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A K3 Surface Associated With Certain Integral Matrices Having Integral Eigenvalues

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Abstract. In this article we will show that there are infinitely many symmetric, integral 3×3 matrices, with zeros on the diagonal, whose eigenvalues are all integral. We will do this by proving that the rational points on a certain non-Kummer, singular K3 surface are dense. We will also compute the entire Néron–Severi group of this surface and find all low degree curves on it.

1 Introduction

In the problem section of Nieuw Archief voor Wiskunde [Be], F. Beukers posed the question whether symmetric, integral 3×3 matrices

$$M_{a,b,c} = \begin{pmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 0 \end{pmatrix} \tag{1}$$

exist with integral eigenvalues and satisfying $q(a,b,c) \neq 0$, where $q(a,b,c)$ is the polynomial $q(a,b,c) = abc(a^2 - b^2)(b^2 - c^2)(c^2 - a^2)$. As it is easy to find such matrices satisfying $q(a,b,c) = 0$, we will call those trivial. R. Vidunas and the author of this article independently proved that the answer to this question is positive, see [BLV]. There are in fact infinitely many nontrivial examples of such matrices. This follows immediately from the fact that for every integer t, if we set

$$\begin{aligned} a &= -(4t-7)(t+2)(t^2-6t+4), \\ b &= (5t-6)(5t^2-10t-4), \\ c &= (3t^2-4t+4)(t^2-4t+6), \\ x &= 2(3t^2-4t+4)(4t-7), \\ y &= (t^2-6t+4)(5t^2-10t-4), \\ z &= -(t+2)(5t-6)(t^2-4t+6), \end{aligned} \tag{2}$$

then the matrix $M_{a,b,c}$ has eigenvalues x, y, and z. This matrix is trivial if and only if we have $t \in \{-2,-1,0,1,2,4,10\}$. For $t = 3$ we get $a = 125$, $b = 99$, and

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$c = 57$ with eigenvalues 190, -55, and -135. By a computer search, we find that this is the second smallest example when ordered by $\max(|a|, |b|, |c|)$. The smallest has $a = 26$, $b = 51$, and $c = 114$. In this article we will show how to find such parametrizations. We will see that there are infinitely many and that the one in (2) has the lowest possible degree.

If the eigenvalues of the matrix $M_{a,b,c}$ are denoted by x, y, and z, then its characteristic polynomial can be factorized as

$$\lambda^3 - (a^2 + b^2 + c^2)\lambda - 2abc = (\lambda - x)(\lambda - y)(\lambda - z).$$

Comparing coefficients, we get three homogeneous equations in x, y, z, a, b, and c. Hence, geometrically we are looking for rational points on the 2-dimensional complete intersection $X \subset \mathbb{P}^5_{\mathbb{Q}}$, given by

$$x + y + z = 0, \quad xy + yz + zx = -a^2 - b^2 - c^2, \quad xyz = 2abc. \tag{3}$$

The points on the curves on X defined by $q(a, b, c) = 0$ correspond to the trivial matrices. Parametrizations as in (2) correspond to curves on X that are isomorphic over $\mathbb{Q}$ to $\mathbb{P}^1$. We will see that X contains infinitely many of them, thereby proving the main theorem of this paper, which states the following.

Theorem 1.1 *The rational points on X are Zariski dense.*

In the next section we will recall the definition and some properties of lattices and elliptic surfaces in the sense of Shioda [Sh]. In Section 3 we will prove Theorem 1.1 using an elliptic fibration of a blow-up Y of X. We will see that Y is a so called elliptic K3 surface. The interaction between the geometry and the arithmetic of K3 surfaces is of much interest. F. Bogomolov and Y. Tschinkel have proved that on every elliptic K3 surface Z over a number field K the rational points are potentially dense, *i.e.,* there is a finite field extension L/K such that the L-points of Z are dense in Z, see [BT, Thm. 1.1]. Key in their analysis of potential density of rational points is the so-called Picard number of a surface, an important geometric invariant. F. Bogomolov and Y. Tschinkel have shown that if the Picard number of a K3 surface is large enough, then the rational points are potentially dense. On the other hand, it is not yet known if there exist K3 surfaces with Picard number 1 on which the rational points are not potentially dense.

After proving the main theorem, we will investigate more deeply the geometry of Y and show in Section 4 that its Picard number equals 20, which is maximal among K3 surfaces in characteristic 0. It is a fact that a K3 surface with maximal Picard number is either a Kummer surface or a double cover of a Kummer surface. These Kummer surfaces are K3 surfaces with a special geometric structure, described in Section 5. As a consequence, their arithmetic can be described more easily. It is therefore natural to ask if Y is a Kummer surface, in which case Y would have had a richer structure that we could have utilized. In Section 5 we will show that this is not the case.

In Section 6 we will describe more of the geometry of X by showing that X contains exactly 63 curves of degree smaller than 4. All points on these curves correspond to matrices that are either trivial or not defined over $\mathbb{Q}$. As the degree of a parametrization as in (2) corresponds to the degree of the curve that it parametrizes, this shows that the one in (2) has the lowest possible degree among parametrizations of nontrivial matrices.

2 Lattices and Elliptic Surfaces

The definition of an elliptic surface and the results in this section can almost all be found in [Sh]. For a more detailed summary of these results and constructions of elliptic surfaces, see also [Lu, §3, 4]. Throughout this paper we will say that a variety V over a field k is smooth if the map $V \to \operatorname{Spec} k$ is smooth.

We will start with the definition of a lattice. Note that for any abelian groups A and G, a symmetric bilinear map $A \times A \to G$ is called *nondegenerate* if the induced homomorphism $A \to \operatorname{Hom}(A, G)$ is injective. Note that we do not require a lattice to be definite, only nondegenerate.

Definition 2.1 A *lattice* is a free $\mathbb{Z}$-module L of finite rank, endowed with a symmetric, bilinear, nondegenerate map $\langle \cdot, \cdot \rangle : L \times L \to \mathbb{Q}$, called the *pairing* of the lattice. An *integral lattice* is a lattice whose pairing is $\mathbb{Z}$-valued. A lattice L is called *even* if $\langle x, x \rangle \in 2\mathbb{Z}$ for every $x \in L$. A *sublattice* of L is a submodule L' of L, such that the induced bilinear pairing on L' is nondegenerate. A sublattice L' of L is called *primitive* if L/L' is torsion-free. The positive or negative definiteness or signature of a lattice is defined to be that of the vector space $L_{\mathbb{Q}}$, together with the induced pairing.

Definition 2.2 For a lattice L with pairing $\langle \cdot, \cdot \rangle$ we denote by $L(n)$ the lattice with the same underlying module as L and the pairing $n \cdot \langle \cdot, \cdot \rangle$.

Definition 2.3 The *Gram matrix* of a lattice L with respect to a given basis $x = (x_1, \dots, x_n)$ is $I_x = \big(\langle x_i, x_j \rangle\big)_{i,j}$. The *discriminant* of L is defined by $\operatorname{disc} L = \det I_x$ for any basis x of L. A lattice L is called *unimodular* if it is integral and $\operatorname{disc} L = \pm 1$.

Lemma 2.4 *Let L' be a sublattice of finite index in a lattice L. Then we have* $\operatorname{disc} L' = [L : L']^2 \operatorname{disc} L$.

Proof This is a well-known fact, see also [Sh, §6]. ■

Definition 2.5 Let C be a smooth, irreducible, projective curve over an algebraically closed field k. An *elliptic surface* over C is a smooth, irreducible, projective surface S, together with a non-smooth, relatively minimal, surjective morphism $f : S \to C$, of which almost all fibers are nonsingular curves of genus 1, and a section $\mathcal{O}$ of f.

Remark 2.6 By Castelnuovo's criterion (see [Ch, Thm. 3.1]), the morphism f is relatively minimal if and only if no fiber contains an exceptional divisor, *i.e.,* a prime divisor E with $E^2 = -1$ and $H^1(E, \mathcal{O}_E) = 0$. By [Ha, Prop. III.9.7], any dominating morphism from an integral variety to a regular curve is flat. Therefore, so is f in the definition above. Also, f is locally of finite presentation. Hence, by [Gr1, Déf. 6.8.1],

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the requirement in the definition above of f not being smooth is equivalent to the requirement that f have a singular fiber.

For the rest of this section, let S be an elliptic surface over a smooth, irreducible, projective curve C over an algebraically closed field k, fibered by $f\colon S \to C$ with a section $\mathcal{O}$. Let $K = k(C)$ denote the function field of C and let $\eta\colon \operatorname{Spec} K \to C$ be its generic point. Then the generic fiber $E = S \times_C \operatorname{Spec} K$ of f is a smooth, projective, geometrically integral curve over K with genus 1. Let ξ denote the natural map $E \to S$.

$$\begin{array}{ccc} E & \xrightarrow{\ \xi\ } & S \\ \downarrow & & \downarrow f \\ \operatorname{Spec} K & \xrightarrow[\ \eta\ ]{} & C \end{array}$$

Lemma 2.7 *Both maps ξ_* and η^* in*

$$E(K) = \operatorname{Hom}_K(\operatorname{Spec} K, E) \xrightarrow{\xi_*} \operatorname{Hom}_C(\operatorname{Spec} K, S) \xleftarrow{\eta^*} \operatorname{Hom}_C(C, S) = S(C)$$

are bijective.

Proof By the universal property of fibered products, we find that every morphism $\sigma\colon \operatorname{Spec} K \to S$ with $f \circ \sigma = \eta$ comes from a unique section of the morphism $E \to \operatorname{Spec} K$. Hence, the map ξ_* is bijective. As C is a smooth curve and S is projective, any morphism from a dense open subset of C to S extends uniquely to a morphism from C, see [Ha, Prop. I.6.8]. As $\operatorname{Spec} K$ is dense in C, the map η^* is bijective as well. ∎

Whenever we implicitly identify the two sets $E(K)$ and $S(C)$, it will be done using the bijection $\xi_*^{-1} \circ \eta^*$ of Lemma 2.7. The section $\mathcal{O}$ of f corresponds to a point on E, giving E the structure of an elliptic curve. This endows $E(K)$ with a group structure, which carries over to $S(C)$, see [Si1, Prop. III.3.4].

Recall that for any proper scheme Y over an algebraically closed field, the Néron–Severi group $\operatorname{NS}(Y)$ of Y is the quotient of $\operatorname{Pic} Y$ by the group $\operatorname{Pic}^0 Y$ consisting of all divisor classes algebraically equivalent to 0, see [Ha, Exer. V.1.7], and [Gr2, Exp. XIII, p. 644, 4.4]. If Y is proper, then $\operatorname{NS}(Y)$ is a finitely generated, abelian group, (see [Ha, Exer. V.1.7-8], for surfaces, or [Gr2, Exp. XIII, Thm. 5.1], in general.) Its rank $\rho = \dim \operatorname{NS}(Y) \otimes \mathbb{Q}$ is called the Picard number of Y. Note that for the rest of this section S is still an elliptic surface.

Proposition 2.8 *On S algebraic equivalence coincides with numerical equivalence. The group $\operatorname{NS}(S)$ is free. The intersection pairing induces a symmetric nondegenerate bilinear pairing on $\operatorname{NS}(S)$, making it into a lattice of signature $(1, \rho - 1)$. If S is a K3 surface, then $\operatorname{NS}(S)$ is an even lattice.*

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Proof The first statement is proved by Shioda [Sh, Thm. 3.1]. It follows immediately that the bilinear intersection pairing is nondegenerate on $\mathrm{NS}(S)$, see [Sh, Thm. 2.1] or [Ha, example V.1.9.1]. The signature is given by the Hodge Index Theorem [Ha, Thm. V.1.9]. If S is a K3 surface, then its canonical sheaf is trivial and the adjunction formula [Ha, Prop. V.1.5] reduces to $D^2 = 2g(D) - 2$ for any irreducible curve D on S with genus $g(D)$. As the irreducible divisors generate $\mathrm{NS}(S)$, the lattice $\mathrm{NS}(S)$ is even. ■

Lemma 2.9 *The induced map* $f^*\colon \mathrm{Pic}^0 C \to \mathrm{Pic}^0 S$ *is an isomorphism.*

Proof See [Sh, Thm. 4.1]. ■

For every point $P \in E(K)$, let (P) denote the prime divisor on S that is the image of the section $C \to S$ corresponding to P by Lemma 2.7. Let $T \subset \mathrm{NS}(S)$ be generated by the classes of the divisor $(\mathcal{O})$ and the irreducible components of the singular fibers of f. For every $v \in C$, let m_v denote the number of irreducible components of the fiber of f at v. Finally, let r denote the rank of the Mordell–Weil group $E(K)$.

Lemma 2.10 *The module T is a sublattice of* $\mathrm{NS}(S)$ *of rank* $\mathrm{rk}\, T = 2 + \sum_v (m_v - 1)$ *and signature* $(1, \mathrm{rk}\, T - 1)$.

Proof See [Sh, Prop. 2.3]. ■

Proposition 2.11 *There is a natural homomorphism* $\varphi\colon \mathrm{NS}(S) \to E(K)$ *with kernel T. It is surjective and maps (P) to P. We have* $\rho = \mathrm{rk}\,\mathrm{NS}(S) = r + 2 + \sum_v (m_v - 1)$.

Proof The map φ is defined in [Sh, §5]. For surjectivity, see [Sh, Lemmas 5.1 and 5.2]. The fact that T is the kernel is [Sh, Thm. 1.3]. The last equality follows from Lemma 2.10 and the fact that the alternating sum of the ranks of finitely generated, abelian groups in an exact sequence equals 0. ■

Corollary 2.12 *There is a unique section ψ of the homomorphism* $\mathrm{NS}(S) \otimes \mathbb{Q} \to E(K) \otimes \mathbb{Q}$ *induced by φ that maps $E(K) \otimes \mathbb{Q}$ onto the orthogonal complement of $T \otimes \mathbb{Q}$ in* $\mathrm{NS}(S) \otimes \mathbb{Q}$. *The homomorphism ψ induces a symmetric bilinear pairing on $E(K)$. The opposite of this pairing induces the structure of a positive definite lattice on* $E(K)/E(K)_{\mathrm{tors}}$.

Proof See [Sh, Thm. 8.4]. ■

Remark 2.13 Shioda gives an explicit formula for the pairing on $E(K)$, based on how the sections intersect the singular fibers and each other, see [Sh, Thm. 8.6].

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3 Proof of the Main Theorem

Let $G \subset \operatorname{Aut} X$ be the group of automorphisms of X generated by permutations of x, y and z, by permutations of a, b, and c, and by switching the sign of two of the coordinates a, b, and c. Then G is isomorphic to $(V_4 \rtimes S_3) \times S_3$ and has order 144. The surface X has 12 singular points, on which G acts transitively. They are all ordinary double points and their orbit under G is represented by $[x:y:z:a:b:c] = [2:-1:-1:1:1:1]$. Let $\pi\colon Y \to X$ be the blow-up of X in these 12 points. Since X is defined over $\mathbb{Q}$, so is Y.

Note that a K3 surface is a smooth, projective, geometrically irreducible surface S, of which the canonical sheaf is trivial and the irregularity $q = q(S) = \dim H^1(S, \mathcal{O}_S)$ equals 0.

Proposition 3.1 *The surface Y is a smooth K3 surface. The exceptional curves above the* 12 *singular points of X are all isomorphic to $\mathbb{P}^1$ and have self-intersection number* -2.

Proof Ordinary double points are resolved after one blow-up, so Y is smooth. The exceptional curves E_i are isomorphic to $\mathbb{P}^1$, see [Ha, Exer. I.5.7]. Their self-intersection number follows from [Ha, example V.2.11.4]. Since X is a complete intersection, it is geometrically connected and $H^1(X, \mathcal{O}_X) = 0$, so $q(X) = 0$, see [Ha, Exer. II.5.5]. From its connectedness it follows that Y is geometrically connected as well. As Y is also smooth, it follows that Y is geometrically irreducible.

To compute the canonical sheaf on Y, note that on the nonsingular part $U = X^{\mathrm{reg}}$ of X the canonical sheaf is given by $\mathcal{O}_X(-5-1+3+2+1)|_U = \mathcal{O}_U$, see [Ha, Prop. II.8.20; Exer. II.8.4]. Note that [Ha, Exer. II.8.4] applies to nonsingular complete intersections in $\mathbb{P}^n$. However, since the canonical sheaf is determined locally, the proof of this exercise shows that outside the singular locus of a complete intersection in $\mathbb{P}^n$ the same formula holds for the canonical sheaf. Hence, the canonical sheaf on Y restricts to the structure sheaf outside the exceptional curves. That implies that there are integers a_i such that $K = \sum_i a_i E_i$ is a canonical divisor. Recall that $E_i^2 = -2$ and $E_i \cdot E_j = 0$ for $i \neq j$. Applying the adjunction formula $2g_C - 2 = C \cdot (C + K)$ (see [Ha, Prop. V.1.5]) to $C = E_i$, we find that $a_i = 0$ for all i, whence $K = 0$.

It remains to show that $q(Y) = q(X)$. It follows immediately from [Ar, Prop. 1] that ordinary double points are rational singularities, *i.e.,* we have $R^1\pi_*\mathcal{O}_Y = 0$. Also, as X is integral, the sheaf $\pi_*\mathcal{O}_Y$ is a sub-$\mathcal{O}_X$-algebra of the constant $\mathcal{O}_X$-algebra $K(X)$, where $K(X) = K(Y)$ is the function field of both X and Y. Since π is proper, $\pi_*\mathcal{O}_Y$ is finitely generated as $\mathcal{O}_X$-module. As X is normal, *i.e.,* $\mathcal{O}_X$ is integrally closed, we get $\pi_*\mathcal{O}_Y \cong \mathcal{O}_X$. Hence, the desired equality $q(Y) = q(X)$ follows from the following lemma, applied to $f = \pi$ and $\mathcal{F} = \mathcal{O}_Y$. ■

Lemma 3.2 *Let $f\colon W \to Z$ be a continuous map of topological spaces. Let $\mathcal{F}$ be a sheaf of groups on W and assume that $R^i f_*(\mathcal{F}) = 0$ for all $i = 1, \dots, t$. Then for all $i = 0, 1, \dots, t$, there are isomorphisms*

$$H^i(W, \mathcal{F}) \cong H^i(Z, f_*\mathcal{F}).$$

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Proof This follows from the Leray spectral sequence. ■

We will now give $\overline{Y}$ the structure of an elliptic surface over $\mathbb{P}^1$. Let $f\colon Y \to \mathbb{P}^1$ be the composition of π with the morphism $f'\colon X \to \mathbb{P}^1$, $[x:y:z:a:b:c] \mapsto [x:a] = [2bc:yz]$. It is easily checked that if $[x:a]$ or $[2bc:yz]$ is not defined, then the other is. This shows that f', and hence f, is well defined everywhere.

If $a = 0$, then clearly $M_{a,b,c}$ in (1) has eigenvalue 0. Geometrically, this reflects the fact that the hyperplane $a = 0$ intersects X in three conics, one in each of the hyperplanes given by $xyz = 0$. Hence, each of the hyperplanes H_t given by $x = ta$ in the family of hyperplanes through the space $x = a = 0$ contains the conic given by $a = x = 0$ on X. The fibers of f consist of the inverse image under π of the other components in the intersection of X with the family of hyperplanes H_t. The fiber above $[t:1]$ is therefore given by the intersection of the two quadrics

$$xy + yz + zx = -a^2 - b^2 - c^2 \quad \text{and} \quad tyz = 2bc \tag{4}$$

within the intersection of two hyperplanes

$$x + y + z = x - ta = 0, \tag{5}$$

which is isomorphic to $\mathbb{P}^3$. The conic C given by $a + b = c - y = 0$ on X maps under f' isomorphically to $\mathbb{P}^1$. The strict transform of C on Y gives a section of f that we will denote by $\mathcal{O}$.

Proposition 3.3 *The morphism f and its section $\mathcal{O}$ give $Y_{\mathbb{C}}$ the structure of an elliptic surface over $\mathbb{P}^1_{\mathbb{C}}$.*

Proof Since Y is a K3 surface, it is minimal. Indeed, by the adjunction formula any smooth curve C of genus 0 on Y would have self-intersection $C^2 = -2$, while an exceptional curve that can be blown down has self-intersection -1, see [Ha, Prop. V.3.]. Hence, f is a relatively minimal fibration by Remark 2.6. The 12 exceptional curves give extra components in the fibers above $t = \pm 1, \pm 2$, so f is not smooth. From the description (4) above, an easy computation shows that the fibers above $t \neq 0, \pm 1, \pm 2, \infty$ are nonsingular. They are isomorphic to the complete intersection of two quadrics in $\mathbb{P}^3$, so by [Ha, Exer. II.8.4g], almost all fibers have genus 1. ■

Let $K \cong \mathbb{Q}(t)$ denote the function field of $\mathbb{P}^1_{\mathbb{Q}}$ and let E/K be the generic fiber of f. It can be given by the same equations (4) and (5). To put E in Weierstrass form, set $\lambda = (t^2-4)\nu+3t$ and $\mu = t(t^2-4)(z-y)(t\nu^2-2\nu+t)/x$, where $\nu = (x-c)/(a+b)$. Then the change of variables

$$u = \left(\mu + (\lambda^2 + t(t^2-1)(t+8))\right)/2,$$
$$v = \left(\mu\lambda + \lambda^3 + (t^2-1)(t^2-8)\lambda - 8t(t^2-1)^2\right)/2$$

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shows that E/K is isomorphic to the elliptic curve over K given by

$$v^2 = u\big(u - 8t(t^2-1)\big)\left(u - (t^2-1)(t+2)^2\right).$$

It has discriminant $\Delta = 2^{10}t^2(t^2-1)^6(t^2-4)^4$ and j-invariant

$$j = \frac{4(t^4+56t^2+16)^3}{t^2(t^2-4)^4}.$$

Lemma 3.4 *The singular fibers of f are at $t = 0, \pm 1, \pm 2$ and at $t = \infty$. They are described in the following table, where m_t (resp., $m_t^{(1)}$) is the number of irreducible components (resp., irreducible components of multiplicity* 1*).*

t	*type*	m_t	$m_t^{(1)}$
$0, \infty$	I_2	2	2
± 1	I_0^*	5	4
± 2	I_4	4	4

Proof This is a straightforward computation. Since we have a Weierstrass form, it also follows easily from Tate's algorithm, see [Ta] and [Si2, IV.9]. ■

Applying the automorphisms $(b,c) \mapsto (-c,-b)$ and $(b,c) \mapsto (-b,-c)$ and $(b,c,y,z) \mapsto (c,b,z,y)$ to the curve $\mathcal{O}$, we get three more sections, which we will denote by P, T_1 and T_2, respectively. By Lemma 2.7, these sections correspond with points on the generic fiber E/K. The Weierstrass coordinates (u,v) of these points are given by

$$T_1 = \left((t^2-1)(t+2)^2, 0\right), \quad T_2 = (0,0), \qquad (6)$$
$$P = \left(2t^3(t+1), 2t^2(t+1)^2(t-2)^2\right),$$

We immediately notice that the T_i are 2-torsion points.

Proposition 3.5 *The section P has infinite order in the group $S(C) \cong E(K)$.*

Proof Note that $S(C)$ and $E(K)$ are isomorphic by the identification of Lemma 2.7. By Corollary 2.12 there is a bilinear pairing on $E(K)$ that induces a nondegenerate pairing on $E(K)/E(K)_{\text{tors}}$. As mentioned in Remark 2.13, Shioda gives an explicit formula for this pairing, see [Sh, Thm. 8.6]. We find that $\langle P, P\rangle = \frac{3}{2} \neq 0$, so P is not torsion. ■

The main theorem now follows immediately.

Proof of Theorem 1.1 By Proposition 3.5 the multiples of P give infinitely many rational curves on Y, so the rational points on Y are dense. As π is dominant, the rational points on X are dense as well. ■

The multiples of P yield infinitely many parametrizations of integral, symmetric 3×3 matrices with zeros on the diagonal and integral eigenvalues. The section $2P$, for example, is a curve of degree 8 on X which can be parametrized by

$$a = t(t^6 - 8t^4 + 20t^2 - 12),$$
$$b = -t(t^6 - 4t^4 + 4),$$
$$c = (t^2 - 2)(t^6 - 6t^4 + 8t^2 - 4),$$

and suitable polynomials for x, y, and z. The parametrization (2) does not come from a section of f. We will see in Section 6 where it does come from.

4 The Mordell–Weil Group and the Néron–Severi Group

As mentioned in the introduction, the geometry and the arithmetic of K3 surfaces are closely related. In the following sections we will further analyze the geometry of Y. Set $L = \mathbb{C}(t) \supset \mathbb{Q}(t) = K$. In this section we will find explicit generators for the Mordell–Weil group $E(L)$ and for the Néron–Severi group of $\overline{Y} = Y_{\mathbb{C}}$. This will be used in Sections 5 and 6.

For any complex surface Z, the Néron-Severi group of Z can be embedded in $H^{1,1}(Z) = H^1(Z, \Omega_Z^1)$, see [BPV, p. 120]. If Z is a complex K3 surface, we have $\dim H^{1,1}(Z) = 20$, see [BPV, Prop. VIII.3.3]. Hence we find that the Picard number $\rho(Z) = \operatorname{rk} \mathrm{NS}(Z)$ is at most 20. If $\rho(Z)$ is equal to 20 we say that Z is a singular K3 surface.

Proposition 4.1 *The Picard group* $\operatorname{Pic} \overline{Y}$ *is isomorphic to* $\mathrm{NS}(\overline{Y})$ *and it is a finitely generated, free abelian group.*

Proof This is true for K3 surfaces in general, but also follows from the theory of elliptic surfaces described in Section 2, see [Sh]. As $\overline{Y}$ has the structure of an elliptic surface over $\mathbb{P}^1$ and $\operatorname{Pic}^0 \mathbb{P}^1 = 0$, the isomorphism follows from Lemma 2.9. The last statement follows from Proposition 2.8. ■

Two of the irreducible components of the singular fibers of $f \colon Y \to \mathbb{P}^1$ above $t = \pm 2$ are defined over $\mathbb{Q}(\sqrt{3})$. They are all in the same orbit under G. In that same orbit we also find a section, given by $z = 2b$ and $2(c - a) = \sqrt{3}(y - x)$. We will denote it by Q. Its Weierstrass coordinates are given by

$$Q = \left(2t(t+1)(t+2), 2\sqrt{3}t(t^2 - 4)(t+1)^2\right).$$

It follows immediately that the Galois conjugate of Q under the automorphism that sends $\sqrt{3}$ to $-\sqrt{3}$ is equal to $-Q$.

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Proposition 4.2 *The surface $\overline{Y}$ is a singular K3 surface. The Mordell–Weil group $E(L)$ is isomorphic to $\mathbb{Z}^2 \times (\mathbb{Z}/2\mathbb{Z})^2$ and generated by P, Q, T_1 and T_2. The Mordell–Weil group $E(K)$ is isomorphic to $\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^2$ and generated by P, T_1 and T_2.*

Proof From Shioda's explicit formula for the pairing on $E(K)$ (see Remark 2.13), we find that $\langle P, P\rangle = \frac{3}{2}$ and $\langle Q, Q\rangle = \frac{1}{2}$ and $\langle P, Q\rangle = 0$. Hence, P and Q are linearly independent and the Mordell–Weil rank $r = \operatorname{rk} E(L)$ is at least 2.

By Lemmas 3.4 and 2.10, the lattice generated by the vertical fibers and $\mathcal{O}$ has rank 18. From Proposition 2.11 it follows that the rank ρ of $\mathrm{NS}(\overline{Y}) = \mathrm{Pic}(\overline{Y})$ is at least $18 + 2 = 20$. As $\overline{Y}$ is a K3 surface (see Proposition 3.1) and 20 is the maximal Picard number for K3 surfaces in characteristic 0, we conclude that $\overline{Y}$ is a singular K3 surface. Using Proposition 2.11 again, we find that the Mordell–Weil rank of $E(L)$ equals 2. Since E has additive reduction at $t = \pm 1$, the order of the torsion group $E(L)_{\text{tors}}$ is at most 4, see [Si2, Remark IV.9.2.2]. Hence we have $E(L)_{\text{tors}} = \langle T_1, T_2\rangle$.

From Shioda's explicit formula for the height pairing it follows that with singular fibers only of type I_2, I_4 and I_0^*, the pairing takes values in $\frac{1}{4}\mathbb{Z}$. Hence, the lattice $\Lambda = (E(L)/E(L)_{\text{tors}})(4)$ is integral, see Definition 2.2. In Λ we have $\langle P, P\rangle = 6$ and $\langle Q, Q\rangle = 2$ and $\langle P, Q\rangle = 0$. Hence, by Lemma 2.4 the sublattice Λ' of Λ generated by P and Q has discriminant $\operatorname{disc} \Lambda' = 12 = n^2 \operatorname{disc} \Lambda$, with $n = [\Lambda : \Lambda']$. Therefore, n divides 2. Suppose $n = 2$. Then there is an $R \in \Lambda \setminus \Lambda'$ with $2R = aP + bQ$. By adding multiples of P and Q to R, we may assume $a, b \in \{0, 1\}$. In Λ we get $4 \mid \langle 2R, 2R\rangle = 6a^2 + 2b^2$. Hence, we find $a = b = 1$, so $2R = P + Q + T$ for some torsion element $T \in E(L)[2]$. Since all the 2-torsion of $E(L)$ is rational over L, it is easy to check whether an element of $E(L)$ is in $2E(L)$. If e is the Weierstrass u-coordinate of one of the 2-torsion points, then there is a homomorphism

$$E(L)/2E(L) \to L^*/L^{*2},$$

given by $S \mapsto u(S) - e$, where $u(S)$ denotes the Weierstrass u-coordinate of the point S, see [Si1, §X.1]. We can use $e = 0$ and find that for none of the four torsion points $T \in E(L)[2]$ the value $u(P + Q + T)$ is a square in L. Hence, we get $n = 1$ and $E(L)$ is generated by P, Q, T_1, and T_2.

Suppose $aP + bQ + \varepsilon_1 T_1 + \varepsilon_2 T_2$ is contained in $E(\mathbb{Q}(t))$ for some integers a, b, ε_i. Then also $bQ \in E(\mathbb{Q}(t))$. As the Galois automorphism $\sqrt{3} \mapsto -\sqrt{3}$ sends Q to $-Q$, we find that $bQ = -bQ$. But Q has infinite order, so $b = 0$. Thus, we have $E(\mathbb{Q}(t)) = \langle P, T_1, T_2\rangle$. ∎

To work with explicit generators of the Néron–Severi group of $\overline{Y}$, in the table below we will name some of the irreducible divisors that we encountered so far. The exceptional curves are given by the point on $\overline{X} = X_{\mathbb{C}}$ above which they lie. Other components of singular fibers are given by their equations on $\overline{X}$. Sections are given by their equations and the names they already have.

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D_1	$x=-2a, b+c=\frac{\sqrt{3}}{2}(y-z)$	D_{11}	$[-1:-1:2:-1:-1:1]$
D_2	$[2:-1:-1:-1:1:-1]$	D_{12}	$(T_1)\colon a-b=c+y=0$
D_3	$(\mathcal{O})\colon a+b=c-y=0$	D_{13}	$[2:-1:-1:1:1:1]$
D_4	$[-1:-1:2:1:-1:-1]$	D_{14}	$x=2a, 2(b-c)=\sqrt{3}(y-z)$
D_5	$a=-x, b=c$	D_{15}	$(Q)\colon z=2b, c-a=\frac{\sqrt{3}}{2}(y-x)$
D_6	$[-1:2:-1:1:-1:-1]$	D_{16}	$x=2a, 2(b-c)=\sqrt{3}(z-y)$
D_7	$(T_2)\colon a+c=b-z=0$	D_{17}	$x=b=0$
D_8	$[-1:2:-1:1:1:1]$	D_{18}	$a=y=0$
D_9	$[-1:2:-1:-1:1:-1]$	D_{19}	$(P)\colon a-c=b+y=0$
D_{10}	$a=x, b=-c$	D_{20}	F (whole fiber)

Proposition 4.3 *The sequence* $\{D_1, D_2, \ldots, D_{20}\}$ *forms an ordered basis for the Néron–Severi lattice* $\mathrm{NS}(\overline{Y})$. *With respect to this basis the Gram matrix of inner products is given by*

$$\begin{pmatrix}
-2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -2 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -2 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -2 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}.$$

Proof By Proposition 2.11 the Néron–Severi group $\mathrm{NS}(\overline{Y})$ is generated by $(\mathcal{O})$, all irreducible components of the singular fibers, and any set of generators of the Mordell–Weil group $E(L)$. Thus, from Lemma 3.4 and Proposition 4.2 we can find a set of generators for $\mathrm{NS}(\overline{Y})$. Using a computer algebra package or even by hand, one checks that $\{D_1, \ldots, D_{20}\}$ generates the same lattice. A big part of the Gram matrix is easy to compute, as we know how all fibral divisors intersect each other. Also, every section intersects each fiber in exactly one irreducible component, with multiplicity 1. The sections are rational curves, so by the adjunction formula they have self-intersection -2. That leaves $\binom{5}{2}$ more unknown intersection numbers among the sections. By applying appropriate automorphisms from $G \subset \operatorname{Aut} X$, we find that they are equal to intersection numbers that are already known by the above. ■

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Remark 4.4 By Proposition 4.3 the hyperplane section H is numerically equivalent to a linear combination of the D_i. This linear combination is uniquely determined by the intersection numbers $H \cdot D_i$ for $i = 1, \dots, 20$ and turns out to be some uninformative linear combination with many nonzero coefficients. The reason for choosing the D_i and their order in this manner is that $D_1, \dots, D_8$ and $D_9, \dots, D_{16}$ generate two orthogonal sublattices, both isomorphic to $E_8(-1)$. In fact, we have the following proposition, which will be used in Section 5.

Proposition 4.5 *The Néron–Severi lattice* $\mathrm{NS}(\overline{Y})$ *has discriminant* -48. *It is isomorphic to the orthogonal direct sum*

$$E_8(-1) \oplus E_8(-1) \oplus \mathbb{Z}(-2) \oplus \mathbb{Z}(-24) \oplus U,$$

where U *is the unimodular lattice with Gram matrix*

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Proof The discriminant of $\mathrm{NS}(\overline{Y})$ is the determinant of the Gram matrix, which equals -48. With respect to the basis $D_1, \dots, D_{20}$, let $C_1, \dots, C_4$ be defined by

$$C_1 = (0, 0, 0, -1, -2, -2, -2, -1, 1, 2, 3, 4, 4, 2, 0, 2, 1, -2, 0, 0),$$
$$C_2 = (6, 12, 26, 29, 32, 19, 6, 16, 9, 18, 27, 36, 34, 23, 12, 17, 7, -3, -8, 4),$$
$$C_3 = (1, 2, 4, 4, 4, 2, 0, 2, 2, 4, 6, 8, 8, 5, 2, 4, 2, -1, -1, 0),$$
$$C_4 = (1, 2, 4, 5, 6, 4, 2, 3, 1, 2, 3, 4, 4, 3, 2, 2, 0, 0, -1, 1),$$

and let $L_1, \dots, L_5$ be the lattices generated by $(D_1, \dots, D_8)$, $(D_9, \dots, D_{16})$, (C_1), (C_2), and (C_3, C_4), respectively. Then one easily checks that $L_1, \dots, L_5$ are isomorphic to $E_8(-1)$, $E_8(-1)$, $\mathbb{Z}(-2)$, $\mathbb{Z}(-24)$, and U respectively. They are orthogonal to each other, and the orthogonal direct sum $L = L_1 \oplus \cdots \oplus L_5$ has discriminant -48 and rank 20. By Lemma 2.4 we find that the index $[\mathrm{NS}(\overline{Y}) : L]$ equals 1, so $\mathrm{NS}(\overline{Y}) = L$. ■

5 The Surface $\overline{Y}$ Is Not Kummer

If A is an abelian surface, then the involution $\iota = [-1]$ has 16 fixed points. The quotient $A/\langle\iota\rangle$ therefore has 16 ordinary double points. A minimal resolution of such a quotient is called a Kummer surface. All Kummer surfaces are K3 surfaces. Because of their rich geometric structure, their arithmetic can be analyzed and described more easily. Every complex singular K3 surface is either a Kummer surface or a double cover of a Kummer surface, see [SI, Thm. 4 and its proof]. It is therefore natural to ask whether our complex singular K3 surface $\overline{Y}$ has the rich structure of a Kummer surface. In Corollary 5.8 we will see that this is not the case.

Shioda and Inose have classified complex singular K3 surfaces by showing that the set of their isomorphism classes is in bijection with the set of equivalence classes of positive definite even integral binary quadratic forms modulo the action of $\mathrm{SL}_2(\mathbb{Z})$, see [SI]. A singular K3 surface S corresponds with the binary quadratic form given by the intersection product on the oriented lattice $T_S = \mathrm{NS}(S)^\perp$ of transcendental cycles on S. Here the orthogonal complement is taken in the unimodular lattice $H^2(S, \mathbb{Z})$ of signature $(3, 19)$ (see [BPV, Prop. VIII.3.2]). To find out which quadratic form the surface $\overline{Y}$ corresponds to, we will use discriminant forms as defined by Nikulin [Ni, §1.3].

Definition 5.1 Let A be a finite abelian group. A *finite symmetric bilinear form* on A is a symmetric bilinear map $b\colon A \times A \to \mathbb{Q}/\mathbb{Z}$.

A *finite quadratic form* on A is a map $q\colon A \to \mathbb{Q}/2\mathbb{Z}$, such that for all $n \in \mathbb{Z}$ and $a \in A$ we have $q(na) = n^2 q(a)$ and such that the unique map $b\colon A \times A \to \mathbb{Q}/\mathbb{Z}$ determined by $q(a+a') - q(a) - q(a') \equiv 2b(a, a') \bmod 2\mathbb{Z}$ for all $a, a' \in A$ is a finite symmetric bilinear form on A. The form b is called *the bilinear form of* q.

Definition 5.2 Let L be an integral lattice. We define the *dual lattice* L^* by

$$\{x \in L_{\mathbb{Q}} \mid \langle x, y\rangle \in \mathbb{Z} \text{ for all } y \in L\}.$$

Lemma 5.3 *Let L be an even lattice and set $A_L = L^*/L$. Then we have $\#A_L = |\operatorname{disc} L|$ and the map*

$$q_L\colon A_L \to \mathbb{Q}/2\mathbb{Z}\colon x \mapsto \langle x, x\rangle + 2\mathbb{Z}$$

is a finite quadratic form on A_L.

Proof The first statement follows from the well-known fact $|\operatorname{disc} L| = [L^* : L]$. The map q_L is well defined, as for $x \in L^*$ and $\lambda \in L$, we have $\langle x + \lambda, x + \lambda\rangle - \langle x, x\rangle = 2\langle x, \lambda\rangle + \langle \lambda, \lambda\rangle \in 2\mathbb{Z}$. The unique map $b\colon A_L \times A_L \to \mathbb{Q}/\mathbb{Z}$ as in Definition 5.2 is given by $(a, a') \mapsto \langle a, a'\rangle + \mathbb{Z}$, which is clearly a finite symmetric bilinear form. Thus, q_L is a finite quadratic form. ■

Definition 5.4 If L is an even lattice, then the map q_L as in Lemma 5.3 is called the *discriminant-quadratic form* associated to L.

Lemma 5.5 *Let L be a primitive sublattice of an even unimodular lattice Λ. Let $L^\perp$ denote the orthogonal complement of L in Λ. Then $q_L \cong -q_{L^\perp}$*, i.e., *there is an isomorphism $A_L \to A_{L^\perp}$ making the following diagram commutative.*

$$\begin{array}{ccc} A_L & \xrightarrow{\cong} & A_{L^\perp} \\ {\scriptstyle q_L}\downarrow & & \downarrow{\scriptstyle q_{L^\perp}} \\ \mathbb{Q}/2\mathbb{Z} & \xrightarrow{[-1]} & \mathbb{Q}/2\mathbb{Z} \end{array}$$

Proof See [Ni, Prop. 1.6.1]. ■

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Lemma 5.6 *The embedding* $\mathrm{NS}(\overline{Y}) \to H^2(\overline{Y}, \mathbb{Z})$ *makes* $\mathrm{NS}(\overline{Y})$ *into a primitive sublattice of the even unimodular lattice* $H^2(\overline{Y}, \mathbb{Z})$*. We have* $\operatorname{disc} T_{\overline{Y}} = 48$.

Proof For the fact that $H^2(\overline{Y}, \mathbb{Z})$ is even and unimodular, see [BPV, Prop. VIII.3.2]. The image of the Néron–Severi group in $H^2(\overline{Y}, \mathbb{Z})$ is equal to $H^{1,1}(\overline{Y}) \cap H^2(\overline{Y}, \mathbb{Z})$, where the intersection is taken in $H^2(\overline{Y}, \mathbb{C})$, see [BPV, p. 120]. Hence, $\mathrm{NS}(\overline{Y})$ is a primitive sublattice. From Lemmas 5.3 and 5.5 we find

$$|\operatorname{disc} T_{\overline{Y}}| = |A_{T_{\overline{Y}}}| = |A_{\mathrm{NS}(\overline{Y})}| = |\operatorname{disc} \mathrm{NS}(\overline{Y})| = 48.$$

As $T_{\overline{Y}}$ is positive definite, we get $\operatorname{disc} T_{\overline{Y}} = 48$. ■

Up to the action of $\mathrm{SL}_2(\mathbb{Z})$, there are only four 2-dimensional positive definite even lattices with discriminant 48. The transcendental lattice $T_{\overline{Y}}$ is equivalent to one of them. They are given by the Gram matrices

$$\begin{pmatrix} 2 & 0 \\ 0 & 24 \end{pmatrix}, \quad \begin{pmatrix} 4 & 0 \\ 0 & 12 \end{pmatrix}, \quad \begin{pmatrix} 8 & 4 \\ 4 & 8 \end{pmatrix}, \quad \begin{pmatrix} 6 & 0 \\ 0 & 8 \end{pmatrix}. \tag{7}$$

Proposition 5.7 *Under the correspondence of Shioda and Inose, the singular K3 surface* $\overline{Y}$ *corresponds to the matrix*

$$\begin{pmatrix} 2 & 0 \\ 0 & 24 \end{pmatrix}.$$

Proof As $E_8(-1)$ and U as in Proposition 4.5 are unimodular, it follows from Proposition 4.5 and Lemma 5.3 that the discriminant-quadratic form of $\mathrm{NS}(\overline{Y})$ is isomorphic to that of $\mathbb{Z}(-2) \oplus \mathbb{Z}(-24)$. By Lemmas 5.5 and 5.6 we find that the discriminant-quadratic form associated to $T_{\overline{Y}}$ is isomorphic to that of $\mathbb{Z}(2) \oplus \mathbb{Z}(24)$, whence it takes on the value $\frac{1}{24} + 2\mathbb{Z}$. Of the four lattices described in (7), the lattice $\mathbb{Z}(2) \oplus \mathbb{Z}(24)$ is the only one for which that is true. ■

Corollary 5.8 *The surface* $\overline{Y}$ *is not a Kummer surface.*

Proof By [In, Thm. 0], a singular K3 surface S is a Kummer surface if and only if its corresponding positive definite even integral binary quadratic form is twice another such form, *i.e.*, if $x^2 \equiv 0 \mod 4$ for all $x \in T_S$. This is not true in our case. ■

6 All Curves on X of Low Degree

Note that so far we have seen 63 rational curves of degree 2 on $\overline{X}$, namely those in the orbits under G of

$$\begin{aligned} D_{10}&\colon x = a, && b = -c, \\ D_{16}&\colon x = 2a, && 2(b-c) = \sqrt{3}(z-y), \\ D_{17}&\colon x = 0, && b = 0. \end{aligned} \tag{8}$$

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These orbits have sizes 18, 36, and 9, respectively. All of these curves correspond to infinitely many matrices that are either trivial or not defined over $\mathbb{Q}$. To find more rational curves of low degree, we look at fibrations of $\overline{Y}$ other than f. The conic ($\mathcal{O}$) given by $a + b = c - y = 0$ on X determines a plane in the four-space in $\mathbb{P}^5$ given by $x + y + z = 0$. The family of hyperplanes in this four-space containing that plane, cuts out another family of elliptic curves on Y. One singular fiber in this family is contained in the hyperplane section $a + b = 2(c - y)$ on X. It is the degree 4 curve corresponding to the parametrization in (2). We will now see that this is the lowest degree of a parametrization of nontrivial matrices defined over $\mathbb{Q}$.

Recall that $G \subset \operatorname{Aut} X$ is the group of automorphisms of X generated by permutations of x, y and z, by permutations of a, b, and c, and by switching the sign of two of the coordinates a, b, and c.

Proposition 6.1 *The union of the three orbits under the action of G of the curves described in* (8) *consists of all* 63 *curves on $\overline{X}$ of degree smaller than* 4.

Arguments similar to the ones used to prove Proposition 6.1 can be found in [Br, p. 302]. To prove this final Proposition 6.1 we will use the following lemma.

Lemma 6.2 *Let S be a minimal, nonsingular, algebraic K3 surface over $\mathbb{C}$. Suppose D is a divisor on S with $D^2 = -2$.*

(i) *If $D \cdot H$ is positive for some ample divisor H on S, then D is linearly equivalent with an effective divisor.*
(ii) *If D is effective and its corresponding closed subscheme is reduced and simply connected, then the complete linear system $|D|$ has dimension* 0.

Proof Since the canonical sheaf on S is trivial and the Euler characteristic χ of $\mathcal{O}_S$ equals 2, the Riemann–Roch theorem for surfaces (see [Ha, Thm V.1.6]) tells us

$$l(D) - s(D) + l(-D) = \frac{1}{2}D^2 + \chi = 1,$$

where $l(D) = \dim H^0(S, \mathcal{L}(D)) = \dim |D| + 1$ and $s(D) = \dim H^1(S, \mathcal{L}(D))$ is the superabundance. Hence we have $l(D) + l(-D) \geq 1$, so D or $-D$ is effective, see also [PS, Lemma 2]. As we have $(-D) \cdot H < 0$, part (i) follows from the fact that effective divisors have nonnegative intersection with ample divisors.

For (ii), D is effective, so we also find $l(-D) = 0$. As the closed subscheme Z associated to D is reduced and connected, we have $\dim H^0(Z, \mathcal{O}_Z) = 1$. From the exact cohomology sequence associated to $0 \to \mathcal{O}_S(-Z) \to \mathcal{O}_S \to \mathcal{O}_Z \to 0$ we find $s(-D) = 0$, see [SD, Lemma 2.2]. By symmetry of Riemann–Roch we get $s(D) = 0$ and thus $l(D) = 1$, which proves (ii). ∎

Proof of Proposition 6.1 Let C be a curve on $\overline{X}$ of degree d and arithmetic genus g_a and let C also denote its strict transform on $\overline{Y}$. Let its coordinates with respect to the basis $\{D_1, \ldots, D_{20}\}$ of $\mathrm{NS}(\overline{Y})$ be given by $m_1, \ldots, m_{20}$. Let H denote a hyperplane section of $\overline{X}$ that does not contain any singular points of $\overline{X}$. By abuse of notation,

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let H also denote the strict transform of H on $\overline{Y}$ and its class in $\mathrm{NS}(\overline{Y})$. If E is any of the 12 exceptional curves on $\overline{Y}$, then we have $H \cdot E = 0$. For any curve D on $\overline{X}$ we also write D for the strict transform of D on $\overline{Y}$. We have $H \cdot D = \deg D$, where the intersection number $H \cdot D$ is taken on $\overline{Y}$. This determines $H \cdot D_i$ for all $i = 1, \dots, 20$ (see Remark 4.4), and we find

$$(9) \quad d = C \cdot H = 2\big(m_1 + m_3 + m_5 + m_7 + m_{10} + m_{12} + m_{14} + \\ + m_{15} + m_{16} + m_{17} + m_{18} + m_{19} + 2m_{20}\big).$$

This implies that d is even, say $d = 2k$. Since we have $H^2 = 6$, we can write the divisor class $[C] \in \mathrm{NS}(\overline{Y})$ as $[C] = \frac{d}{6}H + D = \frac{k}{3}H + D$ for some element $D \in \frac{1}{6}\langle H\rangle^{\perp}$, where the orthogonal complement is taken inside $\mathrm{NS}(\overline{Y})$. From the adjunction formula (see [Ha, Prop. V.1.5]) we find $C^2 = 2g_a - 2$, so from $C^2 = D^2 + (\frac{kH}{3})^2$ we get $D^2 = 2g_a - 2 - \frac{2k^2}{3}$. By the Hodge Index Theorem [Ha, Thm. V.1.9] the lattice $\frac{1}{e}\langle H\rangle^{\perp}$ is negative definite for any $e > 0$, so for fixed k and g_a there are only finitely many elements $D \in \frac{1}{6}\langle H\rangle^{\perp}$ with $D^2 = 2g_a - 2 - \frac{2k^2}{3}$. We will now make this more concrete. Set

$$\begin{aligned}
v_1 &= 2m_2 + m_5 + m_7 + m_{10} + m_{12} + m_{14} + m_{15} + m_{16} + m_{17} + m_{18} + 2m_{20} - k,\\
v_2 &= 4m_3 - m_4 + 2m_5 + 2m_7 + 2m_{10} + 2m_{12} + 2m_{14} + 2m_{15} + 2m_{16} + m_{17} + \\
&\qquad + 2m_{18} + 2m_{19} + 3m_{20} - 2k,\\
v_3 &= 7m_4 - 2m_5 + 2m_7 + 2m_{10} + 2m_{12} + 2m_{14} + 2m_{15} + 2m_{16} + m_{17} + 2m_{18} + \\
&\qquad + 2m_{19} + 3m_{20} - 2k,\\
v_4 &= 33m_5 - 14m_6 + 9m_7 - 14m_8 + 9m_{10} + 9m_{12} + 9m_{14} + 9m_{15} + 9m_{16} + 15m_{17} + \\
&\qquad + 9m_{18} + 16m_{19} + 24m_{20} - 9k,\\
v_5 &= 52m_6 - 24m_7 - 14m_8 + 9m_{10} + 9m_{12} + 9m_{14} + 9m_{15} + 9m_{16} + 15m_{17} + 9m_{18} + \\
&\qquad + 16m_{19} + 24m_{20} - 9k,\\
v_6 &= 24m_7 + m_8 + 4m_{10} + 4m_{12} + 4m_{14} + 4m_{15} + 4m_{16} + 11m_{17} - 9m_{18} - 3m_{19} + \\
&\qquad + 2m_{20} - 4k,\\
v_7 &= 35m_8 + 8m_{10} + 8m_{12} + 8m_{14} + 8m_{15} + 8m_{16} + 13m_{17} + 9m_{18} + 15m_{19} + \\
&\qquad + 22m_{20} - 8k,\\
v_8 &= 2m_9 - m_{10},\\
v_9 &= 211m_{10} - 140m_{11} + m_{12} + m_{14} + m_{15} + m_{16} + 41m_{17} + 23m_{18} + 50m_{19} + \\
&\qquad + 64m_{20} - k,\\
v_{10} &= 282m_{11} - 210m_{12} + m_{14} + m_{15} + m_{16} + 41m_{17} + 23m_{18} + 50m_{19} + 64m_{20} - k,\\
v_{11} &= 119m_{12} - 94m_{13} + m_{14} + m_{15} + m_{16} - 53m_{17} + 23m_{18} + 50m_{19} - 30m_{20} - k,\\
v_{12} &= 144m_{13} - 118m_{14} + m_{15} - 118m_{16} - 53m_{17} + 23m_{18} - 69m_{19} - 30m_{20} - k,
\end{aligned}$$

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$$
\begin{aligned}
v_{13} &= 86m_{14} - 71m_{15} - 58m_{16} - 5m_{17} + 23m_{18} - 9m_{19} + 18m_{20} - k,\\
v_{14} &= 1231m_{15} - 672m_{16} + 249m_{17} - 595m_{18} + 259m_{19} - 346m_{20} - 19k,\\
v_{15} &= 364m_{16} + 19m_{17} + 271m_{18} - 89m_{19} + 290m_{20} - 41k,\\
v_{16} &= 529m_{17} + 361m_{18} + 185m_{19} + 162m_{20} - 107k,\\
v_{17} &= 62m_{18} + m_{19} - 22m_{20} + 8k,\\
v_{18} &= 30m_{19} - 9m_{20} - 8k,\\
v_{19} &= 3m_{20} - 4k.
\end{aligned}
$$

After using (9) to express m_1 in terms of $m_2, \dots, m_{20}$, and k, we can rewrite the equation $C^2 = 2g_a - 2$ as

$$
(10)\quad 112(3 - 3g_a + k^2) = 84v_1^2 + 42v_2^2 + 6v_3^2 + \frac{4v_4^2}{11} + \frac{14v_5^2}{143} + \frac{7v_6^2}{13} + \frac{v_7^2}{5} + 84v_8^2 + \frac{6v_9^2}{1055} + \frac{28v_{10}^2}{9917} + \frac{12v_{11}^2}{799} + \frac{v_{12}^2}{102} + \frac{7v_{13}^2}{258} + \frac{7v_{14}^2}{52933} + \frac{6v_{15}^2}{16003} + \frac{6v_{16}^2}{6877} + \frac{336v_{17}^2}{16399} + \frac{28v_{18}^2}{155} + \frac{28v_{19}^2}{5}.
$$

Suppose k and g_a are fixed. Since the m_i are all integral, so are the v_j. As the right-hand side of (10) is a positive definite quadratic form in the v_j, we find that there are only finitely many integral solutions $(v_1, \dots, v_{19})$ of (10). The m_i being linear combinations of the v_j, there are also only finitely many integral solutions in terms of the m_i. In our case, the even degree d is smaller than 4, so $d = 2$ and $k = 1$. As all curves have even degree, the conic C is irreducible and hence smooth, as all irreducible conics are. Therefore we have $g_a = 0$. A computer search shows that for $k = 1$ and $g_a = 0$ there are exactly 441 solutions of (10) corresponding to integral m_i.

By Lemma 6.2(i) these correspond to 441 effective divisor classes $[D]$ on $\overline{Y}$ with $D^2 = -2$ and $H \cdot D = 2$. We will exhibit 441 of such divisors satisfying the hypotheses of Lemma 6.2(ii). That lemma then implies that each is the only effective divisor in its equivalence class and we conclude that they are the only 441 effective divisors D on $\overline{Y}$ satisfying $D^2 = -2$ and $D \cdot H = 2$.

The first 9 of these 441 divisors correspond to the curves in the orbit of D_{17}. Another 16 correspond to $D_{10} + \varepsilon_1 E_1 + \varepsilon_2 E_2 + \varepsilon_3 E_3 + \varepsilon_4 E_4$ where $\varepsilon_i \in \{0, 1\}$ and the E_i are the four exceptional curves of π that meet D_{10}. Each of these 16 divisors generates an orbit under G of size 18, giving 288 divisors on $\overline{Y}$ altogether. The last 144 divisors correspond to the divisors in the size 36 orbits of $D_{16} + \delta_1 M_1 + \delta_2 M_2$, with $\delta_i \in \{0, 1\}$ and where M_1 and M_2 are the exceptional curves of π in the fiber above $t = 2$. Of these 441 effective divisors, only 63 are the strict transform of a curve on $\overline{X}$, all in an orbit of one of the curves described in (8). ■

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