* $f \in X_{2\pi}$, *and* $\alpha > 0$:

(i) $D^\alpha f$ *exists as an element in* $X_{2\pi}$.

(ii) *There exists* $g_1 \in X_{2\pi}$: $(ik)^\alpha f^\wedge(k) = g_1{}^\wedge(k), k \in \mathbf{Z}, k \neq 0$.

(iii) *There exists* $g_2 \in X_{2\pi}$: $f(x) = (I^\alpha g_2)(x) + f^\wedge(0)$ (*a.e.*).

In this event, the functions g_i, $i = 1, 2$, *are uniquely determined* (*a.e.*) *apart from an additive constant, and*

$$(D^\alpha f)(x) = g_i(x) - g_i{}^\wedge(0) \quad (a.e.).$$

(b) *Set* $W_{X_{2\pi}} = \{f \in X_{2\pi}; D^\alpha f$ *exists as element in* $X_{2\pi}\}$.

(iv) *The operator* D^α: $W_{X_{2\pi}}{}^\alpha \to X_{2\pi}$ *is closed.*

(v) *If* $f \in X_{2\pi}$, $g \in W_{X_{2\pi}}{}^\alpha$, *then* $f * g \in W_{X_{2\pi}}{}^\alpha$ *and*

$$(2.5)\qquad D^\alpha(f*g)(x) = (f * D^\alpha g)(x) \quad (a.e.).$$

(vi) $D^\alpha D^\beta f = D^{\alpha+\beta} f$, *whenever one of the two sides is meaningful.*

(vii) $D^\alpha(I^\alpha f)(x) = f(x) - f^\wedge(0) = (I^\alpha(D^\alpha f))(x)$ (*a.e.*),

the latter equality holding provided $f \in W_{X_{2\pi}}{}^\alpha$.

For the proof, one makes use of the existence of a function $\chi_\alpha(x; h) \in L_{2\pi}{}^1$, $h > 0$ having the properties

$$(2.6)\qquad \|\chi_\alpha(\cdot\,; h)\|_{L_{2\pi}{}^1} \leqq C(\alpha) \quad (h > 0),$$

$$(2.7)\qquad \lim_{h\to 0+}\int_{\delta\leqq|u|\leqq\pi} |\chi_\alpha(u; h)|du = 0 \quad (0 < \delta < \pi),$$

$$(2.8)\qquad [\chi_\alpha(\cdot\,; h)]^\wedge(k) = \begin{cases} (ikh)^{-\alpha}(1-e^{-ikh})^\alpha & , \quad k \neq 0 \\ 1 & , \quad k = 0 \end{cases} \quad (h > 0).$$

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This function will also play an essential role in this paper. Throughout, $C(\alpha)$ denotes the same constant at each occurrence, depending only on α. The same applies to the constants $B(\alpha)$, $J(\alpha)$, $J'(\alpha)$, $F(\alpha)$.

Lemma 2 (ii) shows, in view of [**2**, p. 172], that $D^\alpha f$ coincides with the classical derivative $f^{(\alpha)}$, provided α is integral.

Definition 2. The *integral* or *Steklov means* of $f \in X_{2\pi}$ of order $\alpha > 0$ are given by

$$(2.9) \qquad (A_h^\alpha f)(x) = (f * \chi_\alpha(\cdot\,; h))(x) \quad (h > 0; x \in \mathbf{R}).$$

The A_h^α define a family of operators on $X_{2\pi}$, coinciding in case $\alpha = r \in \mathbf{N}$ with the classical integral means

$$(2.10) \quad (A_h^r f)(x) = h^{-r} \int_0^h \dots \int_0^h f(x - (u_1 + \dots + u_r)) du_1 \dots du_r,$$

and having the following properties.

Lemma 3. *Let $f \in X_{2\pi}$ and $\alpha > 0$.*

(i) $\|A_h^\alpha f\|_{X_{2\pi}} \leqq C(\alpha)\|f\|_{X_{2\pi}} \quad (h > 0)$,

(ii) $\lim_{h\to 0+} \|A_h^\alpha f - f\|_{X_{2\pi}} = 0$,

(iii) $A_h^\alpha f \in W_{X_{2\pi}}^\beta$ *for all* $0 < \beta \leqq \alpha, h > 0$, *and*

$$(2.11) \quad D^\beta(A_h^\alpha f)(x) = h^{-\beta}(A_h^{\alpha-\beta}(\Delta_h^\beta f))(x) \quad (a.e.)\ (0 < \beta < \alpha),$$

$$(2.12) \quad (D^\alpha(A_h^\alpha f))(x) = h^{-\alpha}(\Delta_h^\alpha f)(x) \quad (a.e.).$$

Proof. Part (i) follows by $\|f * \chi_\alpha(\cdot\,; h)\|_{X_{2\pi}} \leqq \|\chi_\alpha(\cdot\,; h)\|_{L^1_{2\pi}} \cdot \|f\|_{X_{2\pi}}$, and (2.6). Part (ii) is a consequence of (2.6)-(2.8) (see [**5**, Proposition 3.2]). Concerning (iii), one has for $0 < \beta < \alpha$

$$(ik)^\beta [A_h^\alpha f]^\wedge(k) = \begin{cases} (ik)^\beta (ikh)^{-\alpha}(1 - e^{-ikh})^\alpha f^\wedge(k), & k \neq 0 \\ 0 & , \ k = 0 \end{cases}$$

$$= h^{-\beta}[A_h^{\alpha-\beta}(\Delta_h^\beta f)]^\wedge(k) \quad (k \in \mathbf{Z}).$$

By Lemma 2, this implies that $f \in W_{X_{2\pi}}^\beta$ together with (2.11). The case $\alpha = \beta$ follows analogously.

3. The fundamental theorem: First half. In the proof of the fundamental theorem of best approximation, Jackson and Bernstein inequalities play the basic role. Bernstein's inequality, in the fractional case due to P. Civin [**6**] (see the review paper [**8**], also [**9**; **12**]), reads

Lemma 4. *One has, for $t_n \in \Pi_n$, $n \in \mathbf{P}$, $\alpha > 0$*

$$(3.1) \qquad \|D^\alpha t_n\|_{X_{2\pi}} \leqq B(\alpha) n^\alpha \|t_n\|_{X_{2\pi}}.$$

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The Jackson inequality for fractional α may derived from the integral case (cf. Remark 1). A self-contained proof, extending that in [**2**, p. 97], is given below.

PROPOSITION 1. *If* $f \in W_{X_{2\pi}}{}^{\alpha}$, *then*

$$E_n(f; X_{2\pi}) \leqq J(\alpha)n^{-\alpha}||D^\alpha f||_{X_{2\pi}} \quad (n \in \mathbf{N}). \tag{3.2}$$

Proof. Let $(K_n f)(x) = (f * \kappa_n)(x)$ be the convolution integral of Fejér-Korovkin (see [**2**, p. 79]), the kernel κ_n being essentially defined by $\kappa_n{}^\wedge(1) = \cos (\pi/(n + 2))$. We first show

$$||K_n f - f||_{X_{2\pi}} \leqq F(\alpha)n^{-\alpha}||D^\alpha f||_{X_{2\pi}} \tag{3.3}$$

for $0 < \alpha \leqq 1$. This will give (3.2) for $0 < \alpha \leqq 1$ since $K_n f \in \Pi_n$.

By Lemma 1 (iv), (v),

$$||\Delta_u{}^1 f||_{X_{2\pi}} = ||\Delta_u{}^{1-\alpha}(\Delta_u{}^\alpha f)||_{X_{2\pi}} \leqq 2^{\{1-\alpha\}}||\Delta_u{}^\alpha f||_{X_{2\pi}},$$

thus it follows by (2.12), (2.5) and Lemma 3 (i), using the fact that κ_n is even,

$$\begin{aligned} ||K_n f - f||_{X_{2\pi}} &\leqq \frac{2^{\{1-\alpha\}}}{\pi}\int_0^\pi ||(\Delta_u{}^\alpha f)(\cdot)||_{X_{2\pi}}\kappa_n(u)du \\ &= \frac{2^{\{1-\alpha\}}}{\pi}\int_0^\pi ||(A_u{}^\alpha(D^\alpha f))(\cdot)||_{X_{2\pi}}u^\alpha\kappa_n(u)du \\ &\leqq \frac{2^{\{1-\alpha\}}C(\alpha)}{\pi}||D^\alpha f||_{X_{2\pi}}\int_0^\pi u^\alpha\kappa_n(u)du. \end{aligned}$$

By a recent result of Taberski (cf. [**14**, Proposition 5]),

$$\frac{1}{\pi}\int_0^\pi u^\alpha\kappa_n(u)du \leqq \left(\frac{\pi}{\sqrt{2}}\right)^\alpha(1 - \kappa_n{}^\wedge(1))^{\alpha/2} \quad (0 < \alpha \leqq 2).$$

This implies (3.3) for $0 < \alpha \leqq 1$, since $1 - \kappa_n{}^\wedge(1) \leqq (\pi^2/2)n^{-2}$.

If $\alpha > 1$ set $\alpha = r + \beta$, $r \in \mathbf{N}$, $0 < \beta \leqq 1$, and

$$U_{r+1,n}f = \sum_{j=1}^{r+1} (-1)^{j+1}\binom{r+1}{j}K_n{}^j f,$$

where $K_n{}^1 = K_n$, $K_n{}^{j+1} = K_n(K_n{}^j)$. Clearly $U_{r+1,n}f \in \Pi_n$ for each $f \in X_{2\pi}$, $r \in \mathbf{N}$. $U_{r+1,n}f$ again is a convolution integral with kernel

$$\sum_{j=1}^{r+1} (-1)^{j+1}\binom{r+1}{j}\underbrace{(\kappa_n * \kappa_n * \ldots * \kappa_n)}_{j\text{-fold}}(x). \tag{3.4}$$

Since $U_{r+1,n}f - f = (-1)^r(K_n - I)^{r+1}f$, it follows by applying (3.3) to $(K_n - I)^r f \in W_{X_{2\pi}}{}^\beta$ that

$$\begin{aligned} E_n(f; X_{2\pi}) &\leqq ||(U_{r+1,n} - I)f||_{X_{2\pi}} = ||(K_n - I)[(K_n - I)^r f]||_{X_{2\pi}} \\ &\leqq F(\beta)n^{-\beta}||D^\beta(K_n - I)^r f||_{X_{2\pi}} = F(\beta)n^{-\beta}||(K_n - I)^r D^\beta f||_{X_{2\pi}} \end{aligned} \tag{3.5}$$

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in view of (2.5) and (3.4). Repeating this procedure one has

$$\|(K_n - I)^r D^\beta f\|_{X_{2\pi}} = \|(K_n - I)[(K_n - I)^{r-1} D^\beta f]\|_{X_{2\pi}}$$
$$\leqq F(1) n^{-1} \|(K_n - I)^{r-1} D^{\beta+1} f\|_{X_{2\pi}} \leqq [F(1)]^r n^{-r} \|D^{\beta+r} f\|_{X_{2\pi}}$$

which, together with (3.5), yields (3.2) with

$$J(\alpha) = F(\beta)[F(1)]^r \leqq 2(\pi/2)^\alpha C(\beta).$$

To establish the first part of our main result, namely Theorems 1 and 3, we apply a general approximation theorem in a normed linear space setting due to Butzer-Scherer [**3**; **4**; **11**]. The necessary facts are briefly summarized in a form needed here.

Let X be a normed linear space with norm $\|\cdot\|_X$ and $\{M_n\}_{n\in \mathbf{P}}$ a sequence of linear subspaces such that $M_n \subset M_{n+1}$, $n \in \mathbf{P}$, and $\lim_{n\to\infty} E_n(f; X) = 0$, where $E_n(f; X) = \inf_{g\in M_n} \|f - g\|_X$. Moreover, it is assumed that for each $f \in X$, $n \in \mathbf{P}$ there exists $g_n^* \equiv g_n^*(f) \in M_n$ such that $E_n(f; X) = \|f - g_n^*\|_X$.

The smoothness properties of functions $f \in X$ are expressed here by means of the K-functional: Given a linear subspace Y of X with seminorm $|\cdot|_Y$, it is defined by

$$K(t, f; X, Y) = \inf_{g\in Y} \{\|f - g\|_X + t|g|_Y\} \quad (t > 0, f \in X).$$

The orders of approximation ϕ are chosen from the set Φ of positive, non-decreasing functions ϕ on $(0, 1]$ for which $\phi(1) < \infty$ and $\lim_{t\to 0+} \phi(t) = 0$.

LEMMA 5. *Let X, Y, $\{M_n\}$ satisfy the above assumptions as well as $M_n \subset Y$' $n \in \mathbf{P}$, a generalized Jackson inequality of order $\alpha > 0$ with respect to Y:*

(J_Y) $\quad E_n(f; X) \leqq J_Y(\alpha) n^{-\alpha} |f|_Y \quad (f \in Y, n \in \mathbf{N})$

and a generalized Bernstein inequality of order $\alpha > 0$

(B_Y) $\quad |g_n|_Y \leqq B_Y(\alpha) n^\alpha \|g_n\|_X \quad (g_n \in M_n, n \in \mathbf{P}).$

Assume further that Z is a second subspace of X with $M_n \subset Z$ and seminorm $|\cdot|_Z$ such that Z is a Banach space under the norm $\|\cdot\|_Z = \|\cdot\|_X + |\cdot|_Z$, and that corresponding Jackson- and Bernstein inequalities (J_Z) and (B_Z) of order $\beta \geqq 0$ with respect to Z are valid.

If $\phi \in \Phi$ is such that

(3.8) $$\int_0^t u^{-1-\beta} \phi(u) du = O(t^{-\beta} \phi(t)) \quad (t \to 0+),$$

(3.9) $$\int_t^1 u^{-1-\alpha} \phi(u) du = O(t^{-\alpha} \phi(t)) \quad (t \to 0+),$$

then the following are equivalent for each $f \in X$:

(i) $E_n(f; X) = O(\phi(n^{-1})) \quad (n \to \infty)$.

(ii) $|g_n^*(f)|_Y = O(n^\alpha \phi(n^{-1})) \quad (n \to \infty)$.

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(iii) $f \in Z$ *and* $|f - g_n^*(f)|_Z = O(n^\beta \phi(n^{-1}))$ $(n \to \infty)$.
(iv) $K(t^\alpha, f; X, Y) = O(\phi(t))$ $(t \to 0+)$.

If condition (3.8) is not satisfied, assertion (iii) connected with the space Z must be dropped, and one has:

COROLLARY 1. *Under the assumption of Lemma 5 for X, Y, and $\{M_n\}$, the following are equivalent for each $\phi \in \Phi$ satisfying* (3.9), $\alpha > 0$, *and* (3.8) *for some $\beta \geqq 0$, and $f \in X$*:
(i) $E_n(f; X) = O(\phi(n^{-1}))$ $(n \to \infty)$,
(ii) $|g_n^*(f)|_Y = O(n^\alpha \phi(n^{-1}))$ $(n \to \infty)$,
(iii) $K(t^\alpha, f; X, Y) = O(\phi(t))$ $(t \to 0+)$.

In order to apply Lemma 5 to $X = X_{2\pi}$ we take $M_n = \Pi_n$, $Y = W_{X_{2\pi}}{}^\alpha$ for an $\alpha > 0$ with $|f|_Y = ||D^\alpha f||_{X_{2\pi}}$. Then $g_n^*(f) = t_n^*(f)$ exists. In view of Proposition 1 and Lemma 4, the Jackson and Bernstein inequalities of order α are valid for $W_{X_{2\pi}}{}^\alpha$. For Z take $W_{X_{2\pi}}{}^\beta$, $0 \leqq \beta < \alpha$, with $|f|_Z = ||D^\beta f||_{X_{2\pi}}$, *so* that (J_Z), (B_Z) are satisfied with order $\beta \geqq 0$. Finally take $\phi(t) = t^\theta$; then (3.8), (3.9) are satisfied, provided $\beta < \theta < \alpha$.

THEOREM 1. *The following assertions are equivalent for $f \in X_{2\pi}$, $0 \leqq \beta < \theta < \alpha$, $\alpha, \beta, \theta \in \mathbf{R}^+$*:
(i) $E_n(f; X_{2\pi}) = O(n^{-\theta})$ $(n \to \infty)$,
(ii) $||D^\alpha t_n^*(f)||_{X_{2\pi}} = O(n^{\alpha-\theta})$ $(n \to \infty)$,
(iii) $f \in W_{X_{2\pi}}{}^\beta$ *and* $||D^\beta f - D^\beta t_n^*(f)||_{X_{2\pi}} = O(n^{\beta-\theta})$ $(n \to \infty)$,
(iv) $K(t^\alpha, f; X_{2\pi}, W_{X_{2\pi}}{}^\alpha) = O(t^\theta)$ $(t \to 0+)$.

For functions ϕ that increase rather rapidly, such as $\phi(t) = \exp t^\alpha$, $\alpha > 0$, one has the counterpart of Theorem 1 only under additional assumptions (see [**7**]). If ϕ increases slowly, such as $\phi(t) = (1 + \log(1/t))^{-1}$, one does not have assertions of type (ii), (iii). Indeed, Corollary 1 yields:

THEOREM 2. *The following are equivalent for $f \in X$, $\alpha > 0$*:
(i) $E_n(f; X_{2\pi}) = O(1/\log n)$ $(n \to \infty)$,
(iv) $K(t^\alpha, f; X_{2\pi}, W_{X_{2\pi}}{}^\alpha) = O(1/\log(1/t))$ $(t \to 0+)$.

Since previous results of this type only deal with integral α, our next interest lies in making concrete conditions (iv) in Theorems 1 and 2 for arbitrary $\alpha > 0$. This leads to the second half of the main theorem, namely Theorem 3. Theorem 1 will also be useful to give a short proof for a reduction theorem for Lipschitz classes of fractional index (see Theorem 3, (v) $\Leftrightarrow$ (vii) and Remark 7).

4. Moduli of continuity of fractional index. The purpose here is to express assertions (iv) in Theorems 1 and 2 in terms of a modulus of continuity of fractional index, just as is the case for natural α. Also, most of the familiar properties of the classical moduli of continuity will be transferred to the fractional case. The present approach is made possible by the fact that the frac-

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tional derivative is defined by the same fractional difference as is the modulus of continuity of fractional index.

Definition 3. The *modulus of continuity of* $f \in X_{2\pi}$ *of index* $\alpha > 0$ is defined by

$$\omega_\alpha(f;\delta;X_{2\pi}) \equiv \omega_\alpha(f;\delta) = \sup_{0<h\leqq\delta} ||\Delta_h^\alpha f||_{X_{2\pi}} \quad (\delta > 0), \tag{4.1}$$

and the associated *Lipschitz class of index* α *and order* $\sigma > 0$ by

$$\mathrm{Lip}_\alpha(\sigma;X_{2\pi}) = \{f \in X_{2\pi};\, \omega_\alpha(f;\delta) = O(\delta^\sigma),\, \delta \to 0+\}. \tag{4.2}$$

Some elementary properties of these two concepts are collected in the following lemma.

LEMMA 6. *Let* $f,\, g \in X_{2\pi}$ *and* $\alpha,\, \beta > 0$.

(i) $\omega_\alpha(f;\,\delta)$ *is a non-negative, increasing function of* δ *on* $(0,\,\infty)$ *with* $\lim_{\delta\to 0+}\omega_\alpha(f;\delta) = 0$.

(ii) $\omega_\alpha(f;\delta) \leqq 2^{\{\alpha-\beta\}}\omega_\beta(f;\delta) \quad (\delta > 0; 0 < \beta \leqq \alpha)$.

(iii) $\omega_\alpha(f+g;\delta) \leqq \omega_\alpha(f;\delta) + \omega_\alpha(g;\delta) \quad (\delta > 0)$.

(iv) $\omega_\alpha(f;\,\delta) = o(\delta^\alpha) \quad (\delta \to 0+)$ *if and only if* $f =$ *constant* (*a.e.*).

If $f \in W_{X_{2\pi}}{}^\alpha$, *then*

(v) $\omega_\alpha(f;\delta) \leqq C(\alpha)\delta^\alpha||D^\alpha f||_{X_{2\pi}} \quad (\delta > 0)$,

(vi) $\omega_{\alpha+\beta}(f;\delta) \leqq C(\alpha)\delta^\alpha\omega_\beta(D^\alpha f;\delta) \quad (\delta > 0)$.

Proof. In (i), the convergence to zero for $\delta \to 0+$ follows by Lemma 1 (vi); (ii) is an obvious consequence of Lemma 1 (iv), (i), and (iii) follows readily by definition. Concerning (iv), $\omega_\alpha(f;\,\delta) = o(\delta^\alpha)$, $\delta \to 0+$, implies $D^\alpha f = 0$ by Definition 1, hence $f =$ constant (a.e.) by Lemma 2; the converse is trivial. For (vi) one has by (2.12), (2.5), Lemma 1 (iii), Lemma 3 (i), and (4.1)

$$\begin{aligned} ||\Delta_h^{\alpha+\beta} f||_{X_{2\pi}} &= h^\alpha||\Delta_h^\beta(D^\alpha(A_h^\alpha f))||_{X_{2\pi}} = h^\alpha||A_h^\alpha(\Delta_h^\beta(D^\alpha f))||_{X_{2\pi}} \\ &\leqq h^\alpha C(\alpha)||\Delta_h^\beta(D^\alpha f)||_{X_{2\pi}} \leqq \delta^\alpha C(\alpha)\omega_\beta(D^\alpha f;\delta) \quad (0 < h \leqq \delta, f \in X), \end{aligned}$$

and (v) is proved similarly.

Remark 1. Lemma 6 enables one to give a short proof of the Jackson inequality (Proposition 1) provided one assumes the validity of the classical Jackson *theorem* for natural α:

$$E_n(f;X_{2\pi}) \leqq J'(\alpha)\omega_\alpha(f;\pi/n) \quad (n \in \mathbf{N}, f \in X_{2\pi}). \tag{4.3}$$

Indeed, if $\alpha \notin \mathbf{N}$, apply (4.3) and use Lemma 6 (ii) to deduce

$$E_n(f;X_{2\pi}) \leqq J'(\{\alpha\})2^{\{\alpha\}-\alpha}\omega_\alpha(f;\pi/n). \tag{4.4}$$

This yields Proposition 1 by Lemma 6(v).

PROPOSITION 2. *For each* $f \in X_{2\pi}$, $\alpha > 0$, *there exist constants* $n(\alpha)$, $N(\alpha) > 0$, *such that for* $0 < t < \infty$

$$n(\alpha)\omega_\alpha(f;t;X_{2\pi}) \leqq K(t^\alpha, f; X_{2\pi}, W_{X_{2\pi}}{}^\alpha) \leqq N(\alpha)\omega_\alpha(f;t;X_{2\pi}). \tag{4.5}$$

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Proof. Setting $f = (f - g) + g$ for an arbitrary $g \in W_{X_{2\pi}}{}^{\alpha}$, we have by Lemma 6(iii), (v) and Lemma 1(i),

$$\omega_\alpha(f; t) \leqq \omega_\alpha(f - g; t) + \omega_\alpha(g; t) \leqq 2^{\{\alpha\}}||f - g||_{X_{2\pi}} + C(\alpha)t^\alpha||D^\alpha g||_{X_{2\pi}}.$$

Taking the infimum over all $g \in W_{X_{2\pi}}{}^{\alpha}$ establishes the left hand side of (4.5) with $[n(\alpha)]^{-1} = \max\{2^{\{\alpha\}}, C(\alpha)\}$.
Setting

$$g_t(x) = -\sum_{j=1}^{r} (-1)^j \binom{r}{j} (A_{tj/r}{}^r f)(x) \quad \text{for some } r > \alpha,\ r \in \mathbf{N},$$

we have $g_t \in W_{X_{2\pi}}{}^{\alpha}$ for all $t > 0$ by Lemma 3(iii). By (2.11),

$$||D^\alpha(A_{tj/r}{}^r f)||_{X_{2\pi}} \leqq C(r - \alpha)\frac{r^\alpha}{(tj)^\alpha}||\Delta_{tj/r}{}^\alpha f||_{X_{2\pi}} \leqq C(r - \alpha)\left(\frac{r}{j}\right)^\alpha \frac{1}{t^\alpha}\omega_\alpha(f; t)$$

for $1 \leqq j \leqq r$. Hence it follows that

$$(4.6) \quad t^\alpha||D^\alpha g_t||_{X_{2\pi}} \leqq C(r - \alpha)r^\alpha\omega_\alpha(f; t)\sum_{j=1}^{r} j^{-\alpha}\binom{r}{j} = C^*(\alpha)\omega_\alpha(f; t) \quad (0 < t < \infty).$$

In view of (2.10) and (2.1) one has

$$||f - g_t||_{X_{2\pi}} = \left\| f + \sum_{j=1}^{r} (-1)^j \times \binom{r}{j}\int_0^1 \cdots \int_0^1 f\left(x - \frac{tj}{r}(u_1 + \ldots + u_r)\right)du_1 \ldots du_r\right\|_{X_{2\pi}}$$

$$\leqq \int_0^1 \cdots \int_0^1 ||\Delta^r{}_{(t/r)(u_1+\ldots+u_r)} f||du_1 \ldots du_r$$

$$\leqq \sup_{0<h\leq t} ||\Delta_h{}^r f||_{X_{2\pi}} = \omega_r(f; t) \leqq 2^{\{r-\alpha\}}\omega_\alpha(f; t),$$

the latter inequality following by Lemma 6(ii). Combining this with (4.6), it follows that

$$K(t^\alpha, f; X_{2\pi}, W_{X_{2\pi}}{}^{\alpha}) \leqq ||f - g_t||_{X_{2\pi}} + t^\alpha||D^\alpha g_t||_{X_{2\pi}} \leqq N(\alpha)\omega_\alpha(f; t),$$

where $N(\alpha) = \max\{C^*(\alpha), 2^{r-\alpha}\}$, which is the right-hand side of (4.3).

Remark 2. If one assumes the validity of (4.3), one can give a very different proof of the right-hand side of (4.5). Here we make use of the fractional M. Riesz inequality (a generalization of Bernstein's inequality) established recently by Nessel-Wilmes [**13**].

LEMMA 7. *For each $t_n \in \Pi_n$, $n \in \mathbf{N}$, $\alpha > 0$*

$$||D^\alpha t_n||_{X_{2\pi}} \leqq (n/2)^\alpha||\Delta_{\pi/n}{}^\alpha t_n||_{X_{2\pi}}.$$

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With regard to this proof, let $0 < t \leqq 2\pi$. Then there is $n \in \mathbf{N}$ such that $(\pi/n) < t \leqq 2(\pi/n)$. Setting $E_n(f; X_{2\pi}) = ||f - t_n^*||_{X_{2\pi}}$, one has for each $\alpha > 0$, by (4.6),

$$||f - t_n^*||_{X_{2\pi}} \leqq J'(\{\alpha\})2^{\{\alpha\}-\alpha}\omega_\alpha(f; \pi/n).$$

On the other hand, by Lemma 7 and Lemma 1(i),

$$\begin{aligned} ||D^\alpha t_n^*||_{X_{2\pi}} &\leqq (n/2)^\alpha ||\Delta_{\pi/n}{}^\alpha t_n^*||_{X_{2\pi}} \\ &\leqq (\pi/t)^\alpha \{2^{\{\alpha\}}||f - t_n^*||_{X_{2\pi}} + ||\Delta_{\pi/n}{}^\alpha f||_{X_{2\pi}}\} \\ &\leqq \pi^\alpha(2^{2\{\alpha\}-\alpha}J'(\{\alpha\}) + 1)t^{-\alpha}\omega_\alpha(f; \pi/n). \end{aligned}$$

This yields, by Lemma 6(i), that

$$K(t^\alpha, f; X_{2\pi}, W_{X_{2\pi}}{}^\alpha) \leqq ||f - t_n^*||_{X_{2\pi}} + t^\alpha||D^\alpha t_n^*||_{X_{2\pi}} \leqq N'(\alpha)\omega_\alpha(f; t)$$

with $N'(\alpha) = (2^{2\{\alpha\}-\alpha}J'(\{\alpha\}) + 1)\pi^\alpha$.

Remark 3. A simple corollary of Proposition 2 and of a property of the K-functional is a corresponding property of the modulus of continuity, namely

$$\omega_\alpha(f; \lambda\delta) \leqq (N(\alpha)/n(\alpha))\lambda^\alpha\omega_\alpha(f; \delta) \quad (\lambda \geqq 1; \delta > 0; \alpha > 0). \tag{4.7}$$

For arbitrary $\alpha > \beta > 0$, $t \in (0, 1)$, $f \in X_{2\pi}$, one has

$$K(t^\beta, f; X_{2\pi}, W_{X_{2\pi}}{}^\beta) \leqq G(\alpha, \beta)t^\beta \times \left\{||f||_{X_{2\pi}} + \int_t^1 u^{-\beta-1}K(u^\alpha, f; X_{2\pi}, W_{X_{2\pi}}{}^\alpha)du\right\}.$$

This is a by-product of the proof of Lemma 5 (cf. [**1**; **7**, Theorem 3]). Combining this with Proposition 2 one has the following fractional analog of Marchaud's inequality (for the integer case cf. [**10**] and the literature cited there).

COROLLARY 2. *For each* $f \in X_{2\pi}$, $\alpha > \beta > 0$, $t \in (0, 1)$:

$$\omega_\beta(f; \delta; X_{2\pi}) \leqq G(\alpha, \beta)[N(\alpha)/n(\alpha)]t^\beta \times \left\{||f||_{X_{2\pi}} + \int_t^1 u^{-\beta-1}\omega_\alpha(f; u; X_{2\pi})du\right\}.$$

5. Lipschitz classes of fractional index.

THEOREM 3. (a) *Let* $f \in X_{2\pi}$. *Assertions* (v)-(viii) *are equivalent to assertions* (i)-(iv) *of Theorem* 1 *for* $0 \leqq \beta < \theta < \alpha$, $\alpha, \beta, \theta, \tau \in \mathbf{R}^+$:

(v) $\omega_\alpha(f; t; X_{2\pi}) = O(t^\theta) \quad (t \to 0+)$, i.e. $f \in \mathrm{Lip}_\alpha(\theta; X_{2\pi})$,

(vi) $D^\beta f \in \mathrm{Lip}_\tau(\theta - \beta; X_{2\pi}) \quad (\theta - \beta < \tau)$,

(vii) $D^\beta f \in \mathrm{Lip}_{\alpha-\beta}(\theta - \beta; X_{2\pi})$,

(viii) $D^\beta f \in \begin{cases} \mathrm{Lip}_1(\theta - \beta; X_{2\pi}), & 0 < \theta - \beta < 1, \\ \mathrm{Lip}_2(\theta - \beta; X_{2\pi}), & 0 < \theta - \beta < 2. \end{cases}$

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(b) *Assertions* (i), (iv) *of Theorem* 2 *for arbitrary* $\alpha > 0$ *are equivalent to*
(v′) $\omega_\alpha(f; t; X_{2\pi}) = O(1/\log(1/t)) \quad (t \to 0+)$.

Proof. (a) The fact that (v) is equivalent to (iv) and so also to (i)-(iii) of Theorem 2 follows by Proposition 2. If (v) holds, then one has by (iii) that $f \in W_{X_{2\pi}}{}^{\beta}$ and

$$E_n(D^\beta f; X_{2\pi}) \leqq ||D^\beta f - D^\beta t_n^*(f)||_{X_{2\pi}} = O(n^{\beta-\theta}) \quad (n \to \infty),$$

that is, (i) is satisfied for $D^\beta f$ and $\theta - \beta$. In view of the equivalence of (i) with (v) there holds $\omega_\tau(D^\beta f; t; X_{2\pi}) = O(t^{\theta-\beta})$, $(t \to 0+)$ for any $\tau > \theta - \beta$. Thus (vi) follows. Then (vii) follows by setting $\tau = \alpha - \beta$ in (vi). Finally, if (vii) holds, then

$$\omega_\alpha(f; t) \leqq C(\beta) t^\beta \omega_{\alpha-\beta}(D^\beta f; t) = O(t^\theta)$$

by Lemma 6 (vi), which is (v). If $0 < \theta - \beta < 1$ or $0 < \theta - \beta < 2$, one can choose $\tau = 1$ and $\tau = 2$ in (vi), which shows that (viii) is equivalent to the other assertions.

(b) Proposition 2 again yields the equivalence of (iv) with (v′).

Remark 4. The cases $\alpha = 1, 2$ of Theorem 3 for the fractional Weyl derivative $f^{(\beta)}$ are due to Hardy-Littlewood and Zygmund (see [**17**, p. 136]). They showed, in the case $\alpha = 1$, that

$$f \in \mathrm{Lip}_1(\theta, C_{2\pi}) \Rightarrow f^{(\beta)} \in \mathrm{Lip}_1(\theta - \beta; C_{2\pi})$$

if $0 < \beta < \theta < 1$, and for $\alpha = 2$, $0 < \beta < 1$ that

$$f \in \mathrm{Lip}_2(1; C_{2\pi}) \Rightarrow f^{(\beta)} \in \mathrm{Lip}_2(1 - \beta; C_{2\pi}).$$

The corresponding converse directions are known for arbitrary integral α and the space $X_{2\pi}$ (see [**2**, p. 427]).

Remark 5. The assertions (i) and (vi) of Theorems 1 and 3, respectively, yield the generalization of the theorems of Jackson and Bernstein to the fractional instance, namely that

$$E_n(f; X_{2\pi}) = O(n^{-\theta}) \quad (n \to \infty) \quad \Leftrightarrow \quad D^\beta f \in \mathrm{Lip}_\tau(\theta - \beta; X_{2\pi})$$

provided $0 \leqq \beta < \theta$ and $\tau > \theta - \beta$, $f \in X_{2\pi}$.

In particular, this gives the existence of the derivatives $D^\beta f$ of order $\beta < \theta$ arbitrarily close to θ. In the classical integral case, β can only be chosen as $[\theta]$ at most if $\theta \notin \mathbf{N}$, and as $\beta = \theta - 1$ if $\theta \in \mathbf{N}$.

Remark 6. Assertion (viii) of Theorem 3 can be used if one wishes to characterize Lipschitz classes with fractional index and fractional order derivatives by integral ones. For, to any $\theta > 0$ there is always $\beta \in \mathbf{P}$ such that $0 < \theta - \beta \leqq 1$.

Remark 7. Clearly Theorem 3 implies that assertion (v) is independent of α for all $\alpha > \theta$, i.e.

$$\mathrm{Lip}_{\alpha_1}(\theta; X_{2\pi}) = \mathrm{Lip}_{\alpha_2}(\theta; X_{2\pi}) \quad (\alpha_1, \alpha_2 > \theta).$$

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If $\alpha_1 > \alpha_2 \geqq \theta$ then, by Lemma 6(ii), generally only

$$\text{Lip}_{\alpha_1}(\theta; X_{2\pi}) \supset \text{Lip}_{\alpha_2}(\theta; X_{2\pi}). \tag{5.1}$$

A similar remark applies to (v').

Remark 8. The authors would like to thank the referee for calling their attention to the question of whether the inclusion (5.1) is strict in case $\alpha_2 = \theta$. Let us sketch a proof for the case $X_{2\pi} = C_{2\pi}$ (setting $\alpha_1 = \alpha$). The $C_{2\pi}$-function $f(x) = \sum_{k=0}^{\infty} 3^{-\theta k} \cos 3^k x$ satisfies $E_n(f; C_{2\pi}) = \sum_{j=k}^{\infty} 3^{-\theta k} = O(n^{-\theta})$, $n \to \infty$ where k is chosen such that $3^k \leqq n < 3^{k+1}$ (cf. [**15**, p. 77; **17**, p. 73]), whence $f \in \text{Lip}_\alpha(\theta; C_{2\pi})$, for each $\alpha > \theta$ in view of Theorem 3. On the other hand, if f would be in $\text{Lip}_\theta(\theta, C_{2\pi})$, by [**5**, Theorem 7.1] there would exist $g \in L_{2\pi}^\infty$ such that $(ik)^\theta f\hat{}(k) = g\hat{}(k)$ for each $k \in \mathbf{Z}$. However, for $n = 3^k$, $k = 1, 2, \ldots,$ one easily verifies that $|(\text{in})^\theta f\hat{}(n)| = 1/2$, which is a contradiction to $\lim_{n\to\infty} g\hat{}(n) = 0$. By the same arguments, using the same function f, one can also show that (5.1) is strict for $L_{2\pi}^p$, $1 < p < \infty$.

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