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ON NONINNER AUTOMORPHISMS OF FINITE NONABELIAN p-GROUPS

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Abstract

A long-standing conjecture asserts that every finite nonabelian p-group has a noninner automorphism of order p. In this paper the verification of the conjecture is reduced to the case of p-groups G satisfying $Z_2^\star(G) \leq C_G(Z_2^\star(G)) = \Phi(G)$, where $Z_2^\star(G)$ is the preimage of $\Omega_1(Z_2(G)/Z(G))$ in G. This improves Deaconescu and Silberberg's reduction of the conjecture: if $C_G(Z(\Phi(G))) \neq \Phi(G)$, then G has a noninner automorphism of order p leaving the Frattini subgroup of G elementwise fixed ['Noninner automorphisms of order p of finite p-groups', *J. Algebra* **250** (2002), 283–287].

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1. Introduction

Let p be a prime and G be a finite nonabelian p-group. A longstanding conjecture asserts that G has a noninner automorphism of order p [12, Problem 4.13]. This conjecture is still open. In fact, the statement of the conjecture is a sharpened version of a well-known and nontrivial property of finite p-groups that, with the exception of groups of order p, they always have a noninner automorphism of p-power order [7].

The conjecture has been established for p-groups of class 2 and 3 [2, 3, 11], for regular p-groups [13], for p-groups G in which $G/Z(G)$ is powerful [1], for p-groups G in which $(G, Z(G))$ is a Camina pair and $p \neq 2$ [9], for 2-groups with a cyclic commutator subgroup [10], and for p-groups of order p^m and exponent p^{m-2} [14]. It is worth noting that most of the noninner automorphisms given in these results leave either $\Phi(G)$ or $Z(G)$ elementwise fixed. Also, Deaconescu and Silberberg have proved that if $C_G(Z(\Phi(G))) \neq \Phi(G)$, then G has a noninner automorphism of order p leaving $\Phi(G)$ elementwise fixed [5]. Hence, they have reduced the verification of the conjecture to the degenerate case in which

$$C_G(Z(\Phi(G))) = \Phi(G). \qquad (*)$$

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The main motivation of the present paper is to reduce the verification of the conjecture further. In addition, our aim is to find a noninner automorphism of order p which acts trivially on a maximal subgroup of G. Let $Z_2^\star(G)/Z(G) = \Omega_1(Z_2(G)/Z(G))$, where for a finite p-group H, $\Omega_1(H) = \langle h \in H \mid h^p = 1\rangle$. Our main results are as follows.

THEOREM 1.1. *Let p be a prime and G be a finite nonabelian p-group. If G fails to fulfil the condition*

$$Z_2^\star(G) \le C_G(Z_2^\star(G)) = \Phi(G), \tag{$**$}$$

then G has a noninner automorphism of order p leaving the Frattini subgroup of G elementwise fixed. Moreover, if p is odd, then the noninner automorphism can be taken such that it acts trivially on a maximal subgroup of G.

Theorem 1.1 reduces the verification of the conjecture to the case of finite p-groups satisfying ($**$). Let $\mathscr{G}_p^*$ and $\mathscr{G}_p^{**}$ denote the sets of all finite p-groups with the properties ($*$) and ($**$), respectively. Then the following theorem holds.

THEOREM 1.2. *For every prime p, $\mathscr{G}_p^{**} \subseteq \mathscr{G}_p^*$ and $\mathscr{G}_p^* \setminus \mathscr{G}_p^{**}$ contains infinitely many p-groups.*

Therefore the result of this paper extends known classes of finite p-groups for which the conjecture holds.

2. Preliminaries

Let G be a finite nonabelian p-group. By $\mathscr{M}(G)$ we denote the set of all maximal subgroups of G. If $x \in G$ and $H \le G$, then $\overline{x}$ and $\overline{H}$ denote the coset $x\Phi(G)$ and the quotient group $H\Phi(G)/\Phi(G)$, respectively. The inner automorphism of G induced by x is denoted by θ_x. Also, we denote the direct product of groups $G_1, G_2, \ldots, G_n$, by $\mathrm{Dr}\Pi_{i=1}^n G_i$. Any unexplained notation is standard and follows that of [8]. We use the following facts in the proofs.

REMARK 2.1. Let $n \in \mathbb{N}$, $x, y \in G$ and $a \in Z_2(G)$.

- $(xa)^n = x^n a^n [a, x]^{\binom{n}{2}}$.
- $[x^n, a] = [x, a]^n = [x, a^n]$.
- $[x, ay] = [x, a][x, y]$.
- Moreover, if $a^p \in Z(G)$ then $[a, \Phi(G)] = 1$.

REMARK 2.2. Let G be a finite p-group, M be a maximal subgroup of G and $g \in G \setminus M$. Let $u \in Z(M)$ such that $(gu)^p = g^p$. Then the map α given by $g \mapsto gu$ and $m \mapsto m$, for all $m \in M$, can be extended to an automorphism of order $|u|$ that acts trivially on M.

REMARK 2.3 [5, Remark 4]. Let G be a central product of subgroups A and B; that is, $G = AB$ and $[A, B] = 1$. Suppose that $\alpha \in \mathrm{Aut}(A)$ and $\beta \in \mathrm{Aut}(B)$ agree on $A \cap B$. Then α and β admit a common extension $\gamma \in \mathrm{Aut}(G)$. In particular, if A has a noninner automorphism of order p which fixes $Z(A)$, then G has a noninner automorphism of order p which fixes $Z(A)$ and B.

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Remark 2.4. Let A and B be two elementary abelian finite p-groups. The set of all homomorphisms from A to B, which is denoted by $\mathrm{Hom}(A, B)$, forms an elementary abelian p-group by $+$ operation (that is, $(f+g)(a) = f(a)g(a)$ for $f, g \in \mathrm{Hom}(A, B)$ and $a \in A$). Let $A = \mathrm{Dr}\Pi_{i=1}^{m}\langle a_i\rangle$ and $B = \mathrm{Dr}\Pi_{i=1}^{n}\langle b_i\rangle$, where $m = d(A)$ and $n = d(B)$. For $1 \le i \le m$ and $1 \le j \le n$, the map $f_{i,j} : A \to B$ defined by $a_k \mapsto b_j^{\delta_{k,i}}$, where δ is the Kronecker delta, can be extended to a homomorphism from A to B. Furthermore $\{f_{i,j} \mid 1 \le i \le m,\ 1 \le j \le n\}$ is a minimal generating set for $\mathrm{Hom}(A, B)$. Thus $\mathrm{Hom}(A, B) \cong \mathrm{Dr}\Pi_{i=1}^{n} A$ is of rank $d(A)d(B)$.

The latter remark becomes obvious when it is realised that A and B are vector spaces over the field of p elements.

Remark 2.5. Let G be a finite nonabelian p-group such that $\Omega_1(Z(G)) \le \Phi(G)$. If $f \in \mathrm{Hom}(\overline{G}, \Omega_1(Z(G)))$ then the map $\sigma_f : G \to G$ defined by $x \mapsto xf(\overline{x})$ is an automorphism of order p. In addition, if $\ker(f) \in \mathscr{M}(\overline{G})$ then σ_f acts trivially on a maximal subgroup of G.

3. Proofs of the main results

Let $Z_2^\star(G)/Z(G) = \Omega_1(Z_2(G)/Z(G))$. In the following lemmas we derive some properties of $Z_2^\star(G)$.

Lemma 3.1. *If G is a finite p-group, then* $[Z_2^\star(G), \Phi(G)] = 1$.

Proof. This follows immediately from Remark 2.1. □

Lemma 3.2. *Let $H \le G$ and $a \in Z_2^\star(G)$. Then the map ${}_H\varphi_a : \overline{H} \to \Omega_1(Z(G))$, given by $\overline{h} \mapsto [h, a]$, for $h \in H$, is a homomorphism. Also, the map*

$$ {}_H\varphi : Z_2^\star(G) \longrightarrow \mathrm{Hom}(\overline{H}, \Omega_1(Z(G))), $$

defined by $a \mapsto {}_H\varphi_a$, for $a \in Z_2^\star(G)$, is a homomorphism and $\ker({}_H\varphi) = Z_2^\star(G) \cap C_G(H)$.

Proof. This is straightforward. □

The following propositions relate $Z_2^\star(G)$ to the automorphisms of order p which act trivially on a maximal subgroup of G.

Proposition 3.3. *Let p be an odd prime and G be a finite nonabelian p-group such that $Z(G)$ is cyclic and $Z_2^\star(G)/Z(G)$ is not cyclic. Then G has a noncentral automorphism of order p leaving a maximal subgroup of G elementwise fixed.*

Proof. By hypothesis, $Z_2^\star(G)/Z(G) = \langle aZ(G)\rangle \times \langle bZ(G)\rangle \times L/Z(G)$, for some $a, b \in Z_2^\star(G) \setminus Z(G)$ and $L \le Z_2^\star(G)$. Since $Z(G)$ is cyclic, we may assume that $b^p = a^{p^i j}$, for some integers i, j. Let $u = ba^{p^{i-1} j}$, $M = C_G(u)$ and ${}_G\varphi_u$ be the homomorphism given in Lemma 3.2. Then it follows that $M = \ker({}_G\varphi_u) \in \mathscr{M}(G)$. Now let $g \in G \setminus M$. Since u is an element of order p in $Z_2^\star(G) \setminus Z(G)$ and $(gu)^p = g^p$, the result follows from Remark 2.2. □

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Proposition 3.4. *Let G be a finite nonabelian p-group and $H \leq G$. If*

$$d(Z_2^\star(G)/Z_2^\star(G) \cap C_G(H)) \neq d(\overline{H})d(Z(G)),$$

then G has a central noninner automorphism of order p which acts trivially on a maximal subgroup of G.

Proof. By a well-known argument (or applying Remark 2.2), we may assume that $Z(G) \leq \Phi(G)$. Now, let ${}_H\varphi$ be the homomorphism given in Lemma 3.2. Then

$$\ker({}_H\varphi) = \frac{Z_2^\star(G)}{Z_2^\star(G) \cap C_G(H)}.$$

By hypothesis, $H \nleq \Phi(G)$ and ${}_H\varphi$ is not an epimorphism. Thus for some $1 \leq i \leq d(\overline{H})$ and $1 \leq j \leq d(Z(G))$, $f_{i,j} \notin Im(\varphi)$, where $f_{i,j}$ is as in Remark 2.4. If necessary, extend $\{x_1, \ldots, x_s\}$ to a minimal generating set $\{x_1, \ldots, x_s, \ldots, x_d\}$ of G. For $1 \leq k \leq d$, set

$$f(\overline{x_k}) = \begin{cases} f_{i,j}(\overline{x_k}) & 1 \leq k \leq s, \\ 1 & s < k \leq d. \end{cases}$$

Then f determines an element of $\mathrm{Hom}(\overline{G}, \Omega_1(Z(G)))$. By Remark 2.5, σ_f is an automorphism of G of order p that fixes a maximal subgroup of G elementwise. If $\sigma_f = \theta_a$ is inner, then one must have $a \in Z_2^\star(G)$. Thus for $x \in H$, $\sigma_f(x) = \theta_a(x)$ and hence

$$f_{i,j}(\overline{x}) = x^{-1}\sigma_f(x) = [x, a] = \varphi_a(\overline{x}).$$

This means that $f_{i,j} \in Im(\varphi)$, a contradiction. Therefore σ_f is noninner and the result follows. □

Proposition 3.5. *Let p be a prime and G be a finite p-group. If $C_G(Z_2^\star(G)) \neq \Phi(G)$, then G has a central noninner automorphism of order p leaving a maximal subgroup of G elementwise fixed.*

Proof. Assume that G is a counterexample to the theorem. Let $M \in \mathscr{M}(G)$ and $g \in G \setminus M$. Let u be an element of order p in $Z(G) \cap M$. Then by Remark 2.2 the map α given by $g \mapsto gu$ and $m \mapsto m$, for all $m \in M$, can be extended to an automorphism of order p that leaves M elementwise fixed. By assumption $\alpha = \theta_{x_M}$, for some $x_M \in G$. Therefore $x_M \in Z_2^\star(G)$ and $M = C_G(x_M)$. By Lemma 3.1, $\Phi(G) \leq C_G(Z_2^\star(G))$. Therefore

$$\Phi(G) \leq C_G(Z_2^\star(G)) \leq \bigcap_{M \in \mathscr{M}(G)} C_G(x_M) = \bigcap_{M \in \mathscr{M}(G)} M = \Phi(G),$$

and the result follows. □

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Second Proof. Let G be a counterexample to the theorem. For $x \in C_G(Z_2^\star(G))$, let $H = \langle \bar{x} \rangle$. Then it follows from Proposition 3.4, that $x \in \Phi(G)$. Therefore $C_G(Z_2^\star(G)) \leq \Phi(G)$. Now the result follows from Lemma 3.1. □

Proposition 3.6. *Let p be a prime and G be a finite p-group of class 2. If either $p > 2$ or $Z(G)$ is not cyclic then* Aut(G) *contains a noninner automorphism of order p leaving a maximal subgroup of G elementwise fixed. In addition, if $Z(G)$ is not cyclic, then the noninner automorphism can be taken to be central.*

Proof. Let G be a counterexample to the proposition. By Theorem 3.5, $C_G(Z_2^\star(G)) = \Phi(G)$. Thus $Z(G) \leq \Phi(G)$ and since G is of class 2, one has $d(Z_2^\star(G)/Z(G)) = d(G/Z(G)) = d(G)$. Now if $d(Z(G)) > 1$, then the result follows from Proposition 3.4, and if $d(Z(G)) = 1$ and $p > 2$, then Proposition 3.3 completes the proof. □

Theorem 3.6 does not hold for 2-groups of class 2 in general. Indeed, there are examples of groups of class 2 in which every automorphism of order two fixing $\Phi(G)$ elementwise is inner [1, 11].

The following result improves [11, Part (1) of Theorem].

Proposition 3.7. *Let p be a prime and G be a finite p-group such that $Z_2^\star(G)$ is not abelian. If p is odd then G has a noninner automorphism of order p leaving a maximal subgroup of G elementwise fixed, and if $p = 2$ then G has a noninner automorphism of order two leaving the Frattini subgroup of G elementwise fixed.*

The proof of Proposition 3.7 requires the following preliminary fact. Recall that a finite nonabelian p-group, all of whose maximal subgroups are abelian, is called a minimal nonabelian p-group or Rédei p-group.

Remark 3.8. Let G be a Rédei p-group. If p is odd then G has a noninner automorphism of order p leaving a maximal subgroup of G elementwise fixed, and if $p = 2$ then G has a noninner automorphism of order p leaving $\Phi(G)$ elementwise fixed. The former follows from Theorem 3.6, since Rédei p-groups have nilpotency class 2, and the latter has been proved by using the classification of Rédei 2-groups [5, Remark 3].

Proof of Proposition 3.7. Assume that G is a counterexample of minimal order to the proposition.

First we prove that $\overline{Z_2^\star(G)}$ is not cyclic. Suppose to the contrary that $\overline{Z_2^\star(G)} = \langle \bar{u} \rangle$, for some $u \in Z_2^\star(G)$. If $x, y \in Z_2^\star(G)$, then $x = u^i a$ and $y = u^j b$ for some $i, j \in \mathbb{N}$ and $a, b \in \Phi(G) \cap Z_2^\star(G)$. Now it follows from Lemma 3.1 that $[x, y] = 1$. But this means that $Z_2^\star(G)$ is abelian, a contradiction.

Then, by Proposition 3.4,

$$d(Z_2^\star(G)/Z_2^\star(G) \cap C_G(Z_2^\star(G))) = d(\overline{Z_2^\star(G)})d(Z(G));$$

and by Proposition 3.5, $C_G(Z_2^\star(G)) = \Phi(G)$. Therefore $Z(G)$ is cyclic.

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Next, suppose that $a, b \in Z_2^\star(G)$ such that $[a, b] \neq 1$. Let $K = \langle a, b\rangle$ and $L = C_G(K)$. Note that $[K, G] = K' = \Omega_1(Z(G)) = \langle [a, b]\rangle$. Hence, if $x \in G$, then $[a, x] = [a, b]^s$ and $[b, x] = [a, b]^t$, for some integers s, t. Thus, $[a, b^{-s}a^t x] = 1$ and $[b, b^{-s}a^t x] = 1$. Therefore $b^{-s}a^t x \in C_G(K)$ and it follows that G is the central product of K and L. Moreover, K is a Rédei p-group. Hence, by Remark 3.8, $K \lneq G$.

Finally, if p is odd, then by assumption K has a noninner automorphism α of order p that acts trivially on a maximal subgroup M of K. By Remark 2.3, α can be extended to a noninner automorphism of G of order p that fixes ML. Since $Z(K) = \langle [a, b], a^p, b^p\rangle = \Phi(K)$, we have $K \cap L = Z(K) = \Phi(K) = M \cap L$ and

$$\frac{|G|}{|ML|} = \frac{|K||L|/|K \cap L|}{|M||L|/|M \cap L|} = \frac{|K|}{|M|} = p.$$

Therefore $ML \in \mathscr{M}(G)$, a contradiction. Also, if $p = 2$, then a similar argument gives a contradiction. □

Proof of Theorem 1.1. This follows immediately form Propositions 3.5 and 3.7. □

To prove Theorem 1.2, we use the following observation.

Lemma 3.9. *If G_1 belongs to $\mathscr{G}_p^* \setminus \mathscr{G}_p^{**}$, then so does $G_1 \times G_2$, for all $G_2 \in \mathscr{G}_p^*$.*

Proof. The result follows immediately from the following elementary facts. Let G_1 and G_2 be two finite p-groups. Let $H_1 \leq G_1$ and $H_2 \leq G_2$. Set $G = G_1 \times G_2$ and $H = H_1 \times H_2$. Then $\Phi(G) = \Phi(G_1) \times \Phi(G_2)$ and $C_G(H) = C_{G_1}(H_1) \times C_{G_2}(H_2)$. □

Proof of Theorem 1.2. Let $G \in \mathscr{G}_p^{**}$. Then by Lemma 3.1, $Z_2^\star(G) \leq Z(\Phi(G))$. Therefore,

$$\Phi(G) = C_G(Z_2^\star(G)) \geq C_G(Z(\Phi(G))) \geq \Phi(G).$$

This proves the first part of the theorem. For the second part, by Lemma 3.9 it suffices to show that for every prime p, $\mathscr{G}_p^* \setminus \mathscr{G}_p^{**} \neq \varnothing$. First, assume that $p > 3$ and let G be a group with the following power-commutator presentation:

$$\begin{aligned} G = \mathrm{Pc}\langle g_1, g_2, g_3, g_4, g_5 | g_1^p = g_2^p = g_3^p = g_4^p = g_5^p = 1,\\ g_3 = [g_2, g_1],\ g_4 = [g_3, g_1],\ g_5 = [g_4, g_1],\\ [g_5, g_1] = 1,\ [g_3, g_2] = g_5,\ [g_4, g_2] = 1,\ [g_5, g_2] = 1\\ [g_4, g_3] = 1,\ [g_5, g_3] = 1,\ [g_5, g_4] = 1\rangle, \end{aligned}$$

To show the consistency of this presentation, it suffices to check that for each of the following pairs of test words the collections of both words coincide (see [15, page 424] and [4, Lemma 2.1]).

(i) $(g_k g_j)g_i$ and $g_k(g_j g_i)$, for $1 \leq i < j < k \leq 5$,
(ii) g_i and $g_j^{p-1}(g_j g_i)$, for $1 \leq i < j \leq 5$,
(iii) g_j and $(g_j g_i)g_i^{p-1}$, for $1 \leq i < j \leq 5$.

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Checking (i) is straightforward and one may use induction to check (ii) and (iii). For instance, by induction on i, we get $g_2g_1^i = (g_2g_1)g_1^{i-1} = g_1^i g_2 g_3^i g_4^{i(i-1)/2} g_5^{i(i-1)(i-2)/6}$. Therefore the collection of $(g_2g_1)g_1^{p-1}$ coincides with g_2.

Now the consistency of the presentation implies that G is of order p^5 and class 4. Thus G is of maximal class. Let $Z_2(G)/Z(G) = \langle uZ(G)\rangle$, for some $u \in Z_2(G)$. Then by an easy argument as in the proof of Proposition 3.3, $C_G(Z_2^\star(G)) = C_G(u) \neq \Phi(G)$. On the other hand, $\Phi(G) = G'$ is abelian and $C_G(Z(\Phi(G))) = \Phi(G)$. Therefore, $G \in \mathcal{G}_p^* \setminus \mathcal{G}_p^{**} \neq \varnothing$.

Now suppose that $p \leq 3$. We use the following code in GAP [6] to complete the proof in this case.

```
f:=function(p,n)
local k,q,g,u,v,t;
k:=NumberSmallGroups(p^n);
q:=0;
for j in [1..k] do
  g:=SmallGroup(p^n,j);
  z:=Center(g);
  u:=Center(FactorGroup(g,z));
  v:= Omega(u,p);
  map:=NaturalHomomorphismByNormalSubgroup( g,z );
  w:=PreImagesSet(map,v);
  phi:=FrattiniSubgroup(g);
  if Centralizer(g,w)<> phi and
     Centralizer(g,Center(phi))=phi
     then q:=q+1; break;
  fi;
od;
return(q);
end;
```

This code accepts prime p and positive integer n. Then it returns 1 if there exists a group G of order p^n in the GAP small groups library such that $G \in \mathcal{G}_p^* \setminus \mathcal{G}_p^{**}$, otherwise it returns 0. We see that `f(2,7)=1` and `f(3,5)=1`, which completes the proof of the theorem. □

We end the paper by answering the natural question that arises here: 'Is there any finite p-group of class two in $\mathcal{G}_p^* \setminus \mathcal{G}_p^{**}$?'

Proposition 3.10. *Let G be a finite nonabelian p-group of class 2. Then $G \in \mathcal{G}_p^*$ if and only if $G \in \mathcal{G}_p^{**}$.*

Proof. By Theorem 1.2, it is enough to prove the 'only if' part. In fact we prove that if $G \in \mathcal{G}_p^*$ is of class 2, then $Z_2^\star(G) = Z(\Phi(G))$. Suppose that G is a finite p-group of class 2 such that $C_G(Z(\Phi(G)) = \Phi(G)$. Then $C_G(\Phi(G)) = Z(\Phi(G))$ and it follows from

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Lemma 3.1 that $Z_2^\star(G) \leq Z(\Phi(G))$. Now let $a \in Z(\Phi(G))$. Thus $1 = [a, x^p] = [a^p, x]$, for every $x \in G$. Therefore $a^p \in Z(G)$ which means that $a \in Z_2^\star(G)$. Hence $Z(\Phi(G)) \leq Z_2^\star(G)$, and the result follows. □

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