Canad. Math. Bull. Vol. **33** (4), 1990

A NOTE ON DUBINS' THEOREM

BY
GUTTI JOGESH BABU

ABSTRACT. Let p be the space of probability measures on a compact Hausdorff space. Recently Dubins has characterized the continuous funcitons on p using operator theory. We shall prove the same using probabilistic arguments.

Let K be a compact Hausdorff space. Let p be the space of probability measures, on the Baire σ-field of K, with the topology given by the weak convergence. Recently Dubins (1983) has shown that for every continuous function g on p, there exists a continuous function f on K^∞ such that $g(P) = \int f\, dP^\infty$, where P^∞ denotes the countably infinite product of P with itself. He uses operator theoretic methods to prove the result. We shall give a purely probabilistic proof based on Bernstein polynomial approximation. Dubins' proof is divided into two parts. We shall prove his Theorem 1 using probabilistic arguments. His Theorem 2 involves straightforward computation of certain limits. For completeness, the proof of Dubins' Lemma 1, which essentially establishes his Theorem 2 is reproduced here, modulo a translation of notation, as part of the proof of the corollary. Let $\underset{\sim}{x} = (x_1, x_2, \ldots)$ denote a point in K^∞ and let $D_n(\underset{\sim}{x})$ denote the probability measure which puts weights $\frac{1}{n}$ at $x_1, \ldots, x_n$.

THEOREM. *(Dubins' Theorem 1). Let g be a continuous function on p. Then $\int g(D_n(\underset{\sim}{x}))\, dP^\infty$ converges uniformly to $g(P)$ on p.*

PROOF. Fix an $\varepsilon > 0$. As p is compact, $|g(P)| \leq M$ for all P in p, for some $M > 0$. Since g is continuous, for each P in p, there exists an open set $U(P)$ such that $|g(P) - g(Q)| < \varepsilon$ for all Q in $U(P)$. Without loss of generality we can take

$$U(P) = \left\{ Q: \left| \int h_i d(P-Q) \right| < 2\delta(P),\ i = 1, \ldots, k \right\},$$

where $h_1, \ldots, h_k$ are continuous functions on K satisfying $|h_i| \leq 1$ for $i = 1, \ldots, k$ and $\delta(P) > 0$. Let

$$V(P) = \left\{ Q : \left| \int h_i d(P-Q) \right| < \delta(P),\ i = 1, \ldots, k \right\}.$$

As p is compact and since $\{ V(P): P \in p\}$ is an open cover of p, there exist $P_1, \ldots, P_r$ such that $\bigcup_{j=1}^r V(P_j) \supset p$. Let $\{ h_{ij} : i = 1, \ldots, k_j,\ j = 1, \ldots, r\}$ be continuous functions on K such that $|h_{ij}| \leq 1$ and

$$V(P_j) = \left\{ Q : \left| \int h_{ij} d(P_j - Q) \right| < \delta(P_j)\ i = 1, \ldots, k_j \right\}.$$

Received June 8, 1988, revised July 7, 1989.
AMS subject classification: 60B99.

Let $\delta = \min\{\delta(P_1),\ldots,\delta(P_r)\}$ and $k = k_1 + \cdots + k_r$. Now if $P \in V(P_j)$, then

$$
\begin{aligned}
\Big|\int g(P) - \int g(D_n(\underline{x})dP^\infty\Big| &\le \int_{U(P_j)} |g(P) - g(D_n(\underline{x}))|\,dP^\infty \\
&\quad + \int_{p-U(P_j)} |g(P) - g(D_n(\underline{x}))|\,dP^\infty \qquad (1)\\
&\le 2\varepsilon + 2MP^\infty\big(D_n(\underline{x}) \notin U(P_j)\big).
\end{aligned}
$$

Since P is in $V(P_j)$, $|\int h_{ij}d(P-P_j)| < \delta(P_j)$ and since $\{h_{ij}(x_s) : s = 1,\ldots,n\}$ are i.i.d. random variables under P^∞, it follows that

$$
\begin{aligned}
P^\infty\big(D_n(\underline{x}) \notin U(P_j)\big) &\le P^\infty\Big(\bigcup_{i,j}\big(|\frac{1}{n}\sum_{s=1}^{n} h_{ij}(x_s) - \int h_{ij}\,dP|\big) > \delta\big)\Big) \\
&\le k \max_{i,j} P^\infty\big(|\frac{1}{n}\sum_{s=1}^{n} h_{ij}(x_s) - \int h_{ij}\,dP| > \delta\big) \qquad (2)\\
&\le k\delta^{-2}n^{-1} \max_{i,j} \mathrm{Var}_P(h_{ij}) \\
&\le 4k\delta^{-2}n^{-1}.
\end{aligned}
$$

From (1) and (2) we have

$$\limsup_{n\to\infty} \sup_{P\in p} |g(P) - \int g(D_n(\underline{x}))\,dP^\infty| \le 2\varepsilon.$$

The result now follows as $\varepsilon > 0$ is arbitrary. Another proof using functional analytic techniques was given by B. V. Rao and S. C. Bagchi.

COROLLARY. *For every continuous function g on p, there exists a continuous function f on K^∞ such that*

$$g(P) = \int f\,dP^\infty.$$

PROOF. For any continuous funciton g on p, let

$$\|g\| = \sup\{|g(P)| : P \in p\}.$$

For any contiuous f on K^∞ and $P \in p$, let

$$Tf(P) = \int f\,dP^\infty.$$

Let f_n be a sequence of continuous functions defined on K^∞ recursively by $f_0 = 0$ and

$$f_{n+1} = f_n + (g - Tf_n)(D_m),$$

where $m = m(g,n)$ is minimal with the property

$$\sup_{P\in p}\Big|g(P) - Tf_n(P) - \int (g - Tf_n)(D_m)\,dP^\infty\Big| \le \frac{1}{2}\|g - Tf_n\|. \qquad (3)$$

https://doi.org/10.4153/CMB-1990-067-x Published online by Cambridge University Press

Such an m exists in view of the theorem. Note that for any continuous function g on p and a positive integer r,

$$\sup_{\underline{x}\in K^\infty} |g(D_r(\underline{x})| \le \|g\|. \tag{4}$$

Clearly by (4) we have

$$\begin{aligned}\sup_{\underline{x}\in K^\infty} |f_{n+1}(\underline{x}) - f_n(\underline{x})| &= \sup_{\underline{x}\in K^\infty} |(g - Tf_n)(D_m)(\underline{x})| \\ &\le \|g - Tf_n\|.\end{aligned} \tag{5}$$

Further by (3)

$$\begin{aligned}\|g - Tf_n\| &= \sup_{P\in p} \left| g(P) - Tf_{n-1}(P) - \int (g - Tf_{n-1})\big(D_{m(g,n-1)}\big)\, dP^\infty \right| \\ &\le \frac{1}{2}\|g - Tf_{n-1}\| \\ &\le \frac{1}{2^2}\|g - Tf_{n-2}\| \\ &\vdots \\ &\le \frac{1}{2^{n-1}}\left\|g - \int g(D_{m(g,0)})\, dP^\infty\right\| \\ &\le \frac{1}{2^n}\|g\|.\end{aligned} \tag{6}$$

From (5) and (6) it follows that f_n is a Cauchy sequence of continuous functions on K^∞, in uniform norm. So the sequence $\{f_n\}$ has a limit, say f. Clearly

$$\begin{aligned}\left|\int f\, dP^\infty - g(P)\right| &\le \left|\int (f - f_n)\, dP^\infty\right| + \left|\int f_n\, dP^\infty - g(P)\right| \\ &\le \sup_{\underline{x}\in K^\infty} |(f - f_n)(\underline{x})| + \frac{1}{2^n}\|g\| \\ &\longrightarrow 0\end{aligned}$$

as $n \longrightarrow \infty$. This establishes the corollary.

REFERENCES

1. L. E. Dubins *Bernstein like polynomial approximation in higher dimensions*, Pacific J. Math. **109** (1983) 305 – 311.

Department of Statistics
219 Pond Laboratory
University Park, PA 16802
U.S.A.

https://doi.org/10.4153/CMB-1990-067-x Published online by Cambridge University Press