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# A Residue Formula for SU(2)-Valued Moment Maps

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*Abstract.* Jeffrey and Kirwan gave expressions for intersection pairings on the reduced space $M_0 = \mu^{-1}(0)/G$ of a Hamiltonian $G$-space $M$ in terms of multiple residues. In this paper we prove a residue formula for symplectic volumes of reduced spaces of a quasi-Hamiltonian SU(2)-space. The definition of quasi-Hamiltonian $G$-spaces was introduced by Alekseev, Malkin and Meinrenken.

## 1 Introduction

Let $(M, \omega)$ be a compact symplectic manifold with Hamiltonian action of a compact Lie group $G$ and $\mu$ be the corresponding moment map. Jeffrey and Kirwan [5] gave expressions for general intersection pairings on the homotopy quotient $M_0 = \mu^{-1}(0)/G$ in terms of multiple residues of certain integrals over connected components of the fixed point set $M^T$ of the maximal torus $T \subset G$.

In [1, 2, 3], Alekseev, Malkin, Meinrenken and Woodward developed a theory of quasi-Hamiltonian $G$-spaces with the moment map taking values in a Lie group $G$. For such spaces, they introduce analogs of "classical" Hamiltonian reduction, Liouville volumes, Duistermaat-Heckmann measure, localization formulas, etc. The goal of this note is to obtain a quasi-Hamiltonian residue localization formula for the special case $G = \mathrm{SU}(2)$.

We first introduce some notation. Let $T = S^1 \subset \mathrm{SU}(2)$ be a choice of a Cartan circle of $G = \mathrm{SU}(2)$ with $\mathfrak{t}, \mathfrak{g}$ the corresponding Lie algebras. Fix a positive Weyl chamber $\mathfrak{t}_+ \subset \mathfrak{t}$. Let $\alpha$ be the unique positive root of $\mathfrak{t}$, and $\lambda \in \mathfrak{t}^*$ the fundamental weight. We identify $\mathfrak{g}$ and $\mathfrak{g}^*$ via a scalar product on $\mathfrak{g}$ such that $(\alpha, \alpha) = 2$, and write $\alpha = 2\lambda$. Note that this choice of the inner product implies that $\operatorname{Vol} T = \sqrt{2}$, $\operatorname{Vol} G = \sqrt{2}/2\pi$. We parameterize the alcove $\mathfrak{A} \cong [0, 1]$, $t \in [0, 1]$ corresponding to $t\lambda \in \mathfrak{t}$.

We shall briefly describe the results of [1, 2]. As we are concerned with the case $G = \mathrm{SU}(2)$, we give below the formulas applied to SU(2) and do not state the results for the general case of a compact connected group $G$. Let $\theta = g^{-1}\, \mathrm{d} g$ and $\overline{\theta} = \mathrm{d} g g^{-1}$ denote the left- and right-invariant Maurer-Cartan forms on $G = \mathrm{SU}(2)$, and let $\chi$ be the canonical closed bi-invariant 3-form on $G$,

$$\chi = \frac{1}{12}(\theta, [\theta, \theta]) = \frac{1}{12}(\overline{\theta}, [\overline{\theta}, \overline{\theta}]).$$

***Definition 1*** ([1]) A quasi-Hamiltonian $G$-space is a $G$-manifold with an invariant 2-form $\omega \in \Omega(M)^G$ and an equivariant map $\Phi \in C^\infty(M, G)^G$ (the "moment map"), such that:

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(i) The differential of $\omega$ is given by $\mathrm{d}\,\omega = -\Phi^*\chi$.
(ii) The moment map satisfies

$$\imath(v_\xi) = \frac{1}{2}\Phi^*(\theta + \overline{\theta}, \xi).$$

(iii) At each $x \in M$, the kernel of $\omega_x$ is given by

$$\ker \omega_x = \{v_\xi, \xi \in \ker(\mathrm{Ad}_{\Phi(x)} + 1)\}.$$

Similarly to the Meyer-Marsden-Weinstein reduction, the quasi-Hamiltonian reduced phase spaces are defined in the following way. Let $g \in G$ be a regular value of the moment map $\Phi$, then the pre-image $\Phi^{-1}(g)$ is a smooth submanifold on which the action of the centralizer $G_g$ is locally free. The reduced space $M_g = \Phi^{-1}(g)/G_g$ is a symplectic orbifold.

As in the Hamiltonian case, there is the volume form $\Gamma$ on $M$, called the Liouville form. For $G = \mathrm{SU}(2)$ it is defined as follows. By equivariance of the moment map it suffices to define $\Gamma$ for points $x \in M$ such that $\Phi(x) \in T$. If $\Phi(x) = \exp(t\lambda)$, and $t \neq \pm 1/2$, then

$$\Gamma_x = \frac{1}{\cos \pi t}(\exp \omega)_{[\mathrm{top}]}.$$

The denominator $\cos \pi t$ cancels the zeroes of $\exp \omega$ arising because of condition (iii), so $\Gamma$ extends smoothly to all points of $\Phi^{-1}(T)$.

The Duistermaat-Heckman measure $\varrho$ on $G$ is now defined as the push-forward of the Liouville form under the moment map,

$$\varrho = \Phi_*\Gamma.$$

We shall consider the DH function, *i.e.*, the density of this measure with respect to the Haar measure, and use the same notation for it, writing

$$\varrho = \varrho(g)\,\mathrm{d}\,\mathrm{Vol}_G\,.$$

As in the Hamiltonian setting, the Duistermaat-Heckmann measure is related to volumes of the reduced spaces. Let $g \in G$ be a regular value of $\Phi$, and $M_g$ the corresponding reduced space. Then $\varrho$ is smooth at $g$. Let $k$ be the cardinality of a generic stabilizer for $G_g$-action on $\Phi^{-1}(g)$. Then for the reduced space at $g = \exp(t\lambda)$ with $t \in (0, 1)$ we have

$$\mathrm{Vol}(M_g) = k\frac{2 \sin \pi t}{\sqrt{2}}\varrho(g), \tag{1}$$

and for central elements $g = \pm e$

$$\mathrm{Vol}(M_g) = k\frac{2\pi}{\sqrt{2}}\varrho(g). \tag{2}$$

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In the case of Hamiltonian torus actions, the localization formula by Berline-Vergne[4] gives an expression for the Fourier-Laplace transform of the DH function in terms of integrals over fixed point manifolds of the torus subgroup generated by $\xi$,

$$\int_M e^{i\pi\langle\Phi,\xi\rangle}\exp\omega = \sum_{F\in\mathcal{F}(\xi)}\int_F \frac{\exp(\omega_F)}{\mathrm{Eul}(\nu_F,\xi)}e^{i\pi\langle\Phi,\xi\rangle},$$

where $\mathrm{Eul}(\nu_F,\xi)$ is the equivariant Euler class. The quasi-Hamiltonian counterpart of this formula, valid for arbitrary compact connected Lie group $G$, deals with the Fourier coefficients of the Duistermaat-Heckmann function, which localize on the fixed point manifolds of certain circle subgroups of $G$.

Let $\mathcal{F}$ be the set of connected components $F \subset M$ of the fixed point set of the Cartan circle $T \subset \mathrm{SU}(2)$. Each $F \in \mathcal{F}$ is a symplectic manifold with the pull-back $\omega_F$ of $\omega$ as a 2-form. By equivariance, the restriction $\Phi|_F$ is constant and sends $F$ to a point of $T$. Write $\Phi_F = \exp(\mu_F\lambda)$, introducing the numbers $\mu_F \in (-1,1]$, and let $\Phi_F^{n\lambda}$ stand for $e^{\pi i n\mu_F}$. We denote by $\mathcal{F}_+$ the set of components $F \in \mathcal{F}$ with $\Phi \in \mathfrak{A}$, that is, $\mu_F \geq 0$. Orientations of $M$ and $F$ induce the orientation on the normal bundle $\nu_F$, and the $T$-equivariant Euler class $\mathrm{Eul}(\nu_F,\,\cdot\,)$ is defined.

Recall that irreducible representations of SU(2) are labelled by their highest weights $n\lambda$, $n = 0,1,2,\ldots$. The dimension of the representation $V_n$ with the highest weight $n\lambda$ is $\dim V_n = n+1$. Let $\chi_n$ denote the character of $V_n$.

***Theorem 1*** (quasi-Hamiltonian localization formula, [2]) *The Fourier coefficients of the DH function are given by*

$$\langle\varrho,\chi_n\rangle = \dim V_n \sum_{F\in\mathcal{F}}\int_F \frac{\exp(\omega_F)}{\mathrm{Eul}(\nu_F,2\pi i(n+1)\lambda)}\,\Phi_F^{(n+1)\lambda}. \tag{3}$$

The DH function is then reconstructed as $\varrho(g) = \frac{1}{\mathrm{Vol}\,G}\sum_n\langle\varrho,\chi_n\rangle\chi_n(g^{-1})$.

## 2 The Residue Formula

The purpose of this note is to obtain a quasi-Hamiltonian residue formula for the Liouville volumes of the reduced phase spaces for SU(2)-actions. Because of (1) and (2), this is equivalent to giving a formula for the DH function $\varrho(g)$. Since $\varrho(g)$ is a function of conjugacy classes, it suffices to evaluate it at the elements of the Cartan circle, $g = \exp(t\lambda)$.

***Theorem 2*** *The function $\varrho(t)$ is a sum of contributions $\varrho_F(t)$ of the components $F \in \mathcal{F}_+$,*

$$\varrho(\exp(t\lambda)) = \sum_{F\in\mathcal{F}_+}\varrho_F(\exp(t\lambda)),$$

*where $\varrho_F(\exp(t\lambda))$ corresponding to $\mu_F \in (0,1)$ are given by*

$$\varrho_F(\exp(t\lambda)) = -\frac{4\pi^2 i}{\sqrt{2}}\frac{1}{\sin\pi t}\,\mathrm{Res}_0\,\frac{z e^{\pi i z\mu_F}\sin(\pi t z)}{e^{2\pi i z}-1}\int_F\frac{\exp(\omega_F)}{\mathrm{Eul}(\nu_F,2\pi i z\lambda)} \tag{4}$$

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*for* $0 < t < \mu_F$, *and*

$$(5)\quad \varrho_F(\exp(t\lambda)) = \frac{4\pi^2 i}{\sqrt{2}} \frac{1}{\sin \pi t} \operatorname{Res}_0 \frac{z e^{\pi i z(\mu_F+1)} \sin\bigl(z\pi(1-t)\bigr)}{e^{2\pi i z}-1} \int_F \frac{\exp(\omega_F)}{\operatorname{Eul}(\nu_F, 2\pi i z\lambda)}$$

*for* $\mu_F < t < 1$.

*If* $\mu_F = 0$ *or* 1, (4) *and* (5) *are valid if the right hand side is multiplied by* $1/2$.

*If* $t = 0$ *or* $t = 1$ *is a regular value of* $\varrho(\exp(t\lambda))$ (i.e., *e or* $-e$ *is a regular value of the moment map), then*

$$(6)\qquad \varrho_F(e) = -\frac{4\pi^2 i}{\sqrt{2}} \operatorname{Res}_0 \frac{z^2 e^{\pi i z \mu_F}}{e^{2\pi i z}-1} \int_F \frac{\exp(\omega_F)}{\operatorname{Eul}(\nu_F, 2\pi i z\lambda)},$$

$$(7)\qquad \varrho_F(-e) = \frac{4\pi^2 i}{\sqrt{2}} \operatorname{Res}_0 \frac{z^2 e^{iz(\mu_F+\pi)}}{e^{2\pi i z}-1} \int_F \frac{\exp(\omega_F)}{\operatorname{Eul}(\nu_F, 2\pi i z\lambda)}.$$

Note that in the case $G = \mathrm{SU}(2)$ the Jeffrey-Kirwan formula for the Liouville volume of $\mu^{-1}(0)/G$ (*i.e.,* the value of the standard DH function at zero) gives

$$\varrho(0) = -\frac{1}{2} \operatorname{Res}_0 z^2 \sum_{F\in\mathcal{F}_+} e^{iz\mu_T(F)} \int_F \frac{e^{i\omega_F}}{\operatorname{Eul}(\nu_F, z)},$$

$\mathcal{F}_+$ denoting the set of the components $F$ for which $\mu_T(F) > 0$. Even in the case of Hamiltonian $G$-spaces our formulas look different as we work with discrete Fourier series instead of continuous Fourier transform used in[5].

In the proof of the Theorem, we shall use the following lemma.

***Lemma 1*** *Let* $f(z)$ *be a rational function with the only pole at zero, such that* $f(z) \to 0$ *as* $z \to \infty$. *Then for* $0 < \gamma < 2\pi$

$$(8)\qquad \sum_{m\in\mathbb{Z}, m\neq 0} e^{im\gamma} f(m) = -2\pi i \operatorname{Res}_0 \frac{f(z)e^{i\gamma z}}{e^{2\pi i z}-1},$$

*and for* $-2\pi < \gamma < 0$

$$(9)\qquad \sum_{m\in\mathbb{Z}, m\neq 0} e^{im\gamma} f(m) = -2\pi i \operatorname{Res}_0 \frac{f(z)e^{i\gamma z}}{1-e^{-2\pi i z}}.$$

This is a one-dimensional case of a lemma of Szenes [7]; although in [7] the rational function $f$ is required to be of order $-2$ or less, the result holds true for functions of order $-1$ as well.

**Proof of Theorem 2** We shall make use of Theorem 1 and sum up the Fourier series. Introduce the functions $\tilde{\varrho}_F(\exp(t\lambda))$ with the Fourier coefficients

$$(10)\qquad \langle \tilde{\varrho}_F, \chi_n\rangle = \dim V_n \int_F \frac{\exp(\omega_F)}{\operatorname{Eul}(\nu_F, 2\pi i(n+1)\lambda)} \Phi_F^{(n+1)\lambda},$$

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so that by (3)

$$\varrho(t) = \sum_{F\in\mathcal{F}} \tilde{\varrho}_F(t).$$

Recall that all the characters of SU(2) are real and have the property $\chi_n(g) = \chi_n(g^{-1})$, so we can just write $\varrho(g) = \frac{2\pi}{\sqrt{2}} \sum \langle \varrho, \chi_n\rangle \chi_n(g)$. Further, the character $\chi_n$ is given at $g = \exp(t\lambda)$ by the Weyl character formula,

$$\chi_n(\exp(t\lambda)) = \frac{\sin(\pi(n+1)t)}{\sin \pi t}.$$

Substituting this into (10) and putting together the Fourier series, we get

$$\tilde{\varrho}_F(\exp(t\lambda)) = \frac{2\pi}{\sqrt{2}} \sum_{n\geq 0} \frac{n+1}{2i \sin \pi t} (e^{\pi i(n+1)t} - e^{\pi i(n+1)t}) \int_F \frac{\exp(\omega_F)}{\operatorname{Eul}(\nu_F, 2\pi i(n+1)\lambda)} \Phi_F^{(n+1)\lambda}.$$

We now gather the components $F \in \mathcal{F}$ of the fixed point set into pairs $F, F' = F^w$ with $F \in \mathcal{F}_+$ by means of the Weyl element $w = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. For $F \in \mathcal{F}$ with $\mu_F \neq 0, 1$ we let $\varrho_F = \tilde{\varrho}_F + \tilde{\varrho}_{F'}$. If $\mu_F = 0$ or 1, then $F = F'$, and we denote $\varrho_F = \tilde{\varrho}_F = \frac{1}{2}(\tilde{\varrho}_F + \tilde{\varrho}_{F'})$. Then

$$\varrho(\exp(t\lambda)) = \sum_{F\in\mathcal{F}_+} \varrho_F\big(\exp(t\lambda)\big).$$

Use the change of variables $y = x^w$ to relate the functions $\tilde{\varrho}_F$ and $\tilde{\varrho}_{F'}$. For $h \in S^1$ we have $hw = wh^{-1}$, so the action of $2\pi in\lambda$ at $\nu_{F'}$ corresponds to the action of $-2\pi in\lambda$ at $\nu_F$. The form $\omega$ is invariant, then

$$\int_{F'} \frac{\exp(\omega_{F'})}{\operatorname{Eul}(\nu_{F'}, 2\pi in\lambda)} = \int_F \frac{\exp(\omega_F)}{\operatorname{Eul}(\nu_F, -2\pi in\lambda)}.$$

By the equivariance of the moment map, $\Phi_{F'} = \Phi_F^{-1}$.

Now pick together the terms with $e^{\pi i(n+1)t}$ from $\tilde{\varrho}_F$ and those with $e^{-\pi i(n+1)t}$ from $\tilde{\varrho}_{F'}$, and vice versa. Substituting $m = n+1$ and $\Phi_F^{n\lambda} = e^{\pi in\mu_F}$, we get

(11) $\varrho_F(\exp(t\lambda))$

$$= \frac{2\pi}{\sqrt{2}} \frac{1}{2i \sin \pi t} \sum_{m\in\mathbb{Z}, m\neq 0} m(e^{\pi im(t+\mu_F)} - e^{-\pi im(t-\mu_F)}) \int_F \frac{\exp(\omega_F)}{\operatorname{Eul}(\nu_F, 2\pi im\lambda)}.$$

(Here we assume that $\Phi_F \neq \pm e$; otherwise a factor 1/2 is needed).

We assume that the moment map has regular points. It follows that the codimension of $F$ in $M$ must be at least 4. Then the rational function $1/\operatorname{Eul}(\nu_F, 2\pi iz\lambda)$ is of order of at least $1/z^2$. It means that (11) consists of two series to which the Lemma applies.

If $t < \mu_F$, both series sum up by means of (8), and we get

$$\varrho_F(t) = -\frac{4\pi^2 i}{\sqrt{2}} \frac{1}{\sin \pi t} \operatorname{Res}_0 \frac{z e^{\pi iz\mu_F} \sin(\pi tz)}{e^{2\pi iz} - 1} \int_F \frac{\exp(\omega_F)}{\operatorname{Eul}(\nu_F, 2\pi iz\lambda)},$$

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which is exactly (4). If $t > \mu_F$, we use both (8) and (9), and obtain the formula (5). To get the formulas for $\varrho_F$ with $\Phi_F = \pm e$ we must divide (4) and (5) by 2.

If $e$ or $-e$ are the regular values of the moment map, we can consider the limits $t \to 0$ and $t \to 1$ to obtain the formulas (6) and (7), respectively. ∎

***Example 1*** We apply the obtained residue formulas to the following example of a quasi-Hamiltonian SU(2)-space from [2]. The space is constructed as follows. Take $\mathbb{C}^2$ equipped with its natural symplectic form and defining SU(2)-action, and let $\Phi_0\colon \mathbb{C}^2 \to \mathrm{su}(2)$ be its classical moment map. Let $Y_1 = Y_2 \subset \mathbb{C}^2$ be the open ball given as the pre-image $\Phi_0^{-1}(G \cdot [0,\lambda))$, and $\Phi_{1,0} = \Phi_{2,0}$ the restrictions of $\Phi_0$. Then

$$Y_3 = \Phi_{1,0}^{-1}((0,\lambda))$$

as a Hamiltonian U(1)-space is equivariantly symplectomorphic to

$$Y_3' = \Phi_{2,0}^{-1}((-\lambda,0))$$

via the isomorphism $(z,\xi) \to (z,\xi-\lambda)$. We now glue the spaces $Y_1$ and $Y_2$ together along their boundaries by means of the embeddings

$$Y_1 \leftarrow \mathrm{SU}(2) \times_{U(1)} Y_3 \to Y_2,$$

and obtain a sphere $S^4$ with SU(2) acting by rotations.

The action has two fixed points, one with $\Phi = e$, the other with $\Phi = -e$.The Euler classes are given by $\mathrm{Eul}(\nu_F,\xi) = \mp\langle\lambda,\xi\rangle^2$ for $\xi \in \mathfrak{t}$, so $\mathrm{Eul}(\nu_F, 2\pi i z\lambda) = \pm\pi^2 z^2$.

Denoting by $\varrho_0$, $\varrho_1$ the contributions of the fixed points, we write

$$\varrho\big(\exp(t\lambda)\big) = \varrho_0\big(\exp(t\lambda)\big) + \varrho_1\big(\exp(t\lambda)\big).$$

Using the formula (4) (with factor 1/2), we get

$$\varrho_1\big(\exp(t\lambda)\big) = \frac{2\pi^2 i}{\sqrt{2}}\frac{1}{\sin\pi t}\operatorname{Res}_0 \frac{ze^{\pi i z t}/\sin(\pi t z)}{e^{2\pi i z}-1}\frac{1}{\mathrm{Eul}(\nu_1, 2\pi z\lambda)}.$$

Computing the residue, we get

$$\varrho_1\big(\exp(t\lambda)\big) = \frac{t}{\sqrt{2}\sin\pi t}.$$

In the same way

$$\varrho_0\big(\exp(t\lambda)\big) = \frac{1-t}{\sqrt{2}\sin\pi t},$$

hence

$$\varrho\big(\exp(t\lambda)\big) = \frac{1}{\sqrt{2}\sin\pi t}.$$

Now use (1), and get $\mathrm{Vol}\, M_g = 1$. This is obviously the correct answer, because the reduced spaces are just points.

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***Example 2*** Let us now apply our residue formula to an SU(2)-space $\mathrm{SU}(2)^{2n}$ obtained by means of a fusion product introduced in [1].

Given a quasi-Hamiltonian $G \times G$-space $M$ with 2-form moment map $\Phi = (\Phi_1, \Phi_2)$, we can consider $M$ as a $G$-space with a diagonal $G$-action. Then $M$ with moment map $\tilde{\Phi} = \Phi_1\Phi_2$ and 2-form $\tilde{\omega} = \omega + \frac{1}{2}(\Phi_1^*\theta, \Phi_2^*\overline{\theta})$ is a quasi-Hamiltonian $G$-space. Taking two quasi-Hamiltonian $G$-spaces $M_1$, $M_2$, we consider the diagonal $G$-action on the $G \times G$-space $M_1 \times M_2$, and obtain the fusion product $M_1 \circledast M_2$.

Now, following [1], consider a double $D(\mathrm{SU}(2))$, that is, the $SU(2) \times \mathrm{SU}(2)$-space $\mathrm{SU}(2) \times \mathrm{SU}(2)$ defined as follows. The group action is given by

$$(a, b)^{(g_1,g_2)} = (g_1 a g_2^{-1}, g_2 b g_1^{-1}),$$

the moment map is $\Phi = (\Phi_1, \Phi_2)$, where

$$\Phi_1(a, b) = ab, \quad \Phi_2(a, b) = a^{-1}b^{-1},$$

and the 2-form is

$$\omega_D = \frac{1}{2}(a^*\theta, b^*\overline{\theta}) + \frac{1}{2}(a^*\overline{\theta}, b^*\theta).$$

We can get a $G$-space $\mathbf{D}(\mathrm{SU}(2))$, applying fusion to $D(\mathrm{SU}(2))$. The group SU(2) acts by conjugation on each factor of $\mathbf{D}(\mathrm{SU}(2)) = \mathrm{SU}(2) \times \mathrm{SU}(2)$, and the moment map is $\Phi(a, b) = aba^{-1}b^{-1} = [a, b]$.

We shall consider the quasi-Hamiltonian SU(2)-space $\mathrm{SU}(2)^{2n}$ obtained as a fusion product of $n$ copies of $\mathbf{D}(\mathrm{SU}(2))$. The fixed point set of the $T$-action has only one component $F$, which is a product of $2n$ copies of $T \subset \mathrm{SU}(2)$. The moment map $\Phi$ sends this set to $e$, so we must insert the factor $1/2$ into the formula (5). To compute the Euler class $\mathrm{Eul}(\nu_F, \cdot)$, note that the normal bundle is trivial and can be identified with $(\mathfrak{g}/\mathfrak{t})^{2n}$, and $\mathrm{Eul}(\nu_F, 2\pi z\lambda) = z^{2n}(\langle 2\lambda, 2\pi i\lambda\rangle\langle -2\lambda, 2\pi i\lambda\rangle)^n = (2z)^{2n}\pi^{2n}$.

For the double $D(\mathrm{SU}(2))$, the form $\exp(\tilde{\omega}_{T\times T})$ coincides with the Riemannian volume form induced by our choice of the inner product on $\mathfrak{g}$. As the restriction of the moment map to the torus in the fusion product is trivial, at points of the torus this form remains intact under fusion. Then the form $\exp(\omega_F)_{[top]}$ on $F = T^{2n}$ is just the Riemannian volume form, and we have

$$\int \exp(\omega_F) = (\mathrm{Vol}\, T)^{2n} = 2^n.$$

Substituting everything into (5), we get

$$\varrho\big(\exp(t\lambda)\big) = \frac{2\pi}{\sqrt{2}} \frac{i}{2^n\pi^{2n-1}\sin\pi t} \operatorname{Res}_0 \frac{e^{\pi i z}\sin\big(z\pi(1-t)\big)}{z^{2n-1}(e^{2\pi i z}-1)}.$$

This can be re-written as

$$\varrho\big(\exp(t\lambda)\big) = \frac{i\sqrt{2}}{2^n\pi^{2n-2}(2n-2)!\sin\pi t} \frac{\partial^{2n-2}}{\partial z^{2n-2}}\Big(\frac{e^{\pi i z}\sin\big(z\pi(1-t)\big)}{e^{2\pi i z}-1}\Big)\Big|_{z=0}.$$

If $n = 1$, for the space $\mathbf{D}(\mathrm{SU}(2))$ we get

$$\varrho(\exp(t\lambda)) = \frac{1-t}{2\sqrt{2}\sin \pi t}$$

for $t \in (0, 1)$.

To find the volumes of the reduced spaces, note that the generic stabilizer is just the center of SU(2), which consists of two points. Using (1), we have

$$\mathrm{Vol}(M_g) = 1 - t.$$

As shown in [3], fusion products can be interpreted as moduli spaces of flat connections on surfaces, so the above computation yields the answer given by Witten's formula for symplectic volumes of the corresponding moduli spaces [8, 6, 3].

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