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A CLASS OF MODULES OVER A LOCALLY FINITE GROUP II

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Introduction

Our main purpose in this paper is to obtain more precise information about two problems which we investigated in Hartley (1971a). They are as follows:

PROBLEM 1. *Let G be a countable locally finite group and π be a set of primes. Suppose that $G = HK$, $H \lhd G$, $H \cap K = 1$, where H is a normal π'-subgroup of G, K is a π-group and $C_K(H) = 1$. If we assume that the Sylow (that is, maximal) π-subgroups of G are conjugate, what can we say about the structure of K?*

More generally, if we wish to consider this problem for uncountable G, it is appropriate to assume that G is Sylow π-sparse Hartley (1972), in the sense that no countable subgroup of G has $2^{\aleph_0}$ Sylow π-subgroups. However, by Hartley (1972) Lemma 3.5, this already implies that K is countable if it is locally soluble, and so in this case we quickly reduce to a problem about countable groups. A discussion of Sylow π-sparseness, and its relationship with other properties such as the conjugacy of the Sylow π-subgroups, can be found in Hartley (1972).

PROBLEM 2. *Let $G \in \mathfrak{U}$. What can be said about the structure of $G/\rho(G)$?*

Here $\rho(G)$ denotes the Hirsch-Plotkin radical of G, and $\mathfrak{U}$ is the class of groups studied in Gardiner, Hartley and Tomkinson (1971), Hartley (1971 and 1971a) and elsewhere. A locally finite group G belongs to $\mathfrak{U}$ if, for every set π of primes, the Sylow π-subgroups of every subgroup of G are conjugate Hartley (1972) (Theorem E and neighbouring remarks). Our previous results relating to these problems are to be found in Hartley (1971a) (Lemmas 4.7–4.8 and Theorem E), and in Theorems C and B respectively of this paper we shall provide answers to Problem 2 and the locally soluble case of Problem 1 which are in a reasonable sense complete.

To obtain our results we have had to introduce some module-theoretic ideas which we have thought it of interest to study in more generality than is required for the strict applications we have in mind.

We introduced in Hartley (1973) the concept of an $\mathfrak{M}_c$-module over kG, where G is a locally finite group and k a field of characteristic $p \geqq 0$ – a (right) kG-module V is called an $\mathfrak{M}_c$-module if each p'-subgroup A of G contains a finite subgroup $F = F(A)$ such that $C_V(A) = C_V(F)$, where $C_V(H)$ denotes the set of elements of V fixed by the subgroup H of G. This is a form of minimal condition on the subsets of V which are centralizers (that is, fixed point sets) of p'-subgroups of G. As was pointed out in Hartley (1973), such modules arise naturally in considering chief factors of $\mathfrak{U}$-groups, and we shall see that they also arise in other contexts as elementary abelian sections of groups whose Sylow subgroups have suitable conjugacy properties.

For our applications in this paper we have to consider the more general notion of an $\mathfrak{M}_c$*-family* for a group G (henceforth, the word "group" will always mean "locally finite group" unless the contrary is stated).

DEFINITION. *An $\mathfrak{M}_c$-family for a group G is a set $\mathbf{X}$ of (right) kG-modules X, where $k = k(X)$ is a field of characteristic $p(X) \geqq 0$, satisfyinng the following condition: For each subgroup A of G, there exists a finite subgroup $F \leqq A$ such that $C_X(F) = C_X(A)$ for all $X \in \mathbf{X}$ such that $p(X) \notin \pi(A)$.*

Here the subgroup F depends only on A and not on the particular module X under consideration, and $\pi(A)$ denotes the set of all primes q such that A contains an element of order q. We shall write $C_G(\mathbf{X}) = \bigcap_{X \in \mathbf{X}} C_G(X)$, and say that $\mathbf{X}$ is *faithful*, if this subgroup is 1. Further, we say that $\mathbf{X}$ is *irreducible* if each $X \in \mathbf{X}$ is irreducible as $k(X)[G]$-module, and *classical* if $p(X) \notin \pi(G)$ for all $X \in \mathbf{X}$.

Let us now consider how such families may arise in practice. Let H be any locally soluble group and let K be any group given together with an action (K on H), that is, a homomorphism from K into Aut H. Then we can consider K as an operator group for H, and then speak of K-composition series of H, using the worg "series" in the general sense of Robinson (1968), p. 5. By a *K-composition factor* of H we shall mean a factor of any such series. Thus a K-composition factor of H is, among other things, a pair $V \lhd U$ of K-invariant subgroups of H such that K normalizes no normal subgroup W of U with $V < W < U$. A straightforward modification of a well known argument due to McLain (Robinson (1968) Theorem 4.31) allows us to deduce from this, using the local solubility of H and the local finiteness of K, that U/V is an elementary abelian q-group for some prime p. Therefore we may view U/V in a natural way as an irreducible $\boldsymbol{Z}_pK$-module. Thus the K-composition factors of H determine a collection of irreducible $\boldsymbol{Z}_pK$-modules, for various primes p, which we shall call the composition factors of the action (K on H).

The relationship between $\mathfrak{M}_c$-families and groups with conjugate Sylow subgroups is now clarified by the results below. In connection with Lemma 6.3, notice that L is certainly Sylow π-sparse if the Sylow π-subgroups of every subgroup of L are conjugate. In fact, by Hartley (1972) Theorem B, these two properties are equivalent for groups like L. We state Lemma 6.3 in terms of the formally weaker property.

LEMMA 6.3. *Let H, K be subgroups of a group $L = HK$ with $H \lhd L$. Suppose that H is a locally soluble π'-group, K is a π-group, and L is Sylow π-sparse. Then the composition factors of $(K$ on $H)$ form a classical irreducible $\mathfrak{M}_c$-family $\mathbf{X}$ for K, and $C_K(\mathbf{X}) = C_K(H)$.*

Conversely, let $\mathbf{X}$ be a classical $\mathfrak{M}_c$-family for a π-group K, and suppose that $p(X) \neq 0$ for all $X \in \mathbf{X}$. Let H denote the direct sum of the modules $X \in \mathbf{X}$, and L the semidirect product HK. Then L is Sylow π-sparse, and $C_K(H) = C_K(\mathbf{X})$.

LEMMA 6.4. *Let $L \in \mathfrak{U}$, let $R = \rho(L)$ and $G = L/R$. Then the set of chief factors of L below R (that is, the composition factors of $(L$ on $R))$ forms in a natural way a faithful irreducible $\mathfrak{M}_c$-family for G.*

Conversely let G be a $\mathfrak{U}$-group admitting a faithful irreducible $\mathfrak{M}_c$-family $\mathbf{X}$ such that $p(X) \neq 0$ for all $X \in \mathbf{X}$. Let R denote the direct sum of the modules $X \in \mathbf{X}$ and L the semidirect product RG. Then $L \in \mathfrak{U}$, and $\rho(L) = R$.

In the second half of Lemma 6.4, the modules $X \in \mathbf{X}$ need not correspond to chief factors of L, since they need not be irreducible over the appropriate prime field.

Thus it is appropriate to study $\mathfrak{M}_c$-families in connection with our Problems 1 and 2.

DEFINITION. *An $\mathfrak{M}_c$-head is a group which admits a faithful irreducible $\mathfrak{M}_c$-family. A classical $\mathfrak{M}_c$-head is a group which admits a faithful classical irreducible $\mathfrak{M}_c$-family.*

As our main result on $\mathfrak{M}_c$-heads we have the following, which characterizes locally soluble $\mathfrak{M}_c$-heads completely and is the central result of the paper.

THEOREM A. *Let G be a locally soluble group. Then G is an $\mathfrak{M}_c$-head if and only if G is almost a subdirect product of a finite number of p'-pinched groups, for various primes p.*

We shall not give the definition of p'-pinched at present (see page 18), but simply remark that it is a more elaborate form of the definition below. A group G "almost" has a certain property, if G has a normal subgroup of finite index with the property.

DEFINITION. *A group G is pinched, if G contains a locally cyclic normal* *subgroup A such that G/A is abelian and A contains every element of prime order of G, and if furthermore every 2-subgroup of G is abelian.*

Notice that the properties of A imply that G contains no elementary abelian subgroup whose order is the square of a prime. Therefore every finite abelian subgroup of G is cyclic, and by Gorenstein (1968) Theorem 5.4.10, every subgroup of prime power order of G is either cyclic or generalized quaternion. The assumption on the 2-subgroups of G is introduced to rule out the latter possibility. For more detailed information about pinched groups, see Lemma 6.5.

From Theorem A we are able, via Lemmas 6.3 and 6.4, to deduce our answers to the problems posed at the outset.

THEOREM B. *Let G be a locally soluble group and let $\pi = \pi(G)$. Then necessary and sufficient conditions that there exist a locally finite and locally soluble Sylow π-sparse group $L = HK$ such that $H \lhd L$, H is a π'-group, $C_K(H) = 1$ and $K \cong G$ are*

(i) *There exists a prime $q \notin \pi$,*
(ii) *G is almost subpropinched,*

where we have used the abbreviation "subpropinched" for "subdirect product of finitely many pinched groups".

It seems quite conceivable that the hypotheses of local solubility are redundant in this result. In this connection, suppose that $L = HK$ is Sylow π-sparse, where H is a normal π'-subgroup of L, K is a π-group, and $C_K(H) = 1$. Then Hartley (1971a) Lemmas 4.7–4.8 show that every locally soluble subgroup of K has finite (Mal'cev special or Prüfer) rank. It follows easily from this and the Feit-Thompson Theorem that whenever a subgroup $Q \cong C_{2^\infty}$ of K normalizes a $2'$-subgroup R of K, then $[Q, R] = 1$. A recent theorem of Šunkov (1970) then shows that K contains a normal 2-subgroup K_0 such that the Sylow 2-subgroups of K/K_0 are finite. This seems to provide a considerable reduction in the problem of describing K in the case $2 \in \pi$.

In answer to Problem 2 we have a theorem with a very similar flavour to Theorem B.

THEOREM C. *Let G be a group. Then necessary and sufficient conditions for the existence of a group $L \in \mathfrak{U}$ such that $L/\rho(L) \cong G$ are*
(i) $G \in \mathfrak{U}$,
(ii) *G is almost subpropinched.*

Some consequences of this can be read off from Lemma 6.6. In particular, an immediate consequence of these results is Corollary C1 below, which may be compared with the well known theorem of Mal'cev that soluble linear groups are nilpotent-by-abelian-by-finite (Robinson (1968) Theorem 2.11). A theorem of

Wehrfritz ((1968) Theorem A1) shows that periodic soluble linear groups belong to $\mathfrak{U}$, and it appears that general $\mathfrak{U}$-groups are in some senses not too far from linear groups.

COROLLARY C1. *Let $G \in \mathfrak{U}$. Then $G/\rho(G)$ contains a normal subgroup of finite index which is metabelian and parasoluble.*

The concept of parasolubility is due to Wehrfritz (1971); a group H is parasoluble if H contains a finite series $1 = H_0 \leqq H_1 \leqq \cdots \leqq H_n = H$ of normal subgroups with abelian factors and such that every subgroup of H_i/H_{i-1} is normal in H/H_{i-1} $(1 \leqq i \leqq n)$.

The paper is organized along the following lines. Section 2 contains some elementary and basic observations about $\mathfrak{M}_c$-families in general. In Section 3 we attack the problem of describing the structure of a locally soluble $\mathfrak{M}_c$-head G and reduce it to the case when G is *finitely radical*, that is, has a finite series with locally nilpotent factors. The next section shows that it suffices to consider so-called *reduced* $\mathfrak{M}_c$-heads, and that finitely radical $\mathfrak{M}_c$-heads are in fact almost metabelian. Section 5 completes the proof of Theorem A and Section 6 the deduction of Theorems B and C.

NOTATION. Much of our notation has already been introduced. If $\{G_\lambda\}$ is a set of subgroups of a group, we write $\langle G_\lambda \rangle$ for the group they generate, and say that $\{G_\lambda\}$ is a *coherent* set if $\pi(\langle G_\lambda \rangle) = \bigcup_\lambda \pi(G_\lambda)$. A set of elements of a group will be called coherent if the cyclic subgroups they generate are coherent. A *Sylow basis* of a group G is a complete set of Sylow p-subgroups of G, one for each prime p, every subset of which is coherent. A Sylow basis of G is said to reduce into a subgroup H of G if the intersections of H with the members of the Sylow basis constitute a Sylow basis for H. Throughout, π denotes a set of primes and π' the complementary set. If G is a group in which the set of π-elements is a subgroup, we denote that subgroup by G_π.

By the *rank* of a group we always understand its Mal'cev special rank, so that a group G has finite rank if there exists a natural number n such that every finitely-generated subgroup of G can be generated by n elements. The least such n is then called the rank of G, and denoted by $\mathrm{rk}(G)$.

We write $\Omega_1(G)$ for the subgroup generated by the elements of prime order of G, and $Z(G)$ for the centre of G. A *section* of G is a factor U/V where $V \lhd U \leqq G$.

All modules considered in this paper will be right modules. If $A \leqq H \leqq G$ are subgroups of G, k is a field, and X is a kH-module, then X_A denotes the module X retricted to A and X^G the induced module.

If $\mathbf{X}$ is an $\mathfrak{M}_c$-family for H, then $\mathbf{X}_A = \{X_A : X \in \mathbf{X}\}$ and $\mathbf{X}^G = \{X^G : X \in \mathbf{X}\}$. The characteristic of a field k will be denoted by $\mathrm{char}\, k$, and we write $\mathrm{char}\, \mathbf{X} = \{\mathrm{char}\, k(X) : X \in \mathbf{X}\} = \{p(X) : X \in \mathbf{X}\}$, the *characteristic* of the $\mathfrak{M}_c$-family $\mathbf{X}$. If π is a set of primes, then $\mathbf{X}_\pi = \{X \in \mathbf{X} : p(X) \in \pi \cup \{0\}\}$.

A *section* of a module X is a factor U/V, where $V \leqq U$ are submodules of X.

2. Constructions with $\mathfrak{M}_c$-families

In this section, we are concerned with methods of constructing new $\mathfrak{M}_c$-families from given ones. Most of the observations are of an elementary nature.

LEMMA 2.1. *Let* $\mathbf{X}$ *be an* $\mathfrak{M}_c$*-family for a group* G. *Then*

(i) *Any subset of* $\mathbf{X}$ *is naturally an* $\mathfrak{M}_c$*-family for* G.

(ii) *If* $H \leqq G$, *then* $\mathbf{X}_H$ *is an* $\mathfrak{M}_c$*-family for* H.

(iii) *If* G *is locally soluble,* $K \lhd G$ *and* $K \leqq C_G(\mathbf{X})$, *then* $\mathbf{X}$ *is naturally an* $\mathfrak{M}_c$*-family for* G/K.

PROOF. (i) and (ii) are immediate from the definitions. As for (iii), let B/K be a subgroup of G/K and let $\pi = \pi(B/K)$. Suppose for a contradiction that for every finite subgroup F/K of B/K there exists a module $X \in \mathbf{X}$ such that $p(X) \notin \pi$ and $C_X(B) < C_X(F)$. Then there exists a tower $1 < F_1/K < F_2/K < \cdots$ of finite subgroups of B/K and sequence $X_1, X_2, \cdots$ of members of $\mathbf{X}$ such that $p(X_i) \notin \pi$ and $C_{X_i}(F_{i+1}) < C_{X_i}(F_i)$ for each i.

If $B^* = \bigcup_{i=1}^{\infty} F_i$ then B^*/K is a countable π-group, and so by well known results (cf Hartley (1971a) Lemma 2.1) we have $B^* = KA$ for some π-subgroup A of G. There exists a finite subgroup F of A such that $C_X(A) = C_X(F)$ for all $X \in \mathbf{X}$ with $p(X) \notin \pi = \pi(A)$.

In particular, choosing i so that $F \leqq F_i$, we obtain $C_{X_i}(F_{i+1}) \geqq C_{X_i}(A) = C_{X_i}(F) \geqq C_{X_i}(F_i)$, the desired contradiction. The result follows.

LEMMA 2.2. *The union of a finite number of* $\mathfrak{M}_c$*-families for a given group* G *is also an* $\mathfrak{M}_c$*-family for* G.

PROOF. This is immediate.

The next two results allow us under many circumstances to replace a given $\mathfrak{M}_c$-family by an irreducible one.

LEMMA 2.3. *Let* X *be a* kG*-module, let* $A \leqq G$ *and suppose that* $\operatorname{char} k \notin \pi(A)$. *Suppose that* F *is a finite subgroup of* A *such that* $C_X(A) = C_X(F)$, *and let* U/V *be a section of* X_A. *Then* $C_{U/V}(A) = C_{U/V}(F)$.

PROOF. Let $u + V$ be any element of $C_{U/V}(F)$. Then as F is finite, u lies in a submodule W of U_F which is finite-dimensional over k. Since $\operatorname{char} k \notin \pi(F)$, W is completely reducible, and $(C_W(F))\phi = C_{W\phi}(F)$ if ϕ is any kF-homomorphism of W. Thus $u + V \in C_W(F) + V/V$ and we find that

$$C_{U/V}(F) = C_U(F) + V/V = C_U(A) + V/V \leqq C_{U/V}(A).$$

Hence $C_{U/V}(F) = C_{U/V}(A)$, as claimed.

LEMMA 2.4. *Let* $\mathbf{X}$ *be an* $\mathfrak{M}_c$*-family for a group* G. *Then*

(i) *Suppose that, for each* $X \in \mathbf{X}$, Σ_X *is a family of* $k(X)[G]$*-sections of* X. *Then* $\bigcup_{X \in \mathbf{X}} \Sigma$ *is an* $\mathfrak{M}_c$*-family for* G.

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(ii) *Suppose* $\mathbf{X}$ *is classical and* Σ_X *is the set of all composition factors of* $X\,(X \in \mathbf{X})$. *Then* $\bigcup_{X \in \mathbf{X}} \Sigma_X = \mathbf{Y}$ *is a classical irreducible* $\mathfrak{M}_c$*-family for* G, *and* $C_G(\mathbf{Y}) = C_G(\mathbf{X})$.

(iii) *If* G *admits a faithful classical* $\mathfrak{M}_c$*-family then* G *is a classical* $\mathfrak{M}_c$*-head.*

PROOF. (i) is immediate from Lemma 2.3 and the definition. For (ii) we have that if $g \in G$ and g centralizes every member of Σ_X, then $\langle g \rangle$ stabilizes a series in the module X. Since the order of g is prime to char k, it follows that g acts trivially on X. Therefore $C_G(X) = C_G(\Sigma_X)$ for each $X \in \mathbf{X}$, and so

$$C_G(\mathbf{X}) = C_G(\mathbf{Y}).$$

(iii) is an immediate consequence of (ii) and the definitions.

We go on to consider field extensions.

LEMMA 2.5. *Let* $\mathbf{X}$ *be an irreducible* $\mathfrak{M}_c$*-family for a group* G. *For each* $X \in \mathbf{X}$, *let* $\bar{k}(X)$ *be an extension field for* $k(X)$, *and let* $\bar{X}$ *be a composition factor of the* $\bar{k}(X)[G]$*-module* $X \otimes_{k(X)} \bar{k}(X)$. *Then* $\bar{\mathbf{X}} = \{\bar{X} : X \in \mathbf{X}\}$ *is an irreducible* $\mathfrak{M}_c$*-family for* G, *and* $C_G(\mathbf{X}) = C_G(\bar{\mathbf{X}})$.

PROOF. Let $X \in \mathbf{X}$ and let $\{w_\lambda\}$ be a basis of $\bar{k} = \bar{k}(X)$ over $k = k(X)$. Then $X \otimes_k \bar{k} = \oplus_\lambda (X \otimes w_\lambda)$ so that, as kG-module, $X \otimes_k \bar{k}$ is a direct sum of copies of the irreducible module X and is in particular completely reducible. Hence every kG-section of $X \otimes_k \bar{k}$ is a direct sum of copies of X, and in particular $\bar{X}$, considered as kG-module, is such a direct sum. Therefore $C_G(X) = C_G(\bar{X})$, and if $F \leqq A \leqq G$ are such that

$$C_X(A) = C_X(F),$$

then $C_{\bar{X}}(A) = C_{\bar{X}}(F)$. From this the result follows.

COROLLARY 2.6. *Let* G *be an* $\mathfrak{M}_c$*-head. Then* G *has a faithful irreducible algebraically closed* $\mathfrak{M}_c$*-family.*

Here the terminology should be clear.

LEMMA 2.7. *Let* $\mathbf{X}$ *be a family of* kG*-modules* X, *where* $k = k(X)$ *is a field. Suppose* G *contains a subgroup* H *of finite index such that* $\mathbf{X}_H$ *is an* $\mathfrak{M}_c$*-family for* H. *Then* $\mathbf{X}$ *is an* $\mathfrak{M}_c$*-family for* G.

PROOF. Let $A \leqq G$. There is a finite subgroup A_0 of A such that $A = \langle A \cap H, A_0 \rangle$. Since $\mathbf{X}_H = \{X_H : X \in \mathbf{X}\}$ is an $\mathfrak{M}_c$-family, there is a finite subgroup F of $A \cap H$ such that $C_X(A \cap H) = C_X(F)$ for all $X \in \mathbf{X}$ such that $p(X) \notin \pi(A \cap H)$. Then $F^* = \langle F, A_0 \rangle$ is a finite subgroup of A, and $C_X(F^*) = C_X(A)$ if $X \in \mathbf{X}$ and $p(X) \notin \pi(A)$.

Finally we need to consider the induced family $\mathbf{X}^G$, where $\mathbf{X}$ is an $\mathfrak{M}_c$-family for a subgroup H of G.

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LEMMA 2.8. *Let H be a subgroup of finite index of a group G, and let $\mathbf{X}$ be an $\mathfrak{M}_c$-family for H. Then $\mathbf{X}^G$ is an $\mathfrak{M}_c$-family for G, and $C_G(\mathbf{X}^G) = \bigcap_{g \in G} C_H(\mathbf{X})^g$.*

PROOF. Let $s_1, \cdots, s_n$ be a right transversal to H in G, so that $G = \bigcup_{i=1}^n Hs_i$. Then, if $X \in \mathbf{X}$, we have $X^G = \oplus_{i=1}^n X \otimes s_i$. Let $K = \bigcap_{g \in G} H^g$. Then $|G: K| < \infty$ and, by Lemma 2.7 it suffices to show that $(\mathbf{X}^G)_K$ is an $\mathfrak{M}_c$-family for K. Let $A \leqq K$. Then Lemma 2.1 shows that $\mathbf{X}_K$ is an $\mathfrak{M}_c$-family and so, if $1 \leqq i \leqq n$, there is a finite subgroup F_i of $s_i A s_i^{-1}$ such that $C_X(F_i) = C_X(s_i A s_i^{-1})$ if $X \in \mathbf{X}$ and $p(X) \notin \pi(s_i A s_i^{-1}) = \pi(A)$. Let $F = \langle s_i^{-1} F_i s_i : 1 \leqq i \leqq n \rangle$. Then F is a finite subgroup of A. If $p(X) \notin \pi(A)$ and $\sum_{i=1}^n x_i \otimes s_i = x \, (x_i \in X)$ is an element of X^G centralized by F, then a direct calculation shows that F_i centralizes x_i. Therefore $s_i A s_i^{-1}$ centralizes x_i, and so A centralizes x. Hence $C_{X^G}(F) = C_{X^G}(A)$ for all $X \in \mathbf{X}$ such that $p(X) \notin \pi(A)$. Thus $(\mathbf{X}^G)_K$ is an $\mathfrak{M}_c$-family for K, as required.

It will often be expedient in the sequel to form the direct sum of the modules in an $\mathfrak{M}_c$-family $\mathbf{X}$ for a group G, and to have a name for the resulting object.

DEFINITION. *Let G be a group and let Y be a ZG-module. We say that Y is a G-mod if there is a set $\{k_\lambda : \lambda \in \Lambda\}$ of fields, and for each $\lambda \in \Lambda$ a $k_\lambda G$-module V_λ such that $Y \cong \oplus_{\lambda \in \Lambda} V_\lambda$ as ZG-modules.*

If the V_λ can be taken as the members of an $\mathfrak{M}_c$-family for G, we say that Y is an *$\mathfrak{M}_c$-mod over G*, and if the $\mathfrak{M}_c$-family can be chosen to be irreducible, we say that Y is a *completely reducible $\mathfrak{M}_c$-mod over G*.

If p is a prime, we write Y_p for the Sylow p-subgroup of Y, and we also write Y_0 for the maximal divisible subgroup of Y, viewing these as ZG–modules. If p is now a prime or zero, then under any isomorphism $Y \cong \oplus_\lambda V_\lambda$ as above, Y_p will correspond to the sum of those modules V_λ for which $\operatorname{char} k_\lambda = p$. Thus Y_p will be an $\mathfrak{M}_c$-mod over G if Y is, and so on. The same is true of $Y_\pi = \oplus_{q \in \pi} Y_q \oplus Y_0$, where π is a set of primes.

LEMMA 2.9. *Let G be a group, let Y be an $\mathfrak{M}_c$-mod over G, let $H \leqq G$ and let $\pi = \pi(H)$. Then $Y_{\pi'}$ is naturally an $\mathfrak{M}_c$-mod over H. and when so viewed, satisfies the minimal condition on centralizers.*

PROOF. Let $Y^* = Y_{\pi'}$. Then the above remarks and Lemma 2.1 show that Y^* is naturally an $\mathfrak{M}_c$-mod over H. From the definitions, it follows that each subgroup A of H contains a finite subgroup F such that $C_{Y^*}(A) = C_{Y^*}(F)$. Thus $(Y^*)_H$ satisfies the minimal condition on centralizers.

The following observation about $\mathfrak{M}_c$-mods will be important in the sequel.

LEMMA 2.10. *Let Y be a completely reducible $\mathfrak{M}_c$-mod for G and suppose that A is a finite non-cyclic abelian subnormal subgroup of G such that $|G: N_G(A)| < \infty$. Then*

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$$Y = \langle C_Y(a) \colon 1 \neq a \in A \rangle.$$

The proof requires the following (certainly well known) version of the weakest form of Clifford's theorem.

LEMMA 2.11. *Let G be any group, not necessarily locally finite, let k be a field, let V be an irreducible kG-module and let $N \lhd G$. Then V_N is completely reducible provided either $|G \colon N|$ is finite or $|N|$ is finite.*

PROOF. Suppose that we know that V_N contains an irreducible submodule U. Then $\sum_{x \in G} Ux$ is a kG-submodule of V, and so $V = \sum_{x \in G} Ux$. Since $N \lhd G$, each Ux is an irreducible kN-module. Thus V_N is a sum of irreducible submodules, and so is completely reducible. When N is finite the existence of U is clear.

If $|G \colon N|$ is finite, let $G = \bigcup_{i=1}^k Ns_i$ and let $0 \neq v \in V$. Then $\sum_{i=1}^k vs_i \cdot kN = \sum_{i=1}^k vkNs_i$ is a non-zero kG-submodule of V, and so is equal to V. Thus V_N is a finitely-generated kN-module, and so has a maximal submodule W. Since $\bigcap_{i=1}^k Ws_i$ is a proper kG-submodule we must have $\bigcap_{i=1}^k Ws_i = 0$. Thus V_N is isomorphic to a submodule of the completely reducible kN-module $\oplus_{i=1}^k V/Ws_i$, and so is completely reducible.

PROOF OF LEMMA 2.10. We may obviously suppose that Y is an irreducible kG-module, where k is a suitable field. There is a normal subgroup N of G such that $|G \colon N| < \infty$ and N normalizes A. Then AN is a subnormal subgroup of finite index of G and so, by applying the first case of Lemma 2.11 repeatedly, we find that Y_{AN} is completely reducible. Hence by the second case of that lemma, so is Y_A. Now it is well known that a finite abelian group which admits a faithful irreducible module is cyclic. Therefore each irreducible summand of Y_A is centralized by some non-trivial element of A. From this the result clearly follows.

3. Locally soluble $\mathfrak{M}_c$-heads

The main result in this section is the following lemma, which is the first step in describing the structure of locally soluble $\mathfrak{M}_c$-heads.

LEMMA 3.1. *Every locally soluble $\mathfrak{M}_c$-head is finitely radical.*

Before beginning its proof we need to draw attention to some well known facts.

LEMMA 3.2. *Let p be a prime. There is a function $f_p(r, n)$ such that if P is a p-group having an abelian normal subgroup of rank $\leqq r$ and index $\leqq n$, then every locally finite subgroup of* Aut P *has rank $\leqq f_p(r, n)$.*

PROOF. Let P be such a p-group and let P_0 be an abelian normal subgroup of P of rank $\leqq r$ and index $\leqq n$. Let $Q = P^n = \langle x^n \colon x \in P \rangle$. Then Q is a characteristic subgroup of P contained in P_0, and so is abelian and of rank $\leqq r$. Furthermore P_0/Q is an abelian group of rank $\leqq r$ and exponent dividing n, and so has order at most n^r. Thus $|P:Q| \leqq n^{r+1}$.

Let A be any finite subgroup of $\operatorname{Aut} P$, $A_0 = C_A(Q)$ and $A_1 = C_A(\Omega_1(Q))$. Then A/A_1 is clearly isomorphic to a subgroup of $GL(r,p)$, and so has order bounded by a number depending only on r and p. Also A_1/A_0 is isomorphic to a finite group of automorphism of Q which acts trivially on $\Omega_1(Q)$. By Gorenstein (1968) Theorem 5.2.4, such a group is necessarily a p-group, and by a result of Hall (Roseblade (1965) Lemma 5), it follows that A_1/A_0 can be generated by $\frac{1}{2}r\,(5r-1)$ elements. Therefore the number of generators required for A/A_0 is bounded by a number depending only on r and p. If $B = C_{A_0}(P/Q)$, then $|A_0/B| \leqq n^{r+1}!$ and so the number of generators needed by A/B is bounded by a number depending only on p, r and n.

Let $\{s_1, \cdots, s_k\}$ be a transversal to Q in P; thus $k \leqq n^{r+1}$. For $1 \leqq i \leqq k$, the map $b \to [s_i, b]$ maps B homomorphically into Q with kernel $C_B(s_i)$. Since these kernels clearly intersect trivially, it follows that B is isomorphic to a finite subgroup of a direct product of k copies of Q. Thus B can be generated by kr elements. Hence the number of generators required by A is bounded by a number depending only on p, r and n, as claimed.

Lemma 3.3. *Let H, K be finite subgroups of a locally soluble group G. Then there is an element $x \in G$ such that H and K^x are coherent.*

Proof. Let $\pi = \pi(H) \cup \pi(K)$ and let S be a Hall π-subgroup of the finite group $\langle H, K\rangle$ with $H \leqq S$. By Hall's theorem we have $K^x \leqq S$ for some $x \in \langle H, K \rangle$, and so $\langle H, K^x \rangle$ is a π-group.

We are now ready to begin the proof of Lemma 3.1, and deal with the classical case separately. In fact, this case has already effectively been dealt with in Hartley (1971a).

Lemma 3.4. *Let G be a classical locally soluble $\mathfrak{M}_c$-head. Then G is a metabelian-by-finite group of finite rank.*

Proof. Let X be a classical faithful irreducible $\mathfrak{M}_c$-family for G and let Y be the G-mod $\oplus_{X \in \mathbf{X}} X$. Then Y satisfies the minimal condition on centralizers, by Lemma 2.9. Let T denote the semidirect product YG. Since $C_G(Y) = 1$, every countable subgroup of T lies in one of the form Y^*G^*, where G^* is a countable subgroup of G normalizing a countable subgroup Y^* of Y such that $C_{G^*}(Y^*) = 1$. If $0 \notin \operatorname{char} \mathbf{X}$ then Hartley (1971a) Lemma 4.3 shows that Y^*G^* only has countably many Sylow π-subgroups. Therefore T is Sylow π-sparse, and the required information about G follows from Hartley (1972) Lemma 3.5.

If $0 \in \operatorname{char} \mathbf{X}$, then the arguments of Hartley (1971a) Lemmas 4.5–4.6 may be applied to Y^*G^* to show that every locally nilpotent subgroup of G^* is almost abelian and of finite rank. It follows that this holds for every locally nilptent subgroup of G. Therefore the result follows from Hartley (1971a) Lemma 4.8.

PROOF OF LEMMA 3.1. We have to consider a locally soluble group G having a faithful irreducible $\mathfrak{M}_c$-family X. As previously described, if σ is a set of primes, we write $X_\sigma = \{X \in X\colon \text{char } k(X) \in \sigma \cup \{0\}\}$, and we also write $K_\sigma = C_G(X_\sigma)$. By Lemma 2.1, X_σ is a faithful irreducible $\mathfrak{M}_c$-family for G/K_σ.

The argument proceeds in stages.

(i) *For each set π of primes, the π-subgroups of $G/K_{\pi'}$ are soluble and of bounded rank.*

For if H is any π-subgroup of $G/K_{\pi'}$, then Lemma 2.1 shows that $(X_{\pi'})_H$ is a faithful classical $\mathfrak{M}_c$-family for H. Therefore by Lemma 2.4 (iii) and Lemma 3.4, H is a soluble group of finite rank. The fact that the π-subgroups of $G/K_{\pi'}$ are of bounded rank follows by a standard argument using the local solubility of $G/K_{\pi'}$ (cf. Hartley (1972), proof of Corollary 3.5).

(ii) *If* char $X = p > 0$ *then G is finitely radical.*

For using (i) and Hartley (1972) Corollary 3.5, we obtain that $G/O_{p\,p',p}(G)$ is finite. But by a result of Kegel (Gardiner, Hartley and Tomkinson (1971) Lemma 3.2), $O_p(G)$ acts trivially on every irreducible module for G over a field of characteristic p. Therefore $O_p(G) = 1$. Since (i) shows that $O_{p'}(G)$ is soluble, (ii) follows

(iii) *The ranks of the groups $P/P \cap K_{p'}$ are bounded, where p runs over all primes and P over the p-subgroups of G.*

Notice that by (i) the ranks of the groups $P/P \cap K_{p'}$ are bounded for each fixed prime p, as P runs over the p-subgroups of G. Suppose that (iii) is false, and that for some natural number $k \geqq 1$ we have finite sets $\{p_1, \cdots, p_k\}$ and σ_k of primes such that $\{p_1, \cdots, p_k\} \cap \sigma_k = \varnothing$. Suppose further that for $1 \leqq i \leqq k$ we have a finite p_i-subgroup P_i of G such that

$$\text{rk}\,(P_i/P_i \cap K_{\sigma_k}) \geqq i \qquad (1 \leqq i \leqq k)$$

and $\{P_1, \cdots, P_k\}$ is a coherent set of subgroups. By (i) and the hypothesis that (iii) is false, we can choose a prime $p_{k+1} \notin \{p_1, \cdots, p_k\} \cup \sigma_k$ and a finite p_{k+1}-subgroup P_{k+1} of G such that

$$\text{rk}\,(P_{k+1}/P_{k+1} \cap K_{p'_{k+1}}) \geqq k + 1 + \sum_{i=1}^{k} n_i,$$

where n_i is the maximum of the ranks of the p_i'-subgroups of G/K_{p_i}, the latter being finite by (i). Furthermore we may assume, by Lemma 3.3, that P_{k+1} is coherent with $\langle P_1, \cdots, P_k \rangle$, so that $\{P_1, \cdots, P_{k+1}\}$ is a coherent set.

We now find that

$$\text{rk}\,(P_{k+1}/P_{k+1} \cap K_{\{p_1,\ldots,p_{k+1}\}'}) \geqq k + 1.$$

Therefore there is a finite subset τ_k of $\{p_1, \cdots, p_{k+1}\}'$ such that

$$\text{rk}\,(P_{k+1}/P_{k+1} \cap K_{\tau_k}) \geqq k + 1.$$

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Letting $\sigma_{k+1} = \sigma_k \cup \tau_k$, we find that the construction can be carried one stage further. Since the construction can clearly be begun, we can eventually obtain two disjoint sets $\{p_1, p_2, \cdots\}$ and $\sigma = \bigcup_{k=1}^{\infty} \sigma_k$ of primes and a coherent set $\{P_1, P_2, \cdots\}$ of finite subgroups of G such that P_i is a p_i-group and

$$\text{(1)} \qquad \mathrm{rk}\,(P_i/P_i \cap K_\sigma) \geqq i$$

for all $i = 1, 2, \cdots$ Let $L = \langle P_1, P_2, \cdots \rangle$. Then by Lemma 2.1, X_σ is a faithful classical $\mathfrak{M}_c$-family for $L/L \cap K_\sigma$ and therefore, by Lemmas 2.4 and 3.4, $L/L \cap K_\sigma$ has finite rank. This contradiction to (1) above establishes (iii).

(iv) *There exists a finite set σ of primes and a natural number t such that if $p \notin \sigma$ and P is a p-subgroup of G then* $\mathrm{rk}(P) \leqq t$.

Again we assume the result false and otbtain a contradiction by carrying out a suitable construction. At the k-th stage of the construction we have two disjoint sets $\{p_1, \cdots, p_k\}$ and $\{q_1, \cdots, q_k\}$ of primes and coherent elements $g_1, \cdots, g_k$ of G such that g_i is of order p_i and belongs to K_{q_i}.

Now it follows from (iii) that there is an integer l such that

$$\text{(2)} \qquad \mathrm{rk}\,(P/P \cap K_{p'}) \leqq l$$

for all primes p and p-subgroups P of G. Thus the ranks of the finite p-subgroups of $G/K_{p'}$ are bounded, and so $G/K_{p'}$ satisfies min-p. It follows from work of Wehrfritz (1971a) that there exist natural numbers (r_i, n_i) $(1 \leqq i \leqq k)$ such that every q_i-subgroup of $G/K_{q_i'}$ contains an abelian subgroup of rank $\leqq r_i$ and index $\leqq n_i$. In fact, this even holds for every q_i-section of $G/K_{q_i'}$.

Since (iv) is assumed false, we can choose a prime $q_{k+1} \notin \{p_1, \cdots, p_k\}$ and a q_{k+1}-subgroup Q of G such that

$$\mathrm{rk}\,(Q) \geqq 1 + l + \sum_{i=1}^{k} f_{q_i}(r_i, n_i)$$

where the functions f are those given by Lemma 3.2. Then by (2), if $Q^* = Q \cap K_{q'_{k+1}}$, then

$$\text{(3)} \qquad \mathrm{rk}\,(Q^*) \geqq 1 + \sum_{i=1}^{k} f_{q_i}(r_i, n_i).$$

Let Y denote the subfamily consisting of those $X \in X$ such that char $k(X) = q_{k+1}$ and let $L = C_G(Y)$. By Lemma 2.1, Y is a faithful irreducible $\mathfrak{M}_c$-family for $\bar{G} = G/L$. Now by Gardener, Hartley and Tomkinson (1971) Lemma 3.2, $O_{q_{k+1}}(\bar{G}) = 1$, and since Q^* acts trivially on $X_{q'_{k+1}}$ the natural homomorphism of G onto $\bar{G}$ maps Q^* isomorphically onto a subgroup $\bar{Q}^*$ of $\bar{G}$ which intersects the Hirsch-Plotkin radical $\rho(\bar{G})$ trivially. By (ii), $\bar{G}$ is finitely radical, and so $\bar{Q}^*$ transforms $\rho(\bar{G})$ faithfully by conjugation. Now as $L \geqq K_{q_i'}$ $(1 \leqq i \leqq k)$, the group $O_{q_i}(\bar{G})$ contains an abelian normal subgroup of rank $\leqq r_i$ and index

$\leqq n_i$. Therefore the group of automorphisms induced on it by $\overline{Q^*}$ has rank at most $f_{q_i}(r_i, n_i)$, by Lemma 3.2. It follows from (3) that there is a prime $p_{k+1} \notin \{q_1, \cdots, q_{k+1}\}$ such that $[Q^*, O_{p_{k+1}}(\bar{G})] \neq 1$. Therefore Q^{*G} contains an element g_{k+1} of order p_{k+1}. Since $Q^* \leqq K_{q'_{k+1}}$, we have $g_{k+1} \in K_{q'_{k+1}}$, and by Lemma 3.3 we can arrange that g_{k+1} is coherent with $g_1, \cdots, g_k$.

Since the above argument also shows how to begin the construction, we obtain in due course disjoint sets $\{p_1, p_2, \cdots\}$ and $\{q_1, q_2, \cdots\}$ of primes and coherent elements $g_1, g_2, \cdots$ such that g_i is of order p_i and belongs to $K_{q'_i}$. Let $H = \langle g_1, g_2, \cdots \rangle$ and $\sigma = \{q_1, q_2, \cdots\}$. Then $\mathbf{X}_\sigma$ is a classical $\mathfrak{M}_c$-family for H, and so there is a finite subgroup F of H such that $C_X(F) = C_X(H)$ for all $X \in \mathbf{X}_\sigma$. Choose n so that $F \leqq \langle g_1, \cdots, g_n \rangle$. Then F centralizes every module $X \in \mathbf{X}$ such that char $k(X) = q_{n+1}$, whereas on the other hand there must be such a module not centralized by g_{n+1}. This contradiction establishes (iv).

Finally we obtain

(v) *G is finitely radical.* Let $L = \cap\, C_G(X)$ over all X such that char $k(X) \in \sigma$, the set given by (iv). Since σ is finite, (ii) gives that G/L is finitely radical. Now $L \cap K_{\sigma'} = 1$ and so $\mathbf{X}_{\sigma'}$ is a faithful $\mathfrak{M}_c$-family for L. It follows from Lemmas 2.1, 2.4 and 3.4 that if p is a prime in σ then every p-subgroup of L has finite rank. By the choice of σ, it follows that every abelian subgroup of L has finite rank. A theorem of Gorčakov (1964) now shows that L itself has finite rank, and a theorem of Kargapolov (1959) yields that L is finitely radical (in fact, $L/\rho(L)$ is abelian-by-finite). Thus G is finitely radical, as asserted.

4. Reduced $\mathfrak{M}_c$-Heads

In this section we shall show that every locally soluble $\mathfrak{M}_c$-head is almost a subdirect product of so-called *reduced* $\mathfrak{M}_c$-heads. Taken in conjunction with the results of the last section, this will allow us to show that locally soluble $\mathfrak{M}_c$-heads are almost metabelian, thus effectively reducing the proof of the main Theorem A to the metabelian case.

DEFINITION. *A group G will be called reduced if whenever A is a subnormal abelian subgroup of G such that $|G: N_G(A)| < \infty$, then A is locally cyclic.*

We have used this perhaps rather unsatisfactory term since reduced $\mathfrak{M}_c$-heads are the end product of a "reduction process", as we shall see.

The following remark is immediate. We recall that H sn G denotes that H is a subnormal subgroup of G.

LEMMA 4.1. *If G is reduced, H sn G and $|G: H| < \infty$, then H is reduced.*

Our aim in this section is to establish

LEMMA 4.2. *Let G be a locally soluble $\mathfrak{M}_c$-head. Then G is almost a subdirect product of finitely many reduced $\mathfrak{M}_c$-heads.*

Before beginning the proof of it we need some information about the Hirsch-Plotkin radical of an $\mathfrak{M}_c$-head. Here, local solubility is not assumed.

LEMMA 4.3. *Let G be an $\mathfrak{M}_c$-head with Hirsch-Plotkin radical R. Then R satisfies Min-p for all primes p.*

PROOF. Let X be a faithful irreducible $\mathfrak{M}_c$-family for G. By Gardiner, Hartley and Tomkinson (1971) Lemma 3.2, R_p acts trivially on every irreducible G-module over a field of characteristic $p > 0$. Hence by Lemma 2.1, $X_{p'}$ can be viewed as a faithful classical $\mathfrak{M}_c$-family for R_p. Therefore R_p satisfies Min, by Lemma 3.4 and a theorem of Černikov (1951).

We could in fact show, by adapting the arguments of Hartley (1971a) Theorem E, that R is almost abelian and of finite rank. However we do not require this fact at present, and in the case when G is locally soluble, it will in due course emerge from our subsequent results.

PROOF OF LEMMA 4.2. Let G be a locally soluble group with a faithful irreducible $\mathfrak{M}_c$-family X. If X denotes the direct sum of the members of X, then X is a completely reducible $\mathfrak{M}_c$-mod over G, and if H is a π-subgroup of G, then the H-mod $(X_{\pi'})_H$ satisfies the minimal condititon on centralizers (Lemma 2.9). However $(X_{\pi'})_H$ may conceivably not be completely reducible.

Now in proving Lemma 4.2 we may clearly suppose that G is not itself reduced. We shall show how to construct two sequences S_n, T_n of finite sets of centralizers in X of subsets of G ($n \geqq 0$) and a sequence $H_0, H_1, \cdots$ of normal subgroups of finite index of G such that the following conditions are satisfied:

(i) H_n normalizes each $K \in S_n \cup T_n$ and $\cap\, C_{H_n}(K) = 1$ as K runs over $S_n \cup T_n$.

(ii) Each $K \in S_n \cup T_n$ is a completely reducible $\mathfrak{M}_c$-mod over H_n.

(iii) If $K \in S_n$ then $H_n/C_{H_n}(K)$ is not reduced.

(iv) If $K \in T_n$ then $H_n/C_{H_n}(K)$ is reduced,

all the actions above being the natural ones. The construction is an elaboration of the argument of Hartley (1971a) Lemma 4.6.

We begin by putting $H_0 = G$, $S_0 = \{X\}$, $T_0 = \phi$. Having obtained H_n, S_n and T_n, we proceed as follows. If $S_n = \phi$ the construction is terminated. Otherwise, choose for each of the finitely many $K \in S_n$ a non-cyclic subnormal elementary abelian subgroup $A_K/C_{H_n}(K)$ of $H_n/C_{H_n}(K)$ normalized by a subgroup of finite index of $H_n/C_{H_n}(K)$. Then $A_K/C_{H_n}(K)$ lies in the Hirsch-Plotkin radical of the $\mathfrak{M}_c$-head $H_n/C_{H_n}(K)$, and so is finite by Lemma 4.3. It follows that there is a normal subgroup H_{n+1} of G contained in H_n and such that

$$|G: H_{n+1}| < \infty \tag{1}$$

and

$$H_{n+1} \text{ centralizes } A_K/C_{H_n}(K) \quad (K \in S_n). \tag{2}$$

Since $A_K/C_{H_n}(K)$ is finite, we have from Lemma 2.10 and (ii) that

(3) $$K = \langle C_K(t)\colon 1 \neq t \in E_K \rangle \qquad (K \in \boldsymbol{S}_n)$$

where E_K is any transversal to $C_{H_n}(K)$ in A_K containing 1.

Now as $A_K/C_{H_n}(K)$ is a subnormal subgroup of $H_n/C_{H_n}(K)$ centralized by H_{n+1}, $H_{n+1}A_K$ is a subnormal subgroup of finite index of H_n. Therefore, by (ii), Lemma 2.11 and Lemma 2.4 (i), $K_{H_{n+1}A_K}$ is a completely reducible $\mathfrak{M}_c$-mod over $H_{n+1}A_K$. If we express it as a direct sum of irreducible $H_{n+1}A_K$-modules, we find, using (2), that the centralizer in it of an element $t \in A_K$ is just the direct sum of those summands in the decomposition which t centralizes. Therefore $C_K(t)$ is a completely reducible $H_{n+1}A_K$-mod, and hence, by Lemmas 2.11 and 2.4 (i) again,

(4) *$C_K(t)$ is a completely reducible $\mathfrak{M}_c$-mod over H_{n+1}.*

It follows from (3) that $\bigcap_{1 \neq t \in E_K} C_{H_{n+1}}(C_K(t)) = C_{H_{n+1}}(K)$ and hence, if we let $\boldsymbol{U}_{n+1}$ consist of the H_{n+1}-mods $C_K(t)$ $(K \in \boldsymbol{S}_n,\ 1 \neq t \in E_K)$, together with the mods $Y_{H_{n+1}}(Y \in \boldsymbol{T}_n)$, then (i) shows that H_{n+1} normalizes each $L \in \boldsymbol{U}_{n+1}$, and

(5) $$\bigcap_{L \in \boldsymbol{U}_{n+1}} C_{H_{n+1}}(L) = 1.$$

We now divide the members of $\boldsymbol{U}_{n+1}$ into two sets $\boldsymbol{S}_{n+1}$, $\boldsymbol{T}_{n+1}$, throwing $L \in \boldsymbol{U}_{n+1}$ into $\boldsymbol{S}_{n+1}$ or $\boldsymbol{T}_{n+1}$ according as $H_{n+1}/C_{H_{n+1}}(L)$ is not or is reduced, respectively. Now if $L \in \boldsymbol{T}_n$ then $H_{n+1}/C_{H_{n+1}}(L) \cong H_{n+1}C_{H_n}(L)/C_{H_n}(L)$, a normal subgroup of finite index of $H_n/C_{H_n}(L)$. It follows from Lemma 4.1 that

$$H_{n+1}/C_{H_{n+1}}(L)$$

is reduced, and hence that

(6) $$\boldsymbol{T}_n \leqq \boldsymbol{T}_{n+1}.$$

Now since H_{n+1} is a normal subgroup of finite index of H_n, Lemmas 2.11 and 2.4 (i) show that $K_{H_{n+1}}$ is a completely reducible $\mathfrak{M}_c$-mod over H_{n+1} if $K \in \boldsymbol{T}_{n+1}$. This, together with (4) and (5), shows that (i) and (ii) of the conditions required by the construction hold; (iii) and (iv) are immediate from the definition of $\boldsymbol{S}_{n+1}$ and $\boldsymbol{T}_{n+1}$.

The lemma will evidently be proved if we can show that our construction terminates, that is, $\boldsymbol{S}_n = \phi$ for some n. For then H_n is a subdirect product of a finite number of reduced $\mathfrak{M}_c$-heads, namely the $H_n/C_{H_n}(K)$ $(K \in \boldsymbol{T}_n)$.

If the construction proceeds indefinitely, then the $\boldsymbol{S}_n$ form a sequence of non-empty finite sets. From (6) and the construction, each member of $\boldsymbol{S}_{n+1}$ has the form $C_K(t)$ for some $K \in \boldsymbol{S}_n$ and $t \in E_K$, and by choosing such a K we obtain a map of $\boldsymbol{S}_{n+1}$ into $\boldsymbol{S}_n$. In view of the fact that an inverse limit of non-empty finite sets is non-empty, it follows that we can select a sequence $K_1, K_2, \cdots$ such that $K_i \in \boldsymbol{S}_i$ and $K_{i+1} = C_{K_i}(t_i)$ for some element $1 \neq t_i \in E_{K_i}$. Let p_i be the prime

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divisor of the order of $A_{K_i}/C_{H_i}(K_i)$. There is a countable subgroup M of G which covers each of the factors $A_{K_i}/C_{H_i}(K_i)$. Let $M_0 \leqq M_1 \leqq \cdots$ be a tower of finite subgroup of M such that $\cup_{i=0}^{\infty} M_i = M$, and let $\boldsymbol{R}$ be a Sylow basis of M which reduces into each M_i. Since A_{K_i} is subnormal in G it follows that $A_{K_i} \cap M_j$ sn M_j for each j and hence that $\boldsymbol{R}$ reduces into each of the groups $A_{K_i} \cap M_j$. Since one of these covers $A_{K_i}/C_{H_i}(K_i)$, it follows that the Sylow p_i-subgroup R_{p_i} of $\boldsymbol{R}$ also covers $A_{K_i}/C_{H_i}(K_i)$. Therefore we may suppose that

$$t_i \in R_{p_i} \qquad (i = 1, 2, \cdots). \tag{7}$$

Now the image of t_i in $H_i/C_{H_i}(K_i)$ lies in a subnormal p_i-subgroup of the latter group, and so by (ii) and Gardiner, Hartley and Tomkinson (1971) Lemma 3.2, we find that t_i acts trivially on the p_i-component of K_i. Therefore there is a prime $q_i \neq p_i$ such that t_i acts non-trivially on $K_{i,q_i} = C_{X_{q_i}}(\langle t_1, \cdots, t_{i-1}\rangle)$. We thus have

$$C_{X_{q_i}}(\langle t_1, \cdots, t_i\rangle) < C_{X_{q_i}}(\langle t_1, \cdots, t_{i-1}\rangle) \tag{8}$$

for each i. It follows that if $i(1) < i(2) < \cdots$ is any strictly increasing sequence of natural numbers, then

$$C_{X_{q_{i(j+1)}}}(\langle t_{i(1)}, \cdots, t_{i(j+1)}\rangle) < C_{X_{q_{i(j+1)}}}(\langle t_{i(1)}, \cdots, t_{i(j)}\rangle). \tag{9}$$

For by (8), the subgroups obtained by intersecting the two sides of (9) with the centralizer in $X_{q_{i(j+1)}}$ of $\langle t_1, \cdots, t_{i(j+1)-1}\rangle$ are distinct.

We next consider the sequences $p_1, p_2, \cdots$ and $q_1, q_2, \cdots$, and claim first that no prime occurs infinitely often in the sequence $\{p_i\}$. Indeed if $i(1) < i(2) < \cdots$ is an infinite sequence such that $p_{i(j)} = p$ for all j, then as $q_{i(j)} \neq p$ for all j and (by (8)) the elements $t_{i(j)}$ generate a p-group, (9) gives a contradiction to the fact that $X_{p'}$, when restricted to any p-subgroup of G, satisfies the minimal condition on centralizers (Lemma 2.9).

Furthermore, no prime q can occur infinitely often among the q_i. For in the contrary case we obtain a sequence $i(1) < i(2) < \cdots$ such that $p_{i(j)} \neq q$ and $q_{i(j)} = q$ for all j. Using (7) and (9), we now obtain a contradiction to the fact that X_q satisfies the minimal condition on centralizers when restricted to any q'-subgroup of G.

An immediate recursive construction now allows us to obtain an infinite sequence $i(1) < i(2) < \cdots$ such that the sets $\sigma = \{p_{i(j)}\}$ and $\tau = \{q_{i(j)}\}$ are disjoint. From (9), we find that the sequence of centralizers $C_{X_\tau}(\langle t_{i(1)}, \cdots, t_{i(j)}\rangle)$ is strictly decreasing. But (7) shows that the elements $t_{i(1)}, t_{i(2)}, \cdots$ generate a τ'-group, and so we have a final contradiction to Lemma 2.9. Therefore Lemma 4.2 is established.

Regarding the structure of reduced $\mathfrak{M}_c$-heads, we can say the following:

LEMMA 4.4. *Let G be a reduced locally soluble $\mathfrak{M}_c$-head with Hirsch-Plotkin radical R. Then $O_{2'}(R)$ is locally cyclic. There is a normal subgroup*

H of finite index in G and a locally cyclic normal subgroup A of H such that $A = C_H(A)$. Further, H/A is abelian.

PROOF. By Lemma 4.3, the Hirsch-Plotkin radical R of G satisfies Min-p for all primes p. Since G contains no non-cyclic elementary abelian normal subgroup, the maximal radicable subgroup R_p^0 of the Sylow p-subgroup R_p of R is locally cyclic. Let $C_p = C_{R_p}(R_p^0)$. Since C_p/R_p^0 is a finite p-group, C_p is nilpotent. If F_p is any finite subgroup of C_p such that $C_p = F_pR_p^0$, then $F_p \lhd C_p$ and C_p/F_p is locally cyclic. It follows that the set of all elements of any given order in C_p generates a finite subgroup. Hence any finite set of elements in C_p lies in a finite characteristic subgroup of C_p and hence in a finite characteristic subgroup of G. Therefore, if E is a finite subgroup of C_p, then E sn G and $|G: C_G(E)| < \infty$. Since G is reduced, we find that every finite abelian subgroup of C_p is cyclic.

Suppose now that p is odd. Then a finite p-group, all of whose abelian subgroups are cyclic, is itself cyclic (Gorenstein (1968) Theorem 5.4.10), and hence C_p is locally cyclic. Furthermore, every non-trivial automorphism of finite order of R_p^0 acts non-trivially on $\Omega_1(R_p^0)$ (Robinson (1968) Lemma 2.36), and hence R_p^0 has no automorphism of order p. Therefore $C_p = R_p$, and R_p is locally cyclic if p is odd. Hence $O_{2'}(R)$ is locally cyclic.

If $R_2^0 \neq 1$ then, since R_2^0 is a direct factor of any larger abelian subgroup of C_2 and every finite abelian subgroup of C_2 is cyclic, we find that R_2 is a maximal abelian subgroup of C_2 and hence that $C_2 = R_2^0$. Since in fact C_2 is the centralizer of the subgroup of order 4 of R_2^0 (Robinson (1968) Theorem 2.36), we have that $|R_2: C_2| = 1$ or 2. Let $A = O_{2'}(R) \times R_2^0$. Then A is locally cyclic. If $D = C_G(A)$, then D also centralizes R/A, and since Lemma 3.1 shows that D is finitely radical, we obtain that $D \leqq R$ by an argument similar to Hartley (1971a) Lemma 5.4. Hence $D \cap R_2 \leqq C_2$, and so $D =$ A. Therefore we may take $H = G$ in this case.

If $R_2^0 = 1$, then $C_2 = R_2$ and R_2 is either cyclic or a generalized quaternion group (Gorenstein (1968) Theorem 5.4.10). Let $H = C_G(R_2)$, $A = H \cap R = O_{2'}(R) \times Z(R_2)$, where $Z(R_2)$ is the centre of R_2. Then $|G: H|$ is finite and $H \lhd G$. Since H is finitely radical and $A = H \cap R$ is the Hirsch-Plotkin radical of H, we have $A = C_H(A)$.

Since the automorphism group of a locally cyclic group is abelian, we have that H/A is abelian in either case.

(A little more argument actually shows that we can take $H = G$ unless R_2 is a quaternion group of order 8, in which case H can be chosen to be of index dividing 6. This is the best that can be expected if we want H to contain a self-centralizing locally cyclic normal subgroup, as the example of GL(2, 3) shows).

We can now state a theorem which gives quite a lot of information on the structure of locally soluble $\mathfrak{M}_c$-heads.

THEOREM 4.5. *Let G be a locally soluble $\mathfrak{M}_c$-head. Then G is almost* *metabelian. The Hirsch-Plotkin radical of G is almost abelian and of finite rank.*

PROOF. Lemma 4.4, together with its proof, shows that the theorem holds for reduced $\mathfrak{M}_c$-heads. Therefore, by Lemma 4.2, it suffices to show that the properties attributed to G are preserved by forming subdirect products with finitely many factors and finite extensions. We leave this straightforward exercise to the reader.

We complete this section by showing how the proof of Theorem A may now be reduced to determining the structure of metabelian reduced $\mathfrak{M}_c$-heads. But first we need to define a p'-pinched group.

DEFINITION. *Let p be a prime. A group G is said to be p'-pinched if G contains a locally cyclic normal subgroup A such that*

(i) *G/A is abelian.*

(ii) *For every prime $q \neq p$, every element of order q of G lies in A.*

(iii) *$O_p(G) \leqq A$.*

(iv) *There is a normal subgroup K of G such that $K \cap O_p(G) = 1$ and G/K satisfies* Min-p.

Notice that, by (ii) and the locally cyclic nature of A, G contains no non-cyclic elementary abelian q-subgroup if $q \neq p$. Hence, if q is an odd prime different from p, every q-subgroup of G is locally cyclic, while if $p \neq 2$ every finite 2-subgroup of G is either cyclic or generalized quaternion. However a p'-pinched group may nevertheless be uncountable, cf Hartley (1972), remarks following Corollary $C1$.

We have

LEMMA 4.6. *Let G be p'-pinched and $N \lhd G$. Then N is p'-pinched.*

PROOF. Let A and K be as in the definition, and let $B = A \cap N$. Then B is locally cyclic, N/B is abelian, and every element of N of prime order $q \neq p$ lies in B. As $O_p(N) = O_p(G) \cap N$, we have $O_p(N) \leqq A \cap N = B$. Furthermore, if $L = N \cap K$, then $L \cap O_p(N) = 1$ since $O_p(N) \leqq O_p(G)$, and $N/L \cong NK/K$ satisfies Min-p.

The necessity of the conditions given in Theorem A for a locally soluble group to be an $\mathfrak{M}_c$-head can now readily be deduced from Lemma 4.7 below, the proof of which we defer to the next section.

LEMMA 4.7. *Let G be a metabelian reduced $\mathfrak{M}_c$-head. Then G is almost a subdirect product of a finite number of p'-pinched groups, for various primes p.*

DEDUCTION OF THEOREM A: *necessity*. By Lemma 4.2, if G is a locally soluble $\mathfrak{M}_c$-head, then G contains a normal subgroup H of finite index and subgroups $K_1, \cdots, K_n \lhd H$ such that each H/K_i is a reduced $\mathfrak{M}_c$-head and $\bigcap_{i=1}^n K_i = 1$. By Lemma 4.4, there exist subgroups L_i with $K_i \leqq L_i \lhd H$ such that $|H : L_i|$ is finite and L_i/K_i is metabelian. There exists a normal subgroup L of G such that $L \leqq \bigcap_{i=1}^{n} L_i$ and $|G: L| < \infty$. Then $L/L \cap K_i \cong LK_i/K_i$, which is a normal subgroup of finite index of H/K_i contained in L_i/K_i. Therefore, by Lemma 4.1, each of the groups $L/L \cap K_i$ is reduced and metabelian. Also, by Lemmas 2.1, 2.4 and 2.11, each of these groups is an $\mathfrak{M}_c$-head.

Let $M_i = L \cap K_i$. By Lemma 4.7, there exists a subgroup T_i of finite index in L such that $M_i \leqq T_i \lhd L$, and finitely many normal subgroups M_{ij} of T_i such that $\bigcap_j M_{ij} = M_i$ and T_i/M_{ij} is p'-pinched for a suitable prime p. Let T be a normal subgroup of finite index of G contained in $\bigcap_{i=1}^{n} T_i$ Then by Lemma 4. 6, each of the groups $T/T \cap M_{ij}$ is p'-pinched for the appropriate prime p. Since $\bigcap_{i,j} M_{ij} = 1$, T is a subdirect product of a finite number of p'-pinched groups, and the deduction of the necessity statement of Theorem A is complete.

We now establish the sufficiency statement of Theorem A, so that all that remains is to prove Lemma 4.7.

LEMMA 4.8. *Let G be a group which is almost a subdirect product of finitely many $\mathfrak{M}_c$-heads. Then G is an $\mathfrak{M}_c$-head.*

PROOF. By hypothesis, G contains a normal subgroup H of finite index and subgroups $K_1, \cdots, K_n \lhd H$ such that each H/K_i is an $\mathfrak{M}_c$-head. Thus H/K_i has a faithful irreducible $\mathfrak{M}_c$-family, $\mathbf{X}_i$ say. We may view $\mathbf{X}_i$ naturally as an irreducible $\mathfrak{M}_c$-family for H with $C_H(\mathbf{X}_i) = K_i$. By Lemma 2.8, $\mathbf{X}_i^G$ is an $\mathfrak{M}_c$-family for G. Now if $X_i \in \mathbf{X}_i$, then $(X_i^G)_H$ is the direct sum of a finite number of irreducible H-submodules, among which a copy of X_i occurs. Therefore we can choose a composition factor Y_i of X_i^G such that $(Y_i)_H$ has a direct summand isomorphic to X_i.

If $\mathbf{Y}_i$ consists of all such modules Y_i, then Lemma 2.4 (i) shows that $\mathbf{Y}_i$ is an irreducible $\mathfrak{M}_c$-family for G, and clearly $C_G(\mathbf{Y}_i) \cap H \leqq K_i$. By Lemma 2.2, $\mathbf{Y} = \bigcup_{i=1}^{n} \mathbf{Y}_i$ is also an $\mathfrak{M}_c$-family for G, and $C_G(\mathbf{Y}) \cap H \leqq \bigcap_{i=1}^{n} K_i = 1$. Finally, G/H is finite and so has a faithful irreducible $\mathfrak{M}_c$-family over any field of characteristic not dividing its order (since G/H has a faithful completely reducible representation over such a field). By viewing such a family as an $\mathfrak{M}_c$-family for G and adjoining it to $\mathbf{Y}$, we obtain the required faithful irreducible $\mathfrak{M}_c$-family over G.

COROLLARY 4.9. *Every almost abelian group with Min is an $\mathfrak{M}_c$-head.*

PROOF. Such a group G contains a normal subgroup of finite index which is a direct product of finitely many groups of type $C_{p\infty}$. Now a group of type $C_{p\infty}$ has a faithful irreducible module over $\mathbf{Z}_q$, is where q is any prime different from p (Robinson (1968) Lemma 2.37). Since such a module is trivially an $\mathfrak{M}_c$-module (as every non-trivial element of the group acts fixed-point-freely), the Corollary follows from Lemma 4.8.

PROOF OF THEOREM A: *sufficiency*. By Lemma 4.8, it suffices to show that every p'-pinched group G has a faithful irreducible $\mathfrak{M}_c$-family. Let A be a locally cyclic normal subgroup of G containing every element of prime order $q \neq p$ of G and such that G/A is abelian. We may suppose that A is chosen maximal subject to these conditions. Let B be any abelian subgroup of G containing A. Then since every abelian p'-subgroup of G is locally cyclic $B_{p'}A = B_{p'} \times A_p$ is locally cyclic. Hence $B_{p'} \leqq A$, by the maximality of A, and so B/A is a p-group. It follows that, if $C = C_G(A)$, then C/A is a p-group. Since C is clearly nilpotent it is the direct product of its Sylow subgroups and we have

$$(10) \qquad C_G(A) \leqq AO_p(G) = A_{p'} \times O_p(G) = A$$

by (iii) of the definition of p'-pinched.

Now A/A_p is a locally cyclic p'-group, and so has faithful irreducible module U over any field k whose characteristic is p or 0. By allowing A_p to act trivially we may view U as a kA-module. Let $W = U^G$ and let V be any composition factor of W. Now W_A is a direct sum of irreducible kA-submodules on each of which the kernel of A is A_p; consequently V is also such a direct sum and

$$(11) \qquad C_A(V) = A_p.$$

Furthermore, since G is p'-pinched, any subgroup L of G which is not a p-group contains a non-trivial p'-element a of A. The latter acts fixed-point freely on V, and so $C_V(L) = C_V(a) = 0$. Therefore

(12) *V is an irreducible $\mathfrak{M}_c$-module over G.*

Since G is p'-pinched, there is a normal subgroup K of G such that G/K satisfies Min-p and $K \cap O_p(G) = 1$. We may evidently suppose that $O_{p'}(G/K) = 1$. Then the Hirsch-Plotkin radical of G/K is a p-group satisfying Min, and contains its centralizer since G/K is soluble. Since a periodic group of automorphisms of a p-group satisfying Min also satisfies min (Robinson (1968) Theorem 2.35), we find that G/K satisfies Min. Then G/K has a faithful irreducible $\mathfrak{M}_c$-family Y, by Corollary 4.9; in fact it is worth pointing out that this family can be taken to consist of modules over any given field k, of characteristic not belonging to $\pi(G/K)$. By viewing Y as a family over G and adding V to it, we obtain an irreducible $\mathfrak{M}_c$-family X over G.

It remains to verify that X is faithful. From (11) we obtain that $C_G(X) \cap A \leqq A_p = O_p(G)$, and hence, as $O_p(G)$ is faithfully represented on Y, we obtain that $C_G(X) \cap A = 1$. Therefore $C_G(X) \leqq C_G(A) = A$ by (10), and so $C_G(X) = 1$, as required.

5. Metabelian $\mathfrak{M}_c$-Heads

In this section we have to investigate the structure of reduced metabelian $\mathfrak{M}_c$-heads, in order to prove that such groups are almost subdirect products of p'-pinched groups. The key lemma in this investigation is Lemma 5.4. In proving this lemma we have to extend the fields over which the modules in our $\mathfrak{M}_c$-family are defined, employing Corollary 2.6 for the purpose. This is the only time in the paper when field extension seems necessary.

Before coming to Lemma 5.4 we need three technical lemmas, the first of which will certainly be well known.

LEMMA 5.1. *Let A be a periodic abelian group and let k be an algebraically closed field. Then*

(i) *Every irreducible kA-module has dimension* 1 *over k.*

(ii) *Let V be an irreducible kA-module, let $K = C_A(V)$, and let $\alpha \in \operatorname{Aut} A$. Let V^α denote the kA-module V with the A-action $(v, a) \to v.a^{\alpha^{-1}}$. Then $V^\alpha \cong V$ if and only if $K^\alpha = K$ and α centralizes A/K.*

PROOF. (i) Let V be an irreducible kA-module. Then V contains a non-zero vector v. If $a \in A$, then v lies in a finite-dimensional subspace U of V invariant under a. Since k is algebraically closed, there is an element $\lambda \in k$ and a non-zero vector $w \in U$ such that $wa = \lambda w$. For fixed λ and a, the set of all vectors $w \in V$ which satisfy this condition is a non-zero kA-submodule of V, and so must be V itself. It follows that every element of A acts on V as multiplication by an element of k, so that every k-subspace of V is a kA-submodule. Hence $\dim_k V = 1$.

(ii) Suppose that $V^\alpha \cong V$. Then as $K^\alpha = C_A(V^\alpha)$, we must have $K^\alpha = K$. By (i), V is 1-dimensional and each element of A acts on V by scalar multiplication. Thus we have a homomorphism $\lambda: A \to k^*$ such that $va = \lambda(a)v$ for all $a \in A$ and $v \in V$. Let $\phi: V \to V^\alpha$ be an isomorphism. Then $(va)\phi = v\phi \cdot a^{\alpha^{-1}}$. Hence $\lambda(a)v\phi = \lambda(a^{\alpha^{-1}})\, v\phi$ for all $v \in V$, $a \in A$, whence $\lambda(a^{-1}a^{\alpha^{-1}}) = 1$ and $a \equiv a^{\alpha^{-1}} \bmod K = \ker \lambda$. Thus α centralizes A/K. The converse is immediate.

LEMMA 5.2. *Let G be a group containing an abelian normal subgrou p A which satisfies* Min-p *for all primes p and is such that G/A is abelian. Suppose that, for infinitely many primes p, G contains an element of order p not belonging to A. Then there is an infinite subgroup B of G such that $B \cap A = 1$ and B is a direct product of cyclic groups of distinct prime orders.*

PROOF. Suppose that C is a cyclic subgroup of G such that $C \cap A = 1$ and $|C|$ is a product of distinct primes. We shall show that C is contained in a larger such subgroup; from this the result will follow.

Let σ be the (finite) set of prime divisors of $|C|$. Since, for each prime p, the Sylow p-subgroup A_p of A is an abelian group satisfying Min, it follows from Robinson (1968) Theorem 2.35 that $G/C_G(A_p)$ is finite. Therefore there exists an element x of prime order $q \notin \sigma$ in G such that x centralizes A_σ and $x \notin A$. If $g \in G$ then we have $[g, x] \in A$ and so

$$1 = [g, x^q] \equiv [g, x]^q \bmod A_{\sigma'}, \text{ as } x \text{ centralizes } A/A_{\sigma'}.$$

Hence, as $q \notin \sigma$, we have $[g, x] \in A_{\sigma'}$ for all $g \in G$.

It follows that $A_{\sigma'}\langle x \rangle \lhd G$. Consider the group $A_{\sigma'}\langle x \rangle C = H$. Now C is a finite maximal σ-subgroup of H, and hence every σ-element of H is conjugate to an element of C. Therefore $A \cap H$ is a σ'-group, and so $A \cap H \leqq A_{\sigma'}\langle x \rangle$. Hence $A \cap H = A_{\sigma'}$, as $x \notin A$. It follows that $H/A_{\sigma'}$ is abelian.

Since $|C|$ is a σ-number, a theorem of Gaschutz (Huppert (1961) Chapter I, Theorem 17.4) shows that H splits over $A_{\sigma'}$. If D is a complement, then D_σ is a maximal σ-subgroup of H, and so is conjugate in H to C. Therefore we can arrange that $D \geqq C$. Since $D \cap A = D \cap A \cap H = D \cap A_{\sigma'} = 1$, D is the required subgroup.

LEMMA. 5.3. *Let G be a group with a faithful irreducible $\mathfrak{M}_c$-family $\mathbf{X}$. Suppose that G contains an infinite normal locally nilpotent subgroup D such that D_q is finite for each prime q. Then there is a subgroup K of G and an infinite set π of primes such that*

(i) *$K \cap D_q < D_q$ for all $q \in \pi$.*

(ii) *For each finite subset σ of π, there exists a module $X_\sigma \in \mathbf{X}$ such that $K \cap D_\sigma = C_{D_\sigma}(X_\sigma)$.*

PROOF. If there is a module $X \in \mathbf{X}$ such that $D/C_D(X)$ is infinite, then we may evidently take $K = C_D(X)$ and π to be the set of all primes q such that $D/C_D(X)$ contains an element of order q. Therefore, during the rest of the proof, we assume that

$$D : C_D(X)| < \infty \text{ for all } X \in \mathbf{X}. \tag{1}$$

We begin by deducing that

(*) *There exists an infinite subset τ of $\pi(D)$ and a subfamily $\mathbf{Y}$ of $\mathbf{X}$ such that $\tau \cap \operatorname{char} \mathbf{Y} = \varnothing$ and $D_q \cap C_G(\mathbf{Y}) < D_q$ for all $q \in \tau$.*

We recall that, if λ is a set of primes, then $\mathbf{X}_\lambda = \{X \in \mathbf{X} : \operatorname{char} k(X) \in \lambda \cup \{0\}\}$. If there is a finite set λ of primes such that $D_q \cap C_G(\mathbf{X}_\lambda) < D_q$ for infinitely many q, then we may evidently take $\mathbf{Y} = \mathbf{X}_\lambda$ and choose τ suitably to obtain (*). Otherwise, we have that if λ is any finite set of primes, then $D_q \leqq C_G(\mathbf{X}_\lambda)$ for all but finitely many q.

In this latter case, suppose we have obtained a finite subset τ_n of $\pi(D)$ and a finite set σ_n of primes such that $\tau_n \cap \sigma_n = \varnothing$ and $D_q \cap C_G(\mathbf{X}_{\sigma_n}) < D_q$ for all $q \in \tau_n$. Then the set of all primes r such that $r \notin \sigma_n \cup \tau_n$ and $1 \neq D_r \leqq C_G(\mathbf{X}_{\sigma_n \cup \tau_n})$ is infinite, as $\pi(D)$ is infinite. Choose such a prime r and let $\tau_{n+1} = \tau_n \cup \{r\}$. Since $\mathbf{X}$ is a faithful $\mathfrak{M}_c$-family for G, there exists a module $X \in \mathbf{X}$ such that $D_r \nleqq C_G(X)$. The choice of r shows that $\operatorname{char} k(X) \notin \sigma_n \cup \tau_n \cup \{0\}$, and furthermore, since $O_r(G)$ acts trivially on every irreducible module for G over a field of characteristic r, (Gardiner, Hartley and Tomkinson (1971) Lemma 3.2) we have char $k(X) \neq r$. Therefore, letting $\sigma_{n+1} = \sigma_n \cup \{\text{char}\, k(X)\}$, we obtain that $\tau_{n+1} \cap \sigma_{n+1} = \varnothing$. Clearly $D_q \cap C_G(X_{\sigma_{n+1}}) < D_q$ for all $q \in \tau_{n+1}$. Proceeding in this way and letting $\tau = \bigcup_{n=1}^{\infty} \tau_n$, $\sigma = \bigcup_{n=1}^{\infty} \sigma_n$ and $\mathbf{Y} = \mathbf{X}_\sigma$, we obtain (*).

We may assume without loss of generality that $D = D_\tau$. By (*), we have that $D/D \cap C_G(\mathbf{Y})$ is infinite. Thus, if we write $K_Y = D \cap C_G(Y)$ $(Y \in \mathbf{Y})$, then

(2) $$D/\bigcap_{Y \in \mathbf{Y}} K_Y \textit{ is infinite}$$

whereas from (1)

(3) $$D/K_Y \textit{ is finite for each } Y \in \mathbf{Y}$$

Since $\pi(D) \cap \text{char}\, \mathbf{Y} = \phi$, each subgroup A of D contains a finite subgroup F such that $C_Y(A) = C_Y(F)$ for all $Y \in \mathbf{Y}$. In particular, we may choose a finite. subset μ_1 of $\pi(D)$ such that $C_Y(D_{\mu_1}) = C_Y(D)$ for all $Y \in \mathbf{Y}$. Thus, if $K_Y \geqq D_{\mu_1}$, then $K_Y = D$. By (2) and (3) the number of distinct subgroups K_Y is infinite, and since D_{μ_1} is finite, we can obtain a subfamily $\mathbf{Y}_1$ of $\mathbf{Y}$ and a subgroup F_1 of D_μ such that the set of subgroups $K_Y\, (Y \in \mathbf{Y}_1)$ is infinite and

(4) $$K_Y \cap D_{\mu_1} = F_1 < D_{\mu_1}$$

for all $Y \in \mathbf{Y}_1$.

We take the situation just obtained as the first stage of a construction, at the n-th stage of which we have pairwise disjoint finite sets $\mu_1, \mu_2, \cdots, \mu_n$ of primes, proper subgroups $F_1, \cdots, F_n$ of $D_{\mu_1}, \cdots, D_{\mu_n}$ respectively, and subfamilies $\mathbf{Y}_1 \geqq \mathbf{Y}_2 \geqq \cdots \geqq \mathbf{Y}_n$ of $\mathbf{Y}$ such that $\{K_Y \colon Y \in \mathbf{Y}_n\}$ is infinite and

(5) $$K_Y \cap (D_{\mu_1} \times \cdots \times D_{\mu_n}) = F_1 \times \cdots \times F_n$$

for all $Y \in \mathbf{Y}_n$. To obtain the next step of the construction, let $\lambda_n = (\mu_1 \cup \cdots \cup \mu_n)'$ There is a finite subset μ_{n+1} of λ_n such that $C_Y(D_{\lambda_n}) = C_Y(D_{\mu_{n+1}})$ for all $Y \in \mathbf{Y}_n$. Thus, if $Y \in \mathbf{Y}_n$ and $K_Y \geqq D_{\mu_{n+1}}$, then $K_Y \geqq D_{\lambda_n}$. Since D_{λ_n} has finite index in D, only finitely many subgroups of D can contain it, and so infinitely many of the subgroups K_Y $(Y \in \mathbf{Y}_n)$ intersect $D_{\mu_{n+1}}$ in a proper subgroup. The finiteness of $D_{\mu_{n+1}}$ then implies that there is a subfamily $\mathbf{Y}_{n+1}$ of $\mathbf{Y}$ and a subgroup F_{n+1} of $D_{\mu_{n+1}}$ such that $\{K_Y \colon Y \in \mathbf{Y}_{n+1}\}$ is infinite and

$$K_Y \cap D_{\mu_{n+1}} = F_{n+1} < D_{\mu_{n+1}} \text{ if } Y \in \mathbf{Y}_{n+1}.$$

Then (5) holds with n replaced by $n + 1$.

Now let $K = F_1 \times F_2 \times \cdots$ and let π consist of all primes q such that some D_{μ_i}/F_i contains an element of order q. Then π is infinite, and it follows from (5) that the required conditions (i) and (ii) are satisfied.

We now come to the result which provides the fundamental step in analysing the structure of metabelian $\mathfrak{M}_c$-heads.

LEMMA 5.4. *Let G be a metabelian $\mathfrak{M}_c$-head. Then there is a finite set σ of primes and a normal abelian subgroup A of G such that G/A is abelian and A contains every element of G whose order is a prime not belonging to σ.*

PROOF. As usual, we assume the lemma false and derive a contradiction. Let R be the Hirsch-Plotkin radical of G. Then by Theorem 4.5, R has finite rank and there is a finite set λ of primes such that $R_{\lambda'}$ is abelian. Let $A = (G' \cap R_\lambda) \times R_{\lambda'}$. Then A and G/A are abelian. Since the lemma is assumed false, there are infinitely many primes p such that G contains an element of order p not lying in A. Therefore, by Lemma 5.2, there is an infinite subgroup B_1 of G such that $B_1 \cap A = 1$ and B_1 is a direct product of cyclic groups of distinct prime orders. Let B the Sylow λ'-subgroup of B_1. Then $B \cap R = 1$, and B is infinite.

Let x be an element of prime order p in B. We claim that

$$[x, A_{p'}] \neq 1. \tag{6}$$

For we have, since G/A is abelian, that $[x, R_{p'}] \leqq A_{p'}$ and so, if $[x, A_{p'}] = 1$, then $[x, R_{p'}] = 1$. It follows from this that $R\langle x\rangle = R_{p'} \times R_p\langle x\rangle$ is a normal locally nilpotent subgroup of G, whence $x \in R$, a contradiction.

We now construct elements $c_1, c_2, \cdots$ of B of distinct prime orders $p_1, p_2, \cdots$ and finite elementary abelian normal subgroups $D_1, D_2, \cdots$ of G of distinct prime exponents $q_1, q_2, \cdots$ such that

$$1 \neq [D_i, c_i] = D_i \qquad (i = 1, 2, \cdots). \tag{7}$$

In fact, suppose $c_1, \cdots, c_n$ and $D_1, \cdots, D_n$ have been obtained. Since A_{q_i} is an abelian group satisfying Min, its centralizer in G has finite index in G. Therefore there exists a prime p_{n+1} different from any of $p_1, \cdots, p_n$ such that B contains an element c_{n+1} of order p_{n+1} which centralizes $A_{q_1} \times \cdots \times A_{q_n}$. By (6), there is a prime $q_{n+1} \neq p_{n+1}$ such that $[A_{q_{n+1}}, c_{n+1}] \neq 1$, and we must have $q_{n+1} \notin \{q_1, \cdots, q_n\}$. Then c_{n+1} does not centralize $\Omega_1(A_{q_{n+1}})$, and we may take $D_{n+1} = [\Omega_1(A_{q_{n+1}}), c_{n+1}]$. Since A and G/A are abelian, we easily see that $D_{n+1} \lhd G$.

Let π be as given by Lemma 5.3, with $D = D_1 \times D_2 \times \cdots$ and X a faithful irreducible algebraically closed $\mathfrak{M}_c$-family for G (see Corollary 2.6). Then by considering D_π instead of D and reindexing, we can obtain new sequences $c_1, c_2, \cdots$ and $D_1, D_2, \cdots$ such that (7) holds; furthermore we have now a subgroup K of D such that

$$K \cap D_i < D_i \qquad (i = 1, 2, \cdots) \tag{8}$$

and for $n = 1, 2, \cdots$ we have a module $X_n \in \mathbf{X}$ such that

$$K \cap (D_1 \times \cdots \times D_n) = C_G(X_n) \cap (D_1 \times \cdots \times D_n). \tag{9}$$

Let the primes $p_1, p_2, \cdots$ be divided in any way into two disjoint infinite subsets ν_1 and ν_2, and write $C = \langle c_1, c_2, \cdots \rangle$, $C_1 = C_{\mu_1}$, $C_2 = C_{\mu_2}$. Since X is an $\mathfrak{M}_c$-family for G, there is a finite subgroup F_1 of C_1 such that $C_X(F_1) = C_X(C_1)$ for all $X \in \mathbf{X}$ such that char $k(X) \notin \pi(C_1) = \nu_1$. Let n be any integer large enough to ensure that $F_1 < C_1 \cap (\langle c_1 \rangle \times \cdots \times \langle c_n \rangle) = H_1$. Now since $E_n = D_1 \times \cdots \times D_n$ is a finite normal subgroup of G, Lemma 2.11 shows that the $k(X_n)[G]$-module X_n becomes completely reducible when restricted to E_n. In fact, let Z be any irreducible submodule of $(X_n)_{E_n}$. Then each Zg $(g \in G)$ is an irreducible E_n-submodule of X_n, and $\sum_{g \in G} Zg = X_n$.

Let $L = C_{E_n}(Z)$. Then $L^g = C_{E_n}(Zg)$ and so $\cap_{g \in G} L^g = E_n \cap C_G(X_n) = K \cap E_n$ by (9). Now $C_G(E_n/L)$ contains A as A is abelian, and hence $C_G(E_n/L) \lhd G$ as G/A is also abelian. Therefore $[E_n, C_G(E_n/L)]$ is a normal subgroup of G contained in L, and so contained in $\bigcap_{g \in G} L^g = K \cap E_n$. It follows from (7) and (8) that $C_G(E_n/L) \cap (\langle c_1 \rangle \times \cdots \times \langle c_n \rangle) = 1$, and in particular that $C_G(E_n/L) \cap H_1 = 1$. Therefore, by Lemma 5.1, the submodules Zg $(g \in H_1)$ are pairwise non-isomorphic.

Therefore the modules Zg $(g \in H_1)$ generate their direct sum, and if $0 \neq z \in Z$, then $y = \sum_{g \in F_1} zg$ is a non-trivial element of X_n centralized by F_1 but not by H_1, as $F_1 < H_1$. Hence C_1 does not centralize y. By the choice of F_1, we must have char $k(X_n) \in \nu_1$; this holds for all large enough n. But by similar considerations, char $k(X_n) \in \nu_2$ for large enough n. Since $\nu_1 \cap \nu_2 = \emptyset$, we have obtained a contradiction and established Lemma 5.4.

A further reduction is needed before we can finally establish Theorem A.

LEMMA 5.5 *Let G be a metabelian $\mathfrak{M}_c$-head containing a normal locally cyclic subgroup A such that G/A is abelian, and let σ be a finite set of primes. Then there is a normal subgroup H of G containing A, such that $|G: H| < \infty$ and, for each prime q, $|H/C_H(A_q)|$ is divisible by at most one prime from σ.*

PROOF. Suppose if possible that the result is false. Let $C = C_G(A_\sigma)$ and suppose that we have obtained n primes $q_1, q_2, \cdots, q_n$ not belonging to σ and such that, if $C_i = C_G(A_\sigma \times A_{q_1} \times \cdots \times A_{q_i})$, then $|C_i/C_{i+1}|$ is divisible by at least two primes in σ for $0 \leqq i \leqq n-1$. Then $|G: C_n| < \infty$ since each Sylow subgroup of A is cyclic or quasi-cyclic, and so by assumption there exists a prime q_{n+1} such that $|C_n/C_{C_n}(A_{q_{n+1}})|$ is divisible by at least two primes in σ. Clearly $q_{n+1} \notin \{q_1, \cdots, q_n\} \bigcup \sigma$ and the construction proceeds, yielding eventually an infinite sequence $q_1, q_2, \cdots$.

Since σ is finite, there must be a pair (p, q) of primes in σ such that $pq \,\|\, C_{i-1}/C_i|$ for infinitely many values of i. Suppose that these values of i form a subsequence $i(1) < i(2) < \cdots$ and let $C_j{}^* = C_G(A_\sigma \times A_{q_{i(1)}} \times \cdots \times A_{q_{i(j)}})$. Then pq divides the order of $C_j{}^* \cap C_{i(j+1)-1}/C_{j+1}^* \cap C_{i(j+1)-1} = C_{i(j+1)-1}/C_{i(j+1)}$. Hence pq divides the order of $(C_j^* \cap C_{i(j+1)-1})C_{j+1}^*/C_{j+1}^*$, a subgroup of C_j^*/C_{j+1}^*. In other words, we may suppose that

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$$(10) \qquad pq \,\big|\, |C_i/C_{i+1}| \text{ for all } i = 1, 2, \cdots$$

Let $B = \Pi_{i+1}^{\infty} \Omega_1(A_{q_i})$. We apply Lemma 5.3 to B to obtain a set π of primes and a subgroup K of B as there described. By passing to a suitable subsequence again as above, we may assume that $\pi = \{q_1, q_2, \cdots\}$. Then as B_{q_i} is cyclic of order q_i and $K \cap B_{q_i} = 1$, we in fact have $K = 1$. For each $n \geqq 1$, we have a module X_n, belonging to a faithful irreducible $\mathfrak{M}_c$-family X for G

$$(11) \qquad C_G(X_n) \cap (B_1 \times \cdots \times B_n) = 1.$$

Let p_n be the characteristic of $k(X_n)$ and suppose there is an infinite subsequence $n(1) < n(2) < \cdots$ such that $p_{n(i)} \neq p$ for all $i = 1, 2, \cdots$. Let $L = \cap_{i=1}^{\infty} C_G(X_{n(i)})$ and consider the group $\bar{G} = G/L$, letting $x \to \bar{x}$ be the natural homomorphism of G onto this group. By Lemma 2.1, $\bar{G}$ has a faithful irreducible $\mathfrak{M}_c$-family $\{X_{n(1)}, X_{n(2)}, \cdots\}$ of characteristic not containing p. For each $i = 1, 2, \cdots$, (10) allows us to choose a p-element x_i which centralizes $A_{q_{n(1)}} \times \cdots \times A_{q_{n(i-1)}}$ but not $A_{q_{n(i)}}$. In fact, using Lemma 3.3, we can also arrange that $\langle x_1, x_2, \cdots \rangle$ is a p-group, P say. Since $q_{n(i)} \neq p$ and $A_{q_{n(i)}}$ is locally cyclic, we have $[B_{q_{n(i)}}, x_i] \neq 1$ and so, from (11) $[B_{q_{n(i)}}, x_i] \nleqq L$. Thus $\bar{x}_i$ belongs to $C_{\bar{P}}(\bar{B}_{q_{n(1)}} \times \cdots \times \bar{B}_{q_{n(i-1)}})$ but not to $C_{\bar{P}}(\bar{B}_{q_{n(1)}} \times \cdots \times \bar{B}_{q_{n(i)}})$, and we find that $\bar{P}$ does not satisfy Min. However as $\bar{P}$ has a faithful $\mathfrak{M}_c$-family of characteristc not containing p, this contradicts Lemma 3.4. It follows that we must have $p_n = p$ for all but finitely many n.

But similarly we must have $p_n = q$ for all but finitely many n, and since these two statements are incompatible, we have obtained a contradiction and proved Lemma 5.5.

Now we are ready to establish Lemma 4.7 and thus conclude the proof of our main theorem, Theorem A.

Proof of Lemma 4.7. We have a metabelian reduced $\mathfrak{M}_c$-head G, which we must show is almost a subdirect product of finitely many p'-pinched groups. By Lemma 5.4, there is a normal abelian subgroup A of G and a finite set σ of primes such that G/A is abelian and every element of G whose order is a prime not lying in σ, belongs to A. It will be convenient to assume $2 \in \sigma$ and $|\sigma| \geqq 2$; this we may clearly do. We may suppose further that A is chosen maximal subject to satisfying the conditions required of it; then A is actually a maximal abelian subgroup of G and so

$$(12) \qquad A = C_G(A).$$

Since G is reduced, A is locally cyclic. By Lemma 5.5 there is a normal subgroup H of G containing A such that $|G:H| < \infty$ and, for each prime q, $|H/C_H(A_q)|$ is divisible by at most one prime from σ. Since A is locally cyclic and σ is finite we have that $|G: C_G(A_\sigma)| < \infty$, and so, by replacing H by $C_H(A_\sigma)$ if necessary, we may suppose that

$$(13)\qquad [H, A_\sigma] = 1.$$

It will suffice to show that H is a subdirect product of a finite number of p'-pinched groups. Let $\sigma = \{p_1, \cdots, p_n\}$. For $1 \leqq i \leqq n$, let $\sigma_i = \sigma - \{p_i\}$ and let π_i be the set of all primes q such that $|H/C_H(A_q)|$ is divisible by some prime in σ_i.

Then

$$(14)\qquad \pi_i \cap \sigma = \varnothing \qquad (1 \leqq i \leqq n),$$

by (13).

Let x be any σ_i-element of H. Then by the choice of π_i, we have $[x, A_q] = 1$ unless $q \in \pi_i$. Therefore $[x, A] \leqq A_{\pi_i}$, and we have $A_{\pi_i}\langle x\rangle \lhd A\langle x\rangle \lhd H$. It follows that the normal closure of $\langle x A_{\pi_i}\rangle$ in H/A_{π_i} is a σ_i-subgroup, and hence that the set of σ_i-elements of H/A_{π_i} is a subgroup U_i/A_{π_i} of H/A_{π_i}. We shall show that

$$(15)\qquad \bigcap_{i=1}^{n} U_i = 1$$

and

$$(16)\qquad H/U_i \textit{ is } p_i'\textit{-pinched},$$

thereby completing the proof.

Now by its construction, U_i is a $\pi_i \cup \sigma_i$-group. It follows from (14) that $\pi(\bigcap_{i=1}^n U_i) \leqq \bigcap_{i=1}^n \pi_i$. Now if q is a prime in this intersection, then $|H/C_H(A_q)|$ is divisible by some prime in σ_n, that is, by some p_i with $1 \leqq i \leqq n-1$. But as $q \in \pi_i$, $|H/C_H(A_q)|$ is divisible by some p_j with $j \neq i$. Thus $p_i p_j \mid |H/C_H(A_q)|$, which contradicts the choice of H. Hence $\bigcap_{i=1}^n \pi_i = \phi$ and (15) is established.

To obtain (16) takes a little more work. First let $q \in \pi_i$. Then there is a σ_i-element $x \in H$ such that $[A_q, x] \neq 1$. Since A_q is cyclic or quasi-cyclic and $q \notin \sigma_i$ (by (13)), we have $[A_q, x] = A_q$. Hence $A_q \leqq H'$. Now it also follows from (13) that $q \notin \sigma$, and so every element of order q of H lies in A. Therefore every elementary abelian q-subgroup of H is cyclic. Hence every abelian q-subgroup of H is cyclic and so, since the assumptions that $2 \in \sigma$ implies that q is odd, every finite q-subgroup of H is cyclic (Gorenstein (1968) Theorem 5.4.10). Since $1 \neq A_q \leqq H'$, every q-element of H lies in a finite subgroup F of H such that $F' \cap A_q \neq 1$. The remarks above show that the Sylow q-subgroups of F are cyclic, and so, by a well-known transfer theorem, q does not divide $|F/F'|$ and every q-element of F lies in F'. Therefore every q-element of H lies in H' and so in A. Since this holds for every $q \in \pi_i$, we have shown that A_{π_i} is the set of π_i-elements of H, and so

(17) $$H/U_i \text{ is a } (\pi_i \cup \sigma_i)'\text{-group},$$

as U_i/A_{π_i} was defined as the set of σ_i-elements of H/A_{π_i}.

Let $x \to \bar{x}$ be the natural homomorphism of H onto $\bar{H} = H/U_i$. Referring to the definition of p'-pinched given before Lemma 4.6, we see that we have that $\bar{A}$ is a locally cyclic normal subgroup of $\bar{H}$ such that $\bar{H}/\bar{A}$ is abelian. Let q be any prime $\neq p_i$ such that $\bar{H}$ contains an element of order q. Then from (17), $q \notin \pi_i \cup \sigma_i$. Since U_i is a $\pi_i \cup \sigma_i$-group, every element of order q of $\bar{H}$ is the natural image of an element of order q of H. Since $q \notin \sigma_i$ and $q \neq p_i$, we have $q \notin \sigma$, and so every element of order q of H lies in A. Hence every element of order q of $\bar{H}$ lies in $\bar{A}$.

Now let $P_i/U_i = O_{p_i}(\bar{H})$. Then $\bar{P}_i$ centralizes $\bar{A}_{p_i'}$, and so P_i centralizes $A_{p_i'}U_i/U_i$, which is H-isomorphic to $A_{p_i'}/A_{p_i'} \cap U_i \cong A_{p_i'}/A_{\pi_i \cup \sigma_i} \cong A_{(\pi_i \cup \sigma)'}$. Let $yU_i \in P_i/U_i$. Then we may choose y to be a p_i-element of H, and the preceding remarks show that y centralizes $A_{(\pi_i \cup \sigma)'}$. But the definition of π_i shows that every p_i-element of H centralizes A_{π_i} and hence, using (13), we find that y centralizes A. Therefore $y \in A$, by (12), and we have $O_{p_i}(\bar{H}) \leqq \bar{A}$.

It remains to establish (iv) of the definition. To this end, we notice that as $\bar{P}_i$ is contained in the locally cyclic subgroup $\bar{A}$ and U_i is a p_i'-group, $\bar{P}_i$ contains a unique minimal subgroup $\bar{Z} = \langle zU_i \rangle$, where z is an element of order p_i of A. Since G is an $\mathfrak{M}_c$-head, there exists an irreducible $\mathfrak{M}_c$-module X for G over some field k such that $\langle z \rangle \cap C_G(X) = 1$. By Gardiner, Hartley and Tomkinson (1971) Lemma 3.2, and the fact that $z \in O_{p_i}(G)$, we have char $k \neq p_i$. Hence, if $K_1 = C_G(X)$, Lemmas 2.1 and 3.4 give that G/K_1 satisfies Min-p_i. Hence, if $K = H \cap K_1$, then H/K satisfies Min-p_i and $\bar{H}/\bar{K}$ satisfies Min-p_i. If $\bar{z} \in \bar{K}$, then we obtain $z \in KU_i$, and hence, as $K \lhd KU_i$ and U_i is a p_i-group, we find that $z \in K$, which is not the case. Therefore $O_{p_i}(\bar{H}) \cap \bar{K} = 1$, and we have shown that $\bar{H}$ is p_i'-pinched. Therefore Lemma 4.7 and Theorem A are established.

6. Consequences of the Main Theorem

Theorem A allows us to answer many questions about the structure of locally soluble $\mathfrak{M}_c$-heads, since the structure of p'-pinched groups is reasonably transparent in many respects. For example, a p'-pinched group clearly contains a self-centralizing locally cyclic normal subgroup. Since the automorphism group of such a group clearly has cardinal at most $2^{\aleph_0}$, the cardinal of a p'-pinched group is at most $2^{\aleph_0}$. Hence we have using Theorem A,

COROLLARY A1. *If G is a locally soluble $\mathfrak{M}_c$-head, then* $|G| \leqq 2^{\aleph_0}$.

It is not hard to see that this bound can be attained — see for example the group QP constructed after Corollary C1 in Hartley (1972). Rather than pursuing locally soluble $\mathfrak{M}_c$-heads in general, however, we revert to those which arise in the context of our main applications, which are Theorems B and C. By Hartley (1971a) Lemmas 4.7-4.8 and Theorem E, such $\mathfrak{M}_c$-heads are of finite rank.

LEMMA 6.1. (i) *A pinched group is p'-pinched for all primes p.*
(ii) *A p'-pinched group of finite rank is almost pinched.*

PROOF. (i) We recall that a group G is pinched, if G contains a locally cyclic normal subgroup A which has abelian factor group and contains every element of prime order of G, and furthermore every 2-subgroup of G is abelian. It follows from the definition and Gorenstein (1968) Theorem 5.4.10, that every subgroup of prime power order of G is cyclic. Thus G satisfies Min-p for all primes p. We may suppose A chosen maximal subject to satisfying the conditions required of it; then $A = C_G(A)$. But $O_p(G)$ centralizes A as $O_p(G)$ is abelian; thus $O_p(G) \leqq A$ and a glance at the definition reveals that G is p'-pinched.

(ii) Let G be p'-pinched, and let A be a locally cyclic normal subgroup of G as in the definition (p. 18). By an argument given in the sufficiency proof of Theorem A, we may assume that $A = C_G(A)$.

Now since G has finite rank, G satisfies Min-q for all primes q. If Q is any q-subgroup of G with maximal radicable subgroup Q^0, then Q^0 centralizes A as A is locally cyclic, and so $Q^0 \leqq A$. Therefore the Sylow q-subgroup of G/A is finite for each prime q. Let H/A be the Sylow $\{p, 2\}'$-subgroup of G/A. Then $H \lhd G$ and $|G:H| < \infty$. Then A is locally cyclic, H/A is abelian, and as A contains every p-element of H and every element of prime order $q \neq p$ of G, A contains every element of prime order of H. Since A also contains every 2-element of H, every 2-subgroup of H is abelian.

Now it is immediate that every subgroup of a pinched group is pinched. A routine argument now allows us to deduce from Theorem A:

THEOREM 6.2. *Let G be a locally soluble group of finite rank. Then G is an $\mathfrak{M}_c$-head if and only if G is almost subpropinched,*

recalling that a subpropinched group is just a subdirect product of finitely many pinched groups.

PROOF. Let G be a locally soluble $\mathfrak{M}_c$-head of finite rank. Then by Theorem A, G contains a normal subgroup H of finite index and finitely many subgroups $K_1, \cdots, K_n \lhd H$ such that each H/K_i is p'-pinched for suitable p and $\bigcap_{i=1}^n K_i = 1$. By Lemma 6.1, H/K_i contains a pinched normal subgroup L_i/K_i of finite index. There is a normal subgroup L of finite index in G and contained in $\bigcap_{i=1}^n L_i$. Then $L/L \cap K_i \cong LK_i/K_i \lhd L_i/K_i$, and so $L/L \cap K_i$ is pinched. Hence L is subpropinched, as required.

Theorem B is now a rather immediate consequence of Theorem 6.2 and Lemma 6.3, which was stated in the introduction and remains to be proved.

PROOF OF LEMMA 6.3. We consider first a Sylow π-sparse group $L = HK$, where H is a normal locally soluble π'-subgroup of L and K is a π-group. If X is the set of composition factors of (K on H), then the remarks preceding the statement of Lemma 6.3 in Section 1 show that $\mathbf{X}$ can be thought of as a family of irreducible K-modules over various fields Z_p. Let A be any subgroup of K. Then a straightforward extension of Hartley (1971a) Lemma 4.3, using the fact that L is Sylow π-sparse, shows that there is a finite subgroup F of A such that $C_H(F) = C_H(A)$. Let $X \in \mathbf{X}$ and identify X with a K-composition factor U/V of H. Arguing as in Lemma 2.3 and using the fact that the fixed points of a finite π-group T acting on a finite π'-group U are preserved by T-homomorphism of U(Gorenstein (1968) Theorem 6.2.2) we find that $C_{U/V}(F) = C_U(F)V/V = C_A(F)V/V \leqq C_{U/V}(A)$, and equality must hold. Thus $\mathbf{X}$ is in fact a classical $\mathfrak{M}_c$-family for K. Furthermore, $C_K(\mathbf{X})$ clearly staailizes a series of H, and hence, using say Gardiner, Hartley and Tomkinson (1971) Lemma 4.11, we obtain $C_K(\mathbf{X}) = C_K(H)$.

Conversely, let $\mathbf{X}$ be a given $\mathfrak{M}_c$-family for a π-group K, and suppose $0 \neq p(X) \notin \pi$ for all $X \in \mathbf{X}$. We form the direct sum H of the modules in $\mathbf{X}$ and the semidirect product HK. Clearly $C_K(\mathbf{X}) = C_K(H)$ in this case. It is also clear, since $\mathbf{X}$ is classical, that each subgroup A of K contains a finite subgroup F such that $C_H(A) = C_H(F)$. Since every countable subgroup of HK is contained in one of the form H_1K_1, where H_1 and K_1 are countable subgroups of H and K respectively and K_1 normalizes H_1, Hartley (1971a) Lemma 4.3 shows that HK is Sylow π-sparse.

Proof of Theorem B. To see the necessity of the given conditions we note that Lemma 6.3 shows that G is a classical locally soluble $\mathfrak{M}_c$-head. Therefore G has finite rank by Lemma 3.4, and so G is almost subpropinched by Theorem 6.2. Clearly there exists a prime $q \notin \pi$.

For the sufficiency, it is enough by Lemma 6.3 to show that a group G which is almost subpropinched admits a faithful irreducible $\mathfrak{M}_c$-family whose characteristic is any given prime $q \notin \pi(G)$. Theorem A shows that G admits some faithful irreducible $\mathfrak{M}_c$-family, and if we recall that a pinched group is p'-pinched for every p, we see that the argument used to show that G has a faithful irreducible $\mathfrak{M}_c$-family (end of Section 4) also shows that the characteristic can be chosen as desired.

To prove Theorem C we must first establish Lemma 6.4.

Proof of Lemma 6.4. If $L \in \mathfrak{U}$ and $R = \rho(L)$, then the set of chief factors of L below R forms in a natural way a family $\mathbf{X}$ of irreducible L-modules over various fields Z_p. Let $A \leqq L$ and let $\pi = \pi(A)$. Then any $X \in \mathbf{X}$ such that char$k(X) \notin \pi$ can be viewed as a chief factor of L below $R_{\pi'}$. From the definition of the class $\mathfrak{U}$, the group $R_{\pi'}A$ is Sylow π-sparse, and so, by the extension of Hartley (1971a) Lemma 4.3 already mentioned, there is a finite subgroup F of A such that $C_{R_{\pi'}}(F) = C_{R_{\pi'}}(A)$. Arguing as in the proof of Lemma 6.3 we find that $C_X(F) = C_X(A)$ for all $X \in \mathbf{X}$ such that char$k(X \notin)\pi$, and so $\mathbf{X}$ is an irreducible $\mathfrak{M}_c$-family for L.

By Hartley (1971) Theorem 2.8, R is the intersection of the centralizers of the chief factors of L below R, and so, by Lemma 2.1, X can be viewed as a faithful irreducible $\mathfrak{M}_c$-family for $G = L/R$.

For the converse, let G, X and R be as given. Then any countable subgroup of $L = RG$ lies in one of the form HK, where H and K are countable subgroups of R and G respectively and K normalizes H. Let A be any π-subgroup of K. Then from the construction of R, there is a finite subgroup F of A such that $C_{R_-'}(A) = C_{R_\pi'}(F)$. Hence $C_{H_-'}(A) = C_{H_-'}(F)$. It follows by the argument of Hartley (1971a) Lemma 7.1 that $HK \in \mathfrak{U}$ and hence, by Hartley (1972) Lemma 2.1 and the obvious fact that L is finitely radical, that $L \in \mathfrak{U}$.

Let $T = \rho(L)$. Then $T \geqq R$ and so $T = R(T \cap G)$. Let $S = T \cap G$. Then S_p is a normal p-subgroup of G, and so, by Gardiner, Hartley and Tomkinson (1971) Lemma 3.2. S_p centralizes every irreducible G-module over a field of characteristic p. However S_p centralizes $R_{p'}$ as $\langle S_p, R_{p'}\rangle$ is locally nilpotent, an so S_p centralizes $X_{p'}$. Hence $S_p \leqq C_G(X) = 1$, and so $\rho(L) = R$ as claimed.

Proof of Theorem C. Suppose that $G \cong L/\rho(L)$, where $L \in \mathfrak{U}$. Then $G \in \mathfrak{U}$ as $\mathfrak{U}$ is image-closed. By Lemma 6.4 G is a locally soluble $\mathfrak{M}_c$-head, and by Hartley (1971a) Theorem E, G has finite rank. Hence by Theorem 6.2, G is almost subpropinched.

Conversely, let G be an almost subpropinched $\mathfrak{U}$-group. By Lemma 6.4, it suffices to show that G admits a faithful irreducible $\mathfrak{M}_c$-family of characteristic not containing zero. This follows by the argument used to construct $\mathfrak{M}_c$-familes in the proof of Theorem A (end of Section 4).

It seems appropriate, in view of Theorems B and C, to conclude with a few remarks about the structure of pinched and almost subpropinched groups. We have seen that if G is pinched, then every subgroup of prime power order of G is cyclic. Thus G' and G/G' are locally cyclic, and G is countable. It is well known that if F is any finite group with cyclic Sylow subgroups, then $|F'|$ and $|F/F'|$ are relatively prime. It follows easily from this that $\pi(G') \cap \pi(G/G') = \varnothing$ and hence, since G is countable, that G splits over G' (e.g. Hartley (1971a) Lemma 2.1). We have established part of

Lemma 6.5. *Let G be a group. Then G is pinched if and only if G can be written as a semidirect product $G = BC$, $B \lhd G$, $B \cap C = 1$ of locally cyclic subgroups B and C such that $\pi(B) \cap \pi(C) = \varnothing$ and $C_C(B)$ contains every element of prime order of C.*

Furthermore, $G \in \mathfrak{U}$ if and only if each subgroup D of C contains a finite subgroup F such that $C_B(D) = C_B(F)$.

Proof. If G is pinched, then taking $B = G'$, we obtain a subgroup C such that $G = BC$ and $\pi(B) \cap \pi(C) = \varnothing$, as described above. Clearly C and B are locally cyclic. Now the fact that every subgroup of prime power order of G is abelian implies that any two abelian normal subgroups of G commute elementwise, and so if A is a maximal abelian normal subgroup of G containing all the elements of prime order of G, then $A \geqq B$. Hence $A = B(A \cap C)$, $A \cap C \leqq C_C(B)$, and so $C_C(B)$ contains every element of prime order of C.

Conversely if $G = BC$ as given, we take $A = BC_C(B)$. Then A is locally cyclic as $\pi(B) \cap \pi(C) = \emptyset$. Clearly G/A is abelian and every element of prime order of G lies in A. Since 2 can belong to at most one of $\pi(B)$ and $\pi(C)$, every 2-subgroup of G is abelian.

Finally, the condition for G to belong to $\mathfrak{U}$ follows from Lemmas 4.3 and 7.1 of Hartley (1971a).

LEMMA 6.6. *Let G be almost subpropinched. Then*

(i) *G is countable.*

(ii) *G is almost metabelian.*

(iii) *G is almost parasoluble.*

(iv) *G has finite rank.*

(v) *For almost all primes q, every q-subgroup of G is abelian.*

(vi) *There is a finite set σ of primes such that the elements of G whose orders are primes not lying in σ generate an abelian subgroup.*

PROOF. It is a straightforward exercise to verify that pinched groups possess all these properties and that they are preserved by taking subgroups, finite extensions and direct products with finitely many factors. The concept of parasolubility was introduced by Wehrfritz (1971); a group G is called parasoluble if G has a finite series $1 = G_0 \leqq G_1 \leqq \cdots \leqq G_n = G$ of normal subgroups with abelian factors and such that every subgroup of G_i/G_{i-1} is normal in G/G_{i-1} $(1 \leqq i \leqq n)$. In showing that pinched groups are parasoluble, notice that every subgroup of a locally cyclic group is characteristic.

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