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POSITIVE HARMONIC FUNCTIONS AND COMPLETE METRICS

BY

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ABSTRACT. We introduce the class of Harnack domains in which a Harnack type inequality holds for positive harmonic functions with bounds given in terms of the distance to the domain's boundary. We give conditions connecting Harnack domains with several different complete metrics. We characterize the simply connected plane domains which are Harnack and discuss associated topics. We extend classical results to Harnack domains and give applications concerning the rate of growth of various functions defined in Harnack domains. We present a perhaps new characterization for quasidisks.

1. **Introduction.** A classical result of Hardy and Littlewood relates the Hölder continuity of a function analytic in the unit disk to the rate of growth of its derivative. In this note we begin a study of Harnack domains. These are the domains in which a Harnack type inequality describes the rate of growth of positive harmonic functions relative to the Euclidean distance to the boundary of the domain.

In Section 2 we define Harnack domains. We give an alternative description for them in terms of the Harnack metric and use this to give a sufficient condition for a domain to be Harnack. In Section 3 we characterize the simply connected plane domains which are Harnack. We discuss Dini domains and show that each of these is Harnack. Then we point out an apparently new representation for the class of quasidisks. In Section 4 we extend some classical results for functions defined in the unit disk to functions defined in Harnack domains and we give various function theoretic applications of our ideas.

Throughout this article $d(\cdot,\cdot)$ denotes Euclidean distance, D denotes a proper subdomain of Euclidean n-dimensional space $\mathbf{R}^n$ and $\mathcal{H}^+(D)$ denotes the class of positive harmonic functions defined in D. Often $n = 2$ and we identify the point (x, y) in $\mathbf{R}^2$ with the complex number $z = x + iy$ in $\mathbf{C}$. We use the symbols α, β, a, b, c to denote constants which may not be the same in different occurrences; we write $c = c(a)$ or $c = c(f)$ to explicitly indicate a constant which may depend on the number a or the function f.

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2. **Harnack domains.** We call $D \subset \mathbf{R}^n$ a β-*Harnack* domain, $\beta > 0$, if there exists a fixed point $x_0 \in D$, called the *center*, and a positive constant c such that

$$d(x, \partial D)^\beta / c \leqq u(x)/u(x_0) \leqq c/d(x, \partial D)^\beta \quad \text{for all } x \in D \tag{2.1}$$

for every function $u \in \mathcal{H}^+(D)$.

REMARKS 2.2. (a) Note that (2.1) really is a condition which describes the boundary behaviour of positive harmonic functions in D. Indeed, if D is any domain and E is any compact connected subset of D, then one can always find (cf. [HV, 2.6 and 3.2]) a constant $b = b(n, \operatorname{dia}(E), d(E, \partial D))$ such that

$$1/b \leqq u(x)/u(x_0) \leqq b \quad \text{for all } x, x_0 \in E$$

and then

$$d(x, \partial D)/c \leqq u(x)/u(x_0) \leqq c/d(x, \partial D) \quad \text{for all } x, x_0 \in E.$$

(b) Whether or not a domain is Harnack does not depend on the center. More precisely, if D is β-Harnack with center x_0 and constant c_0, then D is also β-Harnack with center x_1 and constant $c_1 = c_0^2/d(x_1, \partial D)$ for any x_1 in D. (c) By considering constants we see that the function $d(x, \partial D)$ must be bounded in a Harnack domain D. Thus Harnack domains are Bloch domains, i.e., they cannot contain arbitrarily large balls. (d) By considering the harmonic function $u(x) = 1/|x - y_0|^{n-2}$, where $y_0 \in \partial D$ is chosen so that $d(x_0, \partial D) = |x_0 - y_0|$, we see that for $n \geqq 3$ there exist β-Harnack domains $D \subset \mathbf{R}^n$ only when $\beta \geqq n - 2$. By the proof of Theorem 3.7 below, there exist β-Harnack simply connected proper subdomains $D \subset \mathbf{C}$ only when $\beta \geqq 1$. (e) Consider the following variant of condition (2.1). For all $u \in \mathcal{H}^+(D)$

$$d(\{x_1, x_2\}, \partial D)^\alpha / b \leqq u(x_1)/u(x_2) \leqq b/d(\{x_1, x_2\}, \partial D)^\alpha \quad \text{for all } x_1, x_2 \in D.$$

It is easy to see that this condition implies that the function $d(x, \partial D)$ must be bounded and hence that (2.1) holds with $\beta = \alpha, c = d$. On the other hand, since $d_1 d_2 \geqq (\min\{d_1, d_2\})^2$, we see that (2.1) implies this condition with $\alpha = \beta^2, b = c^2$.

There is an alternative definition for Harnack domains. In 1966 Köhn introduced the *Harnack metric* H_D defined for $x, y \in D \subset \mathbf{R}^n$ by

$$H_D(x, y) = \sup_u |\log u(x)/u(y)|,$$

where the supremum is taken over all functions $u \in \mathcal{H}^+(D)$; see [Kö], [H_1].

PROPOSITION 2.3. *Fix $\beta > 0$. A domain $D \subset \mathbf{R}^n$ is β-Harnack if and only if there exists a point $x_0 \in D$ and a constant a such that*

$$H_D(x, x_0) \leqq \beta \log(a/d(x, \partial D)) \quad \textit{for all } x \in D.$$

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PROOF. It is not difficult to see that the inequality $H_D(x,y) \leqq \log M$ is equivalent to the Harnack type inequality

$$1/M \leqq u(x)/u(y) \leqq M \text{ for all } u \in \mathcal{H}^+(D). \qquad \square$$

Another useful distance function in the *quasihyperbolic metric* k_D, introduced in 1976 by Gehring and Palka [GP], and defined for $x, y \in D \subset \mathbf{R}^n$ by

$$k_D(x,y) = \inf_{\gamma} \int_{\gamma} |dz|/d(z, \partial D),$$

where $|dz|$ is Euclidean arc-length and the infimum is taken over all locally rectifiable arcs $\gamma \subset D$ joining x and y. The next result follows quickly from Proposition 2.3 and gives a sufficient condition for a domain to be Harnack.

THEOREM 2.4. *Fix $\beta > 0$ and $D \subset \mathbf{R}^n$. Suppose there exists a point $x_0 \in D$ and a constant c such that*

$$(2.5) \qquad k_D(x, x_0) \leqq \beta \log(c/d(x, \partial D)) \quad \text{for all } x \in D.$$

Then D is $n\beta$-Harnack.

PROOF. It is known [H_1, Theorem 2] that $H_D \leqq nk_D$ for domains $D \subset \mathbf{R}^n$. $\square$

Gehring and Martio have studied the class of domains which satisfy (2.5) in connection with determining when quasiconformal mappings satisfy local Lipschitz conditions. This class properly contains bounded admissible domains and bounded uniform domains as well as John domains [GM, 3.11 and 3.15], [HV, 2.12]. In fact, this is precisely the class of domains which are bounded and ϕ-John where $\phi(t) = c \log(1+t)$ [HV, 2.5(c)], [H_2, 2.8]. In particular, all of the aforementioned classes of domains are Harnack domains; cf. [HV, 3.8]. We mention that there are Harnack domains which do not satisfy condition (2.5). In fact, there exist unbounded Harnack domains [H_2, 3.3], while every domain satisfying (2.5) is bounded [GM, 3.9].

We use Proposition 2.3 again to point out that bounded Harnack domains can be characterized using the *relative Euclidean distance* which is defined for $x, y \in \mathbf{R}^n$ by

$$j_D(x,y) = \log(1 + |x-y|/\min\{d(x,\partial D), d(y, \partial D)\}).$$

PROPOSITION 2.6. *Fix $\beta > 0$. Let $D \subset \mathbf{R}^n$ be bounded with $d = \operatorname{dia}(D)$. (a) If*

$$H_D(x, x_0) \leqq \beta j_D(x, x_0) \quad \text{for all } x \in D,$$

then D is β-Harnack with center x_0 and constant $c = d^2/d(x_0, \partial D)$. (b) If D is β-Harnack with center x_0 and constant c, then

$$H_D(x, x_0) \leqq \alpha j_D(x, x_0) \quad \text{for all } x \in D,$$

where $\alpha = \max\{2n, \beta(1 + \log(2c/d(x_0, \partial D))/\log(3/2))\}$.

PROOF. (a) Since D is bounded, $j_D(x, x_0) \leqq \log(d^2/(d(x_0, \partial D)d(x, \partial D)))$ for all $x \in D$. (b) Apply [H_1, Theorem 2], [HV, (2.2)] and [H_2, 2.6(b)]. □

EXAMPLE 2.7. (a) An n-ball $D = \{x \in \mathbf{R}^n : |x - x_0| < r\}$ is an $(n-1)$-Harnack domain with center x_0 and constant $c = 2r^{n-1}$. This follows most easily from the fact [Kö, p. 58] that

$$H_D(x, x_0) = \log(r^{n-2}[r + |x - x_0|]/[r - |x - x_0|]^{n-1}).$$

(b) No half-space is Harnack, nor is any non-Bloch domain. (c) The infinite slab $D = \{(x, y) \in \mathbf{R}^{n-1} \times \mathbf{R} : 0 < y < \pi/(n-1)^{1/2}\}$ is not Harnack as seen by considering the positive harmonic function $u(x, y) = \exp(x_1 + \ldots + x_{n-1}) \sin((n-1)^{1/2} y)$ where $(x, y) = (x_1, \ldots, x_{n-1}, y)$. (d) The cylinder $D = \{(x, y) \in \mathbf{R}^{n-1} \times \mathbf{R} : |x| < \pi/2\}$ is not Harnack; consider $u(x, y) = \cos(x_1) \ldots \cos(x_{n-1}) \exp((n-1)^{1/2} y)$. (e) A punctured n-ball is $(n-1)$-Harnack; use (a) and [H_1, Theorem 3].

3. **Plane domains.** Harnack domains are those domains for which a Harnack type inequality describes the rate of growth of positive harmonic functions relative to the Euclidean distance to the boundary of the domain. A classical result of Hardy and Littlewood relates the Hölder continuity of a function analytic in the unit disk to the rate of growth of its derivative, again relative to the Euclidean distance to the boundary of the disk. In this section we characterize the simply connected plane Harnack domains as precisely those domains which are the image of the unit disk under a Hölder continuous conformal mapping. This description also involves the hyperbolic metric of the domain. Then we give a geometric condition which, when satisfied by a simply connected plane domain, guarantees that the domain is Harnack. Finally, we present an apparently new characterization for the class of quasidisks.

A function $f : E \subset \mathbf{R}^n \to \mathbf{R}^m$ belongs to the Lipschitz class $\mathrm{Lip}_\alpha(E)$ if

$$|f(x) - f(y)| \leqq c|x - y|^\alpha \quad \text{for all } x, y \in E$$

for some constant c, where $0 < \alpha \leqq 1$. Gehring and Martio have shown that a K-quasiconformal mapping $f : B^n \to D$ of the unit ball B^n in $\mathbf{R}^n$ belongs to $\mathrm{Lip}_\alpha(B^n)$ for some $0 < \alpha \leqq K^{1/1-n}$ if and only if D satisfies condition (2.5) [GM, 3.24]. We are interested in the domains $D = f(B)$ which are the image of the open unit disk $B = \{\zeta \in \mathbf{C} : |\zeta| < 1\}$ under a conformal mapping f in $\mathrm{Lip}_\alpha(\mathrm{B})$.

The following result is used repeatedly.

LEMMA 3.1. *Fix* $0 < \alpha \leqq 1$. *Let* $f : B \to D$ *be conformal. The following are equivalent.*

$$|f'(\zeta)| \leqq a(1 - |\zeta|)^{\alpha - 1} \quad \text{for all } \zeta \in B, \tag{3.2}$$

$$|f'(\zeta)| \leqq b d(f(\zeta), \partial D)^{1 - 1/\alpha} \quad \text{for all } \zeta \in B, \tag{3.3}$$

$$d(f(\zeta), \partial D) \leqq c(1 - |\zeta|)^\alpha \quad \text{for all } \zeta \in B. \tag{3.4}$$

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Here the constants a, b, c depend only on each other and on α.

PROOF. Koebe's one-quarter theorem [P, 1.4, p. 22] states that

$$d(f(\zeta), \partial D) \leqq |f'(\zeta)|(1-|\zeta|^2) \leqq 4d(f(\zeta), \partial D) \quad \text{for all } \zeta \in B. \tag{3.5}$$

We merely combine (3.5) with the elementary inequalities

$$1-|\zeta| \leqq 1-|\zeta|^2 \leqq 2(1-|\zeta|), \quad \zeta \in B.$$

Fix $\zeta \in B$ and let $z = f(\zeta)$. From (3.5) we have

$$2(1-|\zeta|) \geqq d(z, \partial D)/|f'(\zeta)|,$$

so (3.2) implies

$$|f'(\zeta)| \leqq 2^{1-\alpha} a d(z, \partial D)^{\alpha-1} |f'(\zeta)|^{1-\alpha}$$

which gives (3.3) with $b = 2^{1/\alpha - 1} a^{1/\alpha}$. Using (3.5) again as above with (3.3) we obtain

$$d(z, \partial D) \leqq 2b(1-|\zeta|) d(z, \partial D)^{1-1/\alpha}$$

which gives (3.4) with $c = 2^\alpha b^\alpha$. Finally, the second inequality in (3.5) with (3.4) yields

$$|f'(\zeta)| \leqq 4d(z, \partial D)/(1-|\zeta|) \leqq 4c(1-|\zeta|)^{\alpha-1}. \qquad \square$$

COROLLARY 3.6. *A conformal mapping $f : B \longrightarrow D$ belongs to* $\mathrm{Lip}_\alpha(B)$ *if and only if one (and hence all) of the inequalities* (3.2), (3.3), (3.4) *holds.*

PROOF. Using Cauchy's integral formula one readily verifies that for any domain $G \subset \mathbf{C}$

$$|g(\zeta_1) - g(\zeta_2)| \leqq c|\zeta_1 - \zeta_2|^\alpha \quad \text{for all } \zeta_1, \zeta_2 \in G$$

implies

$$|g'(\zeta)| \leqq c d(\zeta, \partial D)^{\alpha-1} \quad \text{for all } \zeta \in G$$

whenever g is analytic in G. In particular, (3.2) holds as soon as f is in $\mathrm{Lip}_\alpha(B)$. In 1932 Hardy and Littlewood established the reverse implication assuming only that f is analytic in B [HL, Theorems 40 and 41]. $\square$

One more important distance function is *Poincaré's hyperbolic metric* h_D defined for $z, w \in D \subset \mathbf{C}$ by

$$h_D(z, w) = h_B(p^{-1}(z), p^{-1}(w)),$$

where $p : B \longrightarrow D$ is any analytic covering projection and

$$h_B(\zeta, \eta) = \log \frac{|1-\zeta\bar{\eta}| + |\zeta - \eta|}{|1-\zeta\bar{\eta}| - |\zeta - \eta|}$$

is the metric in the unit disk; if no covering exists, we set $h_D = 0$. We now use the hyperbolic metric to present an elementary proof of the following result whose corollary provides the aforementioned characterization of simply connected plane Harnack domains.

THEOREM 3.7. *Fix $\beta > 0$ and let $\alpha = 1/\beta$. Let D be a simply connected proper subdomain of* $\mathbf{C}$. *Fix a point $z_0 \in D$. The following are equivalent.*

(3.8) $d(z, \partial D)^\beta / a \leqq u(z)/u(z_0) \leqq a/d(z, \partial D)^\beta$ for all $z \in D, u \in \mathcal{H}^+(D)$;

(3.9) $h_D(z, z_0) \leqq \beta \log(b/d(z, \partial D))$ for all $z \in D$;

(3.10) *Every conformal $f : B \longrightarrow D$ with $f(0) = z_0$ satisfies* (3.4).

Here the constants a, b, c depend only on each other and on β.

PROOF. (3.8) $\Rightarrow$ (3.9): Fix a point $z \in D$. Let ϕ be the conformal mapping of D onto the right half-plane with $\phi(z_0) = 1$ and $\phi(z) = x > 1$. Then $u = \mathrm{Re}(\phi) \in \mathcal{H}^+(D)$, so

$$h_D(z, z_0) = \log x = \log(u(z)/u(z_0)) \leqq \beta \log(b/d(z, \partial D))$$

where $b = a^\alpha$.

(3.9) $\Rightarrow$ (3.10): Let $f : B \longrightarrow D$ be conformal with $z_0 = f(0)$. Fix $z = f(\zeta)$. Then

$$\log((1 + |\zeta|)/(1 - |\zeta|)) = h_B(\zeta, 0) = h_D(z, z_0) \leqq \beta \log(b/d(z, \partial D)),$$

so (3.4) holds with $c = b$.

(3.10) $\Rightarrow$ (3.8): Fix $u \in \mathcal{H}^+(D)$. Let $f : B \longrightarrow D$ be conformal with $z_0 = f(0)$. Set $v = u \circ f$. Then by Harnack's inequality

$$\frac{1 - |\zeta|}{2} \leqq \frac{1 - |\zeta|}{1 + |\zeta|} \leqq \frac{v(\zeta)}{v(0)} \leqq \frac{1 + |\zeta|}{1 - |\zeta|} \leqq \frac{2}{1 - |\zeta|} \quad \text{for all } \zeta \in B.$$

Letting $z = f(\zeta)$ we see that the above and (3.4) imply (3.8) with $a = 2c^\beta$. □

COROLLARY 3.11. *Fix $\beta > 1$. A simply connected proper subdomain of* $\mathbf{C}$ *is β-Harnack if and only if it is the image of the unit disk B under a conformal mapping in* $\mathrm{Lip}_{1/\beta}(B)$.

REMARKS 3.12. (a) Köhn verified that the Poincaré and Harnack metrics are the same for simply connected hyperbolic Riemann surfaces [Kö, 4.2]. This fact and Proposition 2.3 imply the equivalence of (3.8) and (3.9). (b) It is known that for simply connected proper subdomains $D \subset \mathbf{C}$

$$(1/2)k_D \leqq H_D = h_D \leqq 2k_D,$$

while for general domains $D \subset \mathbf{C}$

$$H_D \leqq h_D \leqq 2k_D;$$

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see, e.g., the references mentioned in $[H_1]$, $[H_2]$. (c) Becker and Pommerenke demonstrated the equivalence of (3.9) and (3.10) [BP, Theorem 1]; note that their result includes a hypothesis that the domain be bounded. (d) From the proof of Theorem 3.7 we infer that there are simply connected β-Harnack plane domains only when $\beta \geqq 1$.

Our next two observations follow at once from the above remarks and Proposition 2.3. The first result gives a sufficient condition for a plane domain to be Harnack and is analagous to Theorem 2.4, while the second result is a converse to Theorem 2.4; see also the corollary in [BP].

COROLLARY 3.13. *Fix $\beta > 0$ and $D \subset \mathbf{C}$. Suppose there exists a point $z_0 \in D$ and a constant b such that* (3.9) *holds. Then D is β-Harnack.*

COROLLARY 3.14. *Fix $\beta > 0$. Let D be a simply connected proper subdomain of* $\mathbf{C}$. *If D is $(\beta/2)$-Harnack with center x_0, then* (2.5) *holds for some constant c.*

Now we give a geometric condition which can be used to show that a simply connected plane domain is Harnack. This condition is essentially one which must hold only at a single boundary point. It helps explain the significance of the constant β for β-Harnack domains.

Following Kuran and Schiff, we call $D \subset \mathbf{C}$ an *α-Dini* domain, $0 < \alpha \leqq 2$, if D is a Jordan domain with a Dini-smooth boundary of angle $\alpha\pi$ at some fixed boundary point; see [KS, p. 196] and [P, p. 298]. This means that D is a bounded Jordan domain and there exists a fixed point $w_0 \in \partial D$ such that ∂D has a Dini-smooth arc-length parametrization $w = w(t)$, $0 \leqq t \leqq l$ with $w(0) = w_0 = w(l)$ and $\alpha\pi$ is the angle between the two tangent vectors

$$w'(0) = \lim_{0<t\to 0} \frac{w(t) - w_0}{t}, w'(l) = \lim_{l>t\to l} \frac{w(t) - w}{l - t}.$$

THEOREM 3.15. *Fix $\beta \geqq 1$ and let $\alpha = 1/\beta$. Suppose that $D \subset \mathbf{C}$ is α-Dini. Then D is β-Harnack.*

PROOF. Assume D has a Dini-smooth boundary of angle $\alpha\pi$ at $0 \in \partial D$. Note that $d(z, \partial D) \leqq |z|$ for all $z \in \partial D$. Let $f : B \to D$ be conformal. Then by [KS, Lemma 1] there exists a constant $b > 0$ such that

$$|f'(\zeta)| \leqq b|f(\zeta)|^{1-\beta} \quad \text{for all } \zeta \in B.$$

Thus (3.3) holds. □

A domain $D \subset \mathbf{C}$ is a *K-quasidisk*, $K \geqq 1$, if $D = f(B)$ is the image of the unit disk under a K-quasiconformal mapping $f : \mathbf{C} \to \mathbf{C}$. We refer to Gehring's exposition $[G]$ for properties of quasidisks. We present here what appears to be a new characterization for this important class of domains.

THEOREM 3.16. *Let D be a simply connected proper subdomain of* $\mathbf{C}$. *Then D is a K-quasidisk if and only if there exists a constant $\beta > 0$ such that for all $u \in \mathcal{H}^+(D)$*

and for all $z_1, z_2 \in D$ *we have*

$$\frac{u(z_1)}{u(z_2)} \leqq \left[\left(\frac{|z_1 - z_2|}{d(z_1, \partial D)} + 1\right)\left(\frac{|z_1 - z_2|}{d(z_2, \partial D)} + 1\right)\right]^{\beta}.$$

Here the constants K *and* β *depend only on each other.*

PROOF. This theorem is an immediate consequence of the proof of Proposition 2.3 and Remark 3.12(a) together with the characterization of quasidisks as being precisely those simply connected plane domains in which the metrics h_D are j_D are equivalent; see [G, p. 36] and [J, Theorems 1, 2 and p. 44]. □

The following corollary shows that bounded quasidisks are Harnack domains. Of course this follows from Theorem 2.4 since quasidisks are uniform domains; see also [HV, 3.6 and 3.8].

COROLLARY 3.17. *Let* $D \subset \mathbf{C}$ *be a bounded* K*-quasidisk. Then there exist constants* $\beta = \beta(K) > 0, c = c(K, \text{dia}(D)) > 0$ *such that*

$$u(z_1)/u(z_2) \leqq [\text{dia}(D)^2/d(z_1, \partial D)d(z_2, \partial D)]^{\beta} \quad \textit{for all } z_1, z_2 \in D$$

and

$$d(\{z_1, z_2\}, \partial D)^{2\beta}/c \leqq u(z_1)/u(z_2) \leqq c/d(\{z_1, z_2\}, \partial D)^{2\beta} \quad \textit{for all } z_1, z_2 \in D$$

for all $u \in \mathcal{H}^+(D)$.

PROOF. The observation $|z_1 - z_2| + d(z_j, \partial D) \leqq \text{dia}(D)$ implies the first set of inequalities, while the second follow by taking $c = \text{dia}(D)^{2\beta}$. □

QUESTION 3.18. Matti Vourinen pointed out that it would be interesting to have information describing the connection between the constants K and β in Theorem 3.16.

4. **Applications.** In this section we extend some classical results for functions defined in the unit disk to functions defined in Harnack domains and we give other typical function theoretic applications. In particular we present results describing the rate of growth of various classes of functions defined in Harnack domains. Finally, we discuss a result related to the "order" of analytic functions and we consider the classes AL(D) and $H^p(D)$.

The next result is known for disks [KS, Theorem 2 and p. 201, (7)]; for related results see also [K], [S] and [HV].

THEOREM 4.1. *Fix* $\beta > 0$. *Let* $D \subset \mathbf{C}$ *be a simply connected* β*-Harnack domain. Suppose that* v *is positive and superharmonic in* D. *Then there exists a constant* $a = a(\beta, v) > 0$ *such that*

$$v(z) \geqq a d(z, \partial D)^{\beta} \quad \textit{for all } z \in D.$$

PROOF. Let $f : B \longrightarrow D$ be conformal. Set $w = v \circ f$. Then w is positive and superharmonic in B, so by [KS, Proposition 1] there exists a constant $b = b(v) > 0$ such that

$$w(\zeta) \geqq b(1 - |\zeta|) \quad \text{for all } \zeta \in B.$$

Letting $z = f(\zeta)$ we have by (3.10) that

$$d(z, \partial D) \leqq c(1 - |\zeta|)^{1/\beta},$$

whence

$$v(z) = w(\zeta) \geqq bc^{-\beta} d(z, \partial D)^{\beta}. \qquad \square$$

COROLLARY 4.2. *Fix $\beta > 0$. Let $D \subset \mathbf{C}$ be a simply connected β-Harnack domain. Suppose that v is non-negative and superharmonic in D. If there exists a point $\zeta \in \partial D$ such that*

$$\liminf_{D \ni z \longrightarrow \zeta} u(z)/d(z, \partial D)^{\beta} = 0,$$

then $u \equiv 0$.

Now we extend a well-known distortion theorem for the class of normalized functions analytic in B with positive real part; see [P, p.40, (11)] and also [HV, 4.1].

THEOREM 4.3. *Fix $\beta > 0$. Let $D \subset \mathbf{C}$ be a simply connected β-Harnack domain with center z_0 and constant c. Suppose that ϕ is analytic in D with* $\mathrm{Re}(\phi) > 0$ *and $\phi(z_0) = 1$. Then for all $z \in D$*

$$d(z, \partial D)^{\beta}/2c \leqq |\phi(z)| \leqq 2c/d(z, \partial D)^{\beta}$$

and

$$|\phi'(z)| \leqq 4c/d(z, \partial D)^{1+\beta}.$$

PROOF. Let $f : B \longrightarrow D$ be conformal with $f(0) = z_0$. Then $F = \phi \circ f$ is analytic in B with $\mathrm{Re}(F) > 0$ and $F(0) = 1$. Thus by [P, p. 40, (11)]

$$\frac{1 - |\zeta|}{2} \leqq \frac{1 - |\zeta|}{1 + |\zeta|} \leqq |F(\zeta)| \leqq \frac{1 + |\zeta|}{1 - |\zeta|} \leqq \frac{2}{1 - |\zeta|}$$

and

$$|F'(\zeta)| \leqq 2/(1 - |\zeta|)^2$$

for all $\zeta \in B$. Letting $z = f(\zeta)$, so $\phi(z) = F(\zeta)$, and combining (3.4) with the above we obtain the desired bounds on $|\phi(z)|$. Next, by (3.5)

$$d(z, \partial D) \leqq 2|f'(\zeta)|(1 - |\zeta|),$$

and thus using (3.4) again we get

$$|\phi'(z)| = |F'(\zeta)|/|f'(\zeta)| \leqq 2/[(1 - |\zeta|)^2 |f'(\zeta)|] \leqq 4c/d(z, \partial D)^{1+\beta}.$$

Our next result generalizes (3.3). □

THEOREM 4.4. *Fix $\beta > 0$. Let $D \subset \mathbf{C}$ be a simply connected β-Harnack domain with center z_0 and constant c. Suppose that $\phi : D' \longrightarrow D$ is conformal. Then*

$$|\phi'(w)| \leqq 4cd(\phi(w), \partial D)^{1-\beta}/d(w, \partial D') \quad \textit{for all } w \in D'.$$

PROOF. Let $f : B \longrightarrow D'$ be conformal with $\phi(f(0)) = z_0$. Since D is Harnack and $F = \phi \circ f : B \longrightarrow D$ is conformal, (3.3) yields

$$|\phi'(w)|\,|f'(\zeta)| = |F'(\zeta)| \leqq 4cd(F(\zeta), \partial D)^{1-\beta} = 4cd(\phi(w), \partial D)^{1-\beta}$$

for all $w = f(\zeta) \in D'$. The result now follows from (3.5). □

The following result is of interest since it shows that for an analytic function f defined in a Harnack domain D, knowledge of a *positive* lower bound for $|f|$ actually gives an upper bound for $|f(z)|$ in terms of $d(z, \partial D)$; a similar result holds when a *finite* upper bound for $|f|$ is known and f is nonvanishing. In particular, we see that analytic functions of "infinite order" defined in Harnack domains must necessarily take on values arbitrarily close to zero.

PROPOSITION 4.5. *Fix $\beta > 0$. Let $D \subset \mathbf{C}$ be a β-Harnack domain with center z_0 and constant c. Set $s = \sup\{d(z, \partial D)^\beta : z \in D\}$. Let f be analytic in D.*

(a) Suppose that $|f| \geqq a > 0$. Then for all $z \in D$

$$a(|f(z_0)|/a)^{d(z,\partial D)^\beta/c} \leqq |f(z)| \leqq a(|f(z_0)|/a)^{c/d(z,\partial D)^\beta}.$$

In particular,

$$|f(z)| \leqq \exp(b/d(z, \partial D)^\beta) \quad \textit{for all } z \in D,$$

where $b = c\log(|f(z_0)|/a) + s\log^+(a)$.

(b) Suppose $0 < |f| \leqq a < +\infty$. Then for all $z \in D$

$$a(|f(z_0)|/a)^{c/d(z,\partial D)^\beta} \leqq |f(z)| \leqq a(|f(z_0)|/a)^{d(z,\partial D)^\beta/c}.$$

In particular,

$$|f(z)| \geqq \exp(-b/d(z, \partial D)^\beta) \quad \textit{for all } z \in D,$$

where $b = c\log(a/|f(z_0)|) + s\log^+(1/a)$.

PROOF. (a) We assume that $u(z) = \log(|f(z)|/a)$ is positive. Then (2.1) yields

$$u(z_0)d(z, \partial D)^\beta/c \leqq u(z) \leqq cu(z_0)/d(z, \partial D)^\beta \quad \text{for all } z \in D$$

from which the first conclusion follows. The second is an immediate consequence of the inequalities

$$\log|f(z)| \leqq \log(a) + cu(z_0)/d(z, \partial D)^\beta \leqq [s\log^+(a) + cu(z_0)]/d(z, \partial D)^\beta.$$

(b) This follows by considering $1/f$. □

We now obtain a result similar to the above for the class $AL(D)$ of *analytic Lindelöfian* functions. We have $f \in AL(D)$ if f is analytic in D and if $\log^+|f|$ has a harmonic majorant in D, i.e., if there exists a function u harmonic in D with $\log^+|f(z)| \leqq u(z)$ for all $z \in D$. We note that the Hardy class $H^p(D)$ is a subclass of $AL(D)$ for $0 < p < +\infty$, and $AL(B) = N$ is Nevanlinna's class of functions of bounded characteristic.

PROPOSITION 4.6. *Fix $\beta > 0$. Let $D \subset \mathbf{C}$ be a β-Harnack domain with center z_0 and constant c. For each $f \in AL(D)$ there exists a constant $b = b(c,f) \geqq 0$ such that*

$$|f(z)| \leqq \exp(b/d(z,\partial D)^\beta) \quad \textit{for all } z \in D.$$

PROOF. Choose u harmonic in D with $\log^+|f| \leqq u$. If $u = 0$, set $b = 0$. Otherwise, $u \in \mathcal{H}^+(D)$ and thus for all $z \in D$

$$\log|f(z)| \leqq \log^+|f(z)| \leqq u(z) \leqq cu(z_0)/d(z,\partial D)^\beta$$

which gives the conclusion with $b = cu(z_0)$. □

Finally, we give a result describing the possible rate of growth of H^p functions defined in simply connected plane Harnack domains.

PROPOSITION 4.7. *Fix $\beta > 0$. Let $D \subset \mathbf{C}$ be a simply connected β-Harnack domain. For each $f \in H^p(D)$ there exist constants $a \geqq 0, b \geqq 0$ such that*

$$|f(z)|^p \leqq a/d(z,\partial D)^\beta - bd(z,\partial D)^\beta \quad \textit{for all } z \in D.$$

PROOF. Let $w = |f|^p$. Then w is non-negative and subharmonic in D, so w has a least harmonic majorant u. Since either $u = 0$ or $u \in \mathcal{H}^+(D)$, there exists a constant $a \geqq 0$ such that

$$u(z) \leqq a/d(z,\partial d)^\beta \quad \text{for all } z \in D. \square$$

Then $v = u - w$ is non-negative and superharmonic in D, so by Theorem 4.1 there exists a constant $b \geqq 0$ such that

$$v(z) \geqq bd(z,\partial D)^\beta \quad \text{for all } z \in D.$$

Thus

$$|f(z)|^p = w(z) = u(z) - v(z) \leqq a/d(z,\partial D)^\beta - bd(z,\partial D)^\beta \text{ for all } z \in D. \quad \square$$

ADDED. We thank the referee for numerous helpful suggestions and for sending us the preprint of Anderson and Hinkkanen. We point out that [AH, Theorem 1] generalizes the implication (3.10) ⇒ (3.8) of Theorem 3.7 combined with Theorem 4.1 and [AH, Theorem 3] is essentially the implication (3.8) ⇒ (3.10) in Theorem 3.7.

Of greater significance is the fact that their work presents a substantial step towards answering Question 3.18..

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