Canad. J. Math. Vol. 68 (1), 2016 pp. 129–149
http://dx.doi.org/10.4153/CJM-2015-014-x

Lower Escape Rate of Symmetric Jump-diffusion Processes

Yuichi Shiozawa

Abstract. We establish an integral test on the lower escape rate of symmetric jump-diffusion processes generated by regular Dirichlet forms. Using this test, we can find the speed of particles escaping to infinity. We apply this test to symmetric jump processes of variable order. We also derive the upper and lower escape rates of time-changed processes by using those of underlying processes.

1 Introduction

In [27], we studied the upper escape rate of symmetric jump-diffusion processes generated by regular Dirichlet forms (see [25] and the references given there for symmetric diffusion processes and [12, 14] for Markov chains on weighted graphs). This notion expresses how far particles can go for all sufficiently large time, and is thus regarded as a quantitative version of conservativeness (see [7, 10, 13, 18, 19, 26, 28] for conservativeness criteria of symmetric jump-diffusion processes). The result in [27] shows how the upper escape rate can be affected by the rates of volume growth, coefficient growth, and big jump. In this paper, we are concerned with the lower escape rate of symmetric jump-diffusion processes, that is, the speed of particles escaping to infinity. We can regard this notion as a quantitative version of transience. The purpose of this paper is to establish an integral test on the lower escape rate of symmetric jump-diffusion processes (Theorem 2.1 and Corollary 3.3). We also apply this test to symmetric jump processes of variable order. This application ensures the sharpness of the test.

Dvoretzky and Erdös [6] determined the speed of Brownian particles on $\mathbb{R}^d$ escaping to infinity, and Takeuchi [30] extended this result to symmetric stable processes on $\mathbb{R}^d$. More precisely, let $(\{X_t\}_{t\geq 0}, P)$ be the symmetric α-stable process on $\mathbb{R}^d$ starting from the origin for $0<\alpha\leq 2$. This process is nothing but the Brownian motion for $\alpha=2$. It is well known that for $d>\alpha$, this process is transient and escapes to infinity as $t\to\infty$ almost surely. Furthermore, if we define $r_{\alpha,p}(t) = t^{1/\alpha}/(\log t)^{\frac{1+p}{d-\alpha}}$ for a constant p, then

$$P\big(|X_t|\geq r_{\alpha,p}(t) \text{ for all sufficiently large } t\big) = \begin{cases} 1, & p>0,\\ 0, & p\leq 0.\end{cases}$$

Received by the editors September 15, 2014; revised March 16, 2015.
Published electronically September 15, 2015.
Supported in part by the Grant-in-Aid for Scientific Research (C) 26400135.
AMS subject classification: **60G17**, 31C25, 60J25.
Keywords: lower escape rate, Dirichlet form, Markov process, time change,

This result shows that the smaller the index α is, the faster the escape speed of particles is. For $p > 0$, the function $r_{\alpha,p}(t)$ is called a *lower rate function* for the symmetric α-stable process on $\mathbb{R}^d$.

Ichihara [15, Theorem E] extended the result of Dvoretzky and Erdös [6] to symmetric diffusion processes on $\mathbb{R}^d$ generated by uniformly elliptic operators with smooth coefficients. For the proof of this result, full heat kernel estimates and martingale theory are utilized. Grigor'yan [9] obtained an integral test on the lower escape rate of Brownian motions on Riemannian manifolds (see also [2]). This result means that lower rate functions can be determined by the upper bounds of the volume growth rate and the heat kernel on-diagonal part. In the proof of this result, the capacitary upper estimate by Sturm [29] played an important role in estimating hitting probabilities to compact sets. On the other hand, Hendricks [11] and Khoshnevisan [17] extended the result of Takeuchi [30] to direct products of stable processes with different indices.

Our result is applicable to more general symmetric Markov processes. In fact, we can generalize the result of Grigor'yan [9] to symmetric jump-diffusion processes. This generalization reveals that the scaling order of big jumps determines the speed of particles escaping to infinity. Our approach here is similar to Grigor'yan [9]. Namely, we first give an upper estimate of the hitting probability to a compact set after a fixed time in terms of the capacity in a similar way to Bendikov and Saloff-Coste [2]. We then use the capacitary upper estimate for Dirichlet forms of non-local type as developed by Ôkura [23] (see also the recent result of Ôkura and Uemura [24]). Our result seems to be the first application of his estimate to the transient case.

We finally note that the upper and lower escape rates of time changed processes can be determined by using those of underlying processes. For instance, let $\{Y_t\}_{t\ge0}$ be a Markov process on $\mathbb{R}^d$ generated by the operator

$$\mathcal{L} = -\frac{m(x)}{2}(-\Delta)^{\alpha/2}$$

for $0 < \alpha \le 2$. Here, $m(x)$ is a positive measurable function on $\mathbb{R}^d$ such that $m(x) \asymp (1+|x|^2)^p$ for some $p \ge 0$. This process is nothing but a time changed symmetric α-stable process such that, if we take large p, then particles move speedily in space. If we assume that $0 \le p < \alpha/2$ and $d > \alpha$, then $\{Y_t\}_{t\ge0}$ is conservative and transient. Moreover, we can find the upper and lower escape rates of $\{Y_t\}_{t\ge0}$, and thus

$$\lim_{t\to\infty} \frac{\log|Y_t|}{\log t} = \frac{1}{\alpha-2p}$$

almost surely; see Section 5 for details. For $\alpha = 2$, even though Metafune and Spina [20] obtained the upper bound of the heat kernel for $\mathcal{L}$, this bound with Theorem 2.1 does not seem to imply the lower escape rate with sharp polynomial growth.

Throughout this paper, the letters c and C (with subscript) denote finite positive constants that may vary from place to place. For nonnegative functions $f(x)$ and $g(x)$ on a space S, we write $f(x) \asymp g(x)$ if there exist $c_1 > 0$ and $c_2 > 0$ such that

$$c_1 g(x) \le f(x) \le c_2 g(x) \quad \text{for any } x \in S.$$

https://doi.org/10.4153/CJM-2015-014-x Published online by Cambridge University Press

2 The Result

2.1 Preliminaries

We recall the notions of Dirichlet forms from [3] and [8]. Let $(\mathcal{X}, d)$ be a locally compact separable metric space and let m be a positive Radon measure on $\mathcal{X}$ with full support. We write $C(\mathcal{X})$ for the totality of continuous functions on $\mathcal{X}$, and $C_0(\mathcal{X})$ for that of continuous functions on $\mathcal{X}$ with compact support. Let $(\mathcal{E}, \mathcal{F})$ be a Dirichlet form on $L^2(\mathcal{X}; m)$; that is, $(\mathcal{E}, \mathcal{F})$ is a closed Markovian symmetric form on $L^2(\mathcal{X}; m)$. We assume that $(\mathcal{E}, \mathcal{F})$ is *regular*: $\mathcal{F} \cap C_0(\mathcal{X})$ is dense both in $\mathcal{F}$ with respect to the norm $\sqrt{\mathcal{E}_1}$, and in $C_0(\mathcal{X})$ with respect to the uniform norm. Here for $\alpha > 0$,

$$\mathcal{E}_\alpha(u, u) := \mathcal{E}(u, u) + \alpha \|u\|^2_{L^2(\mathcal{X}; m)}, \quad u \in \mathcal{F}.$$

By the Beurling–Deny formula ([8, Theorem 3.2.1, Lemma 4.5.4]),

$$\mathcal{E}(u, v) = \mathcal{E}^{(c)}(u, v) + \iint_{\mathcal{X} \times \mathcal{X} \setminus \text{diag}} (u(x) - u(y))(v(x) - v(y)) \, J(\mathrm{d}x\mathrm{d}y) + \int_{\mathcal{X}} u(x)v(x) \, k(\mathrm{d}x)$$

for $u, v \in \mathcal{F} \cap C_0(\mathcal{X})$, where

- $(\mathcal{E}^{(c)}, \mathcal{F} \cap C_0(\mathcal{X}))$ is a symmetric form with the strong local property (see [8, p. 120] for definition);
- J is a symmetric positive Radon measure on $\mathcal{X} \times \mathcal{X} \setminus \text{diag}$, where
$$\text{diag} = \{(x, y) \in \mathcal{X} \times \mathcal{X} \mid x = y\};$$
- k is a positive Radon measure on $\mathcal{X}$.

In particular, these three factors are determined uniquely for $(\mathcal{E}, \mathcal{F})$. We call J and k the *jumping measure* and the *killing measure*, respectively, associated with $(\mathcal{E}, \mathcal{F})$.

We can extend $\mathcal{E}^{(c)}$ uniquely to $\mathcal{F}$. Furthermore, for $u \in \mathcal{F}$, there exists a positive Radon measure $\mu^c_{\langle u \rangle}$ on $\mathcal{X}$ such that

$$\mathcal{E}^{(c)}(u, u) = \frac{1}{2} \mu^c_{\langle u \rangle}(\mathcal{X})$$

(see [8, p. 123]). We call $\mu^c_{\langle u \rangle}$ the *local part of the energy measure* of u.

We first introduce the notion of transience. Let $\{T_t\}_{t>0}$ be a strongly continuous Markovian semigroup on $L^2(\mathcal{X}; m)$ and

$$S_t f = \int_0^t T_s f \, \mathrm{d}s, \quad f \in L^2(\mathcal{X}; m).$$

Here, the integral is defined as the Bochner integral in $L^2(\mathcal{X}; m)$. We can then extend T_t and S_t on $L^1(\mathcal{X}; m) \cap L^2(\mathcal{X}; m)$ to $L^1(\mathcal{X}; m)$, uniquely. Let

$$L^1_+(\mathcal{X}; m) = \{u \in L^1(\mathcal{X}; m) \mid u \geq 0, m\text{-a.e.}\} \quad \text{and} \quad Gf = \lim_{N \to \infty} S_N f, \quad f \in L^1_+(\mathcal{X}; m).$$

We say that $\{T_t\}_{t>0}$ is *transient* if

$$Gf < \infty \quad m\text{-a.e. for any } f \in L^1_+(\mathcal{X}; m).$$

This condition is equivalent to the existence of a function $f \in L^1(\mathcal{X}; m)$ strictly positive m-a.e. on $\mathcal{X}$ such that $Gf < \infty$ m-a.e. ([8, Lemma 1.5.1]). We also know that, if

$\{T_t\}_{t>0}$ is transient, then there exists a bounded and m-integrable function g strictly positive m-a.e. on $\mathcal{X}$ such that $\int_{\mathcal{X}} g \cdot Gg \, \mathrm{d}m \le 1$ ([8, p. 40]). This function is called a *reference function* of $\{T_t\}_{t>0}$. We say that $(\mathcal{E}, \mathcal{F})$ is *transient* if there exists a bounded m-integrable function g strictly positive m-a.e. on $\mathcal{X}$ such that

$$\int_{\mathcal{X}} |u| g \, \mathrm{d}m \le \sqrt{\mathcal{E}(u,u)} \quad \text{for any } u \in \mathcal{F}.$$

The function g is called a *reference function* of $(\mathcal{E}, \mathcal{F})$. Let $\{T_t\}_{t>0}$ be a strongly continuous Markovian semigroup on $L^2(\mathcal{X}; m)$ associated with $(\mathcal{E}, \mathcal{F})$. Then by [8, Theorem 1.5.1], $(\mathcal{E}, \mathcal{F})$ is transient if and only if $\{T_t\}_{t>0}$ is transient. Moreover, there exists a common reference function of $\{T_t\}_{t>0}$ and $(\mathcal{E}, \mathcal{F})$.

Let $\mathcal{F}_e$ be the totality of m-measurable functions u on $\mathcal{X}$ such that $|u| < \infty$ m-a.e. and there exists a sequence $\{u_n\} \subset \mathcal{F}$ such that $\lim_{n\to\infty} u_n = u$, m-a.e. on $\mathcal{X}$ and

$$\lim_{m,n\to\infty} \mathcal{E}(u_n - u_m, u_n - u_m) = 0.$$

This sequence is called an *approximating sequence* of u. For any $u \in \mathcal{F}_e$ and its approximating sequence $\{u_n\}$, the limit

$$\mathcal{E}(u,u) := \lim_{n\to\infty} \mathcal{E}(u_n, u_n)$$

exists and does not depend on the choice of $\{u_n\}$ ([8, Theorem 1.5.2]). We call $(\mathcal{F}_e, \mathcal{E})$ the *extended Dirichlet space* of $(\mathcal{E}, \mathcal{F})$ ([8, p. 41]). In particular, if $(\mathcal{E}, \mathcal{F})$ is transient, then $\mathcal{F}_e$ is complete with respect to $\mathcal{E}$ ([8, Lemma 1.5.5]).

We next introduce the notion of capacity. In what follows, we assume that $(\mathcal{E}, \mathcal{F})$ is transient. Let $\mathcal{O}$ be the totality of open sets in $\mathcal{X}$. For $A \in \mathcal{O}$, define

$$\mathcal{L}_A = \{u \in \mathcal{F}_e \mid u \ge 1 \ m\text{-a.e. on } A\}$$

and

$$\mathrm{Cap}_{(0)}(A) = \begin{cases} \inf_{u \in \mathcal{L}_A} \mathcal{E}(u,u), & \mathcal{L}_A \ne \emptyset \\ \infty, & \mathcal{L}_A = \emptyset. \end{cases}$$

For any $B \subset \mathcal{X}$, we define the 0-*order capacity* by

$$\mathrm{Cap}_{(0)}(B) = \inf_{A \in \mathcal{O},\, B \subset A} \mathrm{Cap}_{(0)}(A).$$

We see by [8, p. 74] that if $\mathcal{L}_B \ne \emptyset$, then there exists a unique element $e_B^{(0)} \in \mathcal{L}_B$ such that

$$\mathrm{Cap}_{(0)}(B) = \mathcal{E}(e_B^{(0)}, e_B^{(0)}).$$

The function $e_B^{(0)}$ is called the *equilibrium potential* of B.

For $A \subset \mathcal{X}$, a statement depending on $x \in A$ is said to hold *quasi everywhere* (q.e. for short) on A if there exists a set $N \subset A$ of zero capacity such that the statement holds for every $x \in A \setminus N$.

A function $u \in \mathcal{F}$ is said to be *quasi continuous* if for any $\varepsilon > 0$, there exists $O \in \mathcal{O}$ with $\mathrm{Cap}_{(0)}(O) < \varepsilon$ such that $u|_{\mathcal{X}\setminus O}$ is finite continuous, where $u|_{\mathcal{X}\setminus O}$ is the restriction of u on $\mathcal{X} \setminus O$. It is known that every $u \in \mathcal{F}$ admits its quasi continuous m-version; see, for instance, [8, Theorem 2.1.3].

We say that a positive Radon measure μ on $\mathcal{X}$ is of (0-*order*) *finite energy integral* ($\mu \in S_0^{(0)}$ in notation) if there exists $C > 0$ such that

$$\int_{\mathcal{X}} |f| \, \mathrm{d}\mu \leq C\sqrt{\mathcal{E}(f,f)} \quad \text{for any } f \in \mathcal{F} \cap C_0(\mathcal{X}).$$

Then any measure $\mu \in S_0^{(0)}$ charges no set of zero capacity and associates a unique element $U\mu \in \mathcal{F}_e$, which is called the (0-*order*) *potential* of μ, such that

$$\mathcal{E}(U\mu, v) = \int_{\mathcal{X}} \widetilde{v} \, \mathrm{d}\mu \quad \text{for any } v \in \mathcal{F}_e$$

([8, p. 85]). For any compact set K, there exists a unique measure $\nu_K \in S_0^{(0)}$ with $\operatorname{supp}[\nu_K] \subset K$ such that $e_K^{(0)} = U\nu_K$ and

$$\operatorname{Cap}_{(0)}(K) = \mathcal{E}(e_K^{(0)}, e_K^{(0)}) = \nu_K(K) \tag{2.1}$$

(0-order version of [8, Lemma 2.2.6]). The measure ν_K is called the (0-*order*) *equilibrium measure* of K.

We write $\mathcal{B}(\mathcal{X})$ for the family of all Borel measurable subsets of $\mathcal{X}$. Let $\mathcal{X}_\Delta = \mathcal{X} \cup \{\Delta\}$ be the one point compactification of $\mathcal{X}$ and

$$\mathcal{B}(\mathcal{X}_\Delta) = \mathcal{B}(\mathcal{X}) \cup \left\{ B \cup \{\Delta\} : B \in \mathcal{B}(\mathcal{X}) \right\}.$$

Let $\mathbf{M} = (\{X_t\}_{t\geq 0}, \{P_x\}_{x\in\mathcal{X}})$ be an m-symmetric Hunt process on $\mathcal{X}$ generated by $(\mathcal{E}, \mathcal{F})$, and let $\{p_t\}_{t\geq 0}$ be the transition function of $\mathbf{M}$ given by

$$p_t(x, A) = P_x(X_t \in A), \quad x \in \mathcal{X},\ t \geq 0,\ A \in \mathcal{B}(\mathcal{X}).$$

A set $B \subset \mathcal{X}$ is called *nearly Borel measurable* if for any probability measure μ on $\mathcal{X}_\Delta$, there exist $B_1, B_2 \in \mathcal{B}(\mathcal{X}_\Delta)$ such that $B_1 \subset B_2$ and

$$P_\mu(X_t \in B_2 \setminus B_1 \text{ for some } t \geq 0) = 0.$$

We say that a set $N \subset \mathcal{X}$ is *properly exceptional* if N is nearly Borel measurable such that $m(N) = 0$ and $\mathcal{X} \setminus N$ is $\mathbf{M}$-*invariant*, that is,

$$P_x\Big(X_t \in (\mathcal{X} \setminus N)_\Delta \text{ and } X_{t-} \in (\mathcal{X} \setminus N)_\Delta \text{ for any } t > 0 \Big) = 1, \quad x \in \mathcal{X} \setminus N.$$

Here, $(\mathcal{X} \setminus N)_\Delta = (\mathcal{X} \setminus N) \cup \{\Delta\}$ and $X_{t-} = \lim_{s\uparrow t} X_s$. Note that any properly exceptional set N is exceptional, and thus $\operatorname{Cap}_{(0)}(N) = 0$ by [8, Theorem 4.2.1].

We now impose the following assumption on $\mathbf{M}$.

Assumption 1 (Absolute continuity) There exist a properly exceptional Borel set $N \subset \mathcal{X}$ and a nonnegative symmetric kernel $p_t(x,y)$ on $(0,\infty) \times (\mathcal{X} \setminus N) \times (\mathcal{X} \setminus N)$ such that $p_t(x, \mathrm{d}y) = p_t(x,y)\, m(\mathrm{d}y)$ and

$$p_{t+s}(x,y) = \int_{\mathcal{X}\setminus N} p_t(x,z) p_s(z,y)\, m(\mathrm{d}z), \quad x, y \in \mathcal{X} \setminus N,\ t, s > 0.$$

If there exists a positive left continuous function $M(t)$ on $(0,\infty)$ such that

$$\|T_t f\|_\infty \leq M(t) \|f\|_1, \quad \text{for any } f \in L^1(\mathcal{X}; m) \text{ and } t > 0,$$

then Assumption 1 holds with

$$p_t(x,y) \leq M(t) \quad \text{for } x, y \in \mathcal{X} \setminus N \text{ and } t > 0$$

(see [1, Theorem 3.1]). We define $p_t(x,y)=0$ for $(x,y)\notin(\mathcal{X}\setminus N)\times(\mathcal{X}\setminus N)$ and $t>0$ so that

$$p_{t+s}(x,y)=\int_{\mathcal{X}}p_t(x,z)p_s(z,y)\,m(\mathrm{d}z),\quad x,y\in\mathcal{X},\ t,s>0. \tag{2.2}$$

Assume that $(\mathcal{E},\mathcal{F})$ is transient. Let $r(x,y)$ be the Green kernel defined by

$$r(x,y)=\int_0^\infty p_t(x,y)\,\mathrm{d}t,$$
$$R\mu(x)=\int_{\mathcal{X}}r(x,y)\,\mu(\mathrm{d}y)$$

for $\mu\in S_0^{(0)}$. We then see that $R\mu$ is a quasi continuous and excessive version of $U\mu$ in the same way as in [3, Lemma 6.1.1].

2.2 Result

Let $(\mathcal{E},\mathcal{F})$ be a regular Dirichlet form on $L^2(\mathcal{X};m)$. We say that a function u on $\mathcal{X}$ belongs to $\mathcal{F}$ locally ($u\in\mathcal{F}_{\mathrm{loc}}$ in notation), if for any relatively compact open set $G\subset\mathcal{X}$, there exists a function $u_G\in\mathcal{F}$ such that $u=u_G$ m-a.e. on G. We can then define $\mu^c_{\langle u\rangle}$ for any $u\in\mathcal{F}_{\mathrm{loc}}$ ([8, p. 130]).

Let $\mathcal{A}$ be the totality of functions ρ in $\mathcal{F}_{\mathrm{loc}}\cap C(\mathcal{X})$ such that

(a) $\mu^c_{\langle\rho\rangle}$ is absolutely continuous with respect to m;
(b) $\lim_{x\to\Delta}\rho(x)=\infty$;
(c) for each $r>0$, the set $B_\rho(r):=\{x\in\mathcal{X}\mid\rho(x)<r\}$ is relatively compact.

We impose the next assumption on $(\mathcal{E},\mathcal{F})$.

Assumption 2 (i) $\mathcal{A}$ is non-empty.
(ii) The jumping measure $J(\mathrm{d}x\mathrm{d}y)$ satisfies

$$J(\mathrm{d}x\mathrm{d}y)=J(x,\mathrm{d}y)m(\mathrm{d}x)$$

for some kernel $J(x,\mathrm{d}y)$ that associates a positive Radon measure on $\mathcal{B}(\mathcal{X}\setminus\{x\})$ for each $x\in\mathcal{X}$ and depends on $x\in\mathcal{X}$ in a measurable way.
(iii) The killing measure k vanishes.

Fix $\rho\in\mathcal{A}$ and define

$$w^{(c)}(R)=\operatorname*{ess\,sup}_{x\in B_\rho(R)}\Gamma^c(\rho)(x),$$
$$w^{(j)}(R)=\operatorname*{ess\,sup}_{x\in\mathcal{X}}\int_{\mathcal{X}\setminus\{x\}}\big\{(\rho(x)-\rho(y))^2\wedge R^2\big\}\,J(x,\mathrm{d}y),$$

where $\Gamma^c(\rho)$ is the density function of $\mu^c_{\langle\rho\rangle}$ with respect to m. Let f be a strictly positive and nondecreasing function on $(0,\infty)$ such that

$$f(r)\ge m(B_\rho(r))\quad\text{for any } r>0.$$

Let g be a strictly positive, nonincreasing, and differentiable function on $(0,\infty)$ such that

$$g(r)\ge\frac{1}{r^2}\big(w^{(c)}(r)+w^{(j)}(r)\big)\quad\text{for any } r>0.$$

Define $h(r) = 1/g(r)$ and

$$I(R) = \int_R^\infty \frac{h'(t)}{f(t)}\,\mathrm{d}t.$$

Note that the function $h(t)$ expresses the scaling order of $(\mathcal{E},\mathcal{F})$. For instance, we assume that $(\mathcal{E},\mathcal{F})$ is associated with the independent sum of a Brownian motion and a symmetric α-stable process on $\mathbb{R}^d$ for some $\alpha \in (0,2)$. Namely,

$$\begin{aligned}\mathcal{E}(u,v) &= \frac{1}{2}\int_{\mathbb{R}^d} \nabla u(x)\cdot\nabla v(x)\,\mathrm{d}x\\ &\quad + \frac{1}{2}c_{d,\alpha}\iint_{\mathbb{R}^d\times\mathbb{R}^d\setminus\mathrm{diag}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{d+\alpha}}\,\mathrm{d}x\mathrm{d}y\\ \mathcal{F} &= \Big\{u\in L^2(\mathbb{R}^d)\ \Big|\ \frac{\partial u}{\partial x_i}\in L^2(\mathbb{R}^d),\ 1\le i\le d\Big\}\end{aligned}$$

with

$$c_{d,\alpha} = \frac{\alpha 2^{\alpha-2}\Gamma((d+\alpha)/2)}{\pi^{d/2}\Gamma(1-\alpha/2)}.$$

Then by letting $\rho(x) = |x|$, we get

$$w^{(c)}(r) \le c_1 \quad\text{and}\quad w^{(j)}(r)\le c_2 r^{2-\alpha}$$

for some $c_1 > 0$ and $c_2 > 0$. Hence, we can take $h(r) = c_3 r^\alpha$ for some $c_3 > 0$. This implies that if $d > \alpha$, then $I(R) = c_4 R^{\alpha-d}$ for some $c_4 > 0$.

We finally impose the next assumption on the volume growth of the underlying measure.

Assumption 3 (Volume doubling condition) There exists $c_V > 0$ such that

$$m(B_\rho(2R)) \le c_V\cdot m(B_\rho(R)) \quad \text{for any } R > 0.$$

Let $\mathbf{M} = (\{X_t\}_{t\ge 0}, \{P_x\}_{x\in\mathcal{X}})$ be an m-symmetric Hunt process on $\mathcal{X}$ generated by $(\mathcal{E},\mathcal{F})$. Then $\mathbf{M}$ has no killing inside because the killing measure vanishes by Assumption 2(iii). The main result in this paper is the following integral test on the lower escape rate of $\mathbf{M}$.

Theorem 2.1 *Let Assumptions 1–3 hold. Assume that $(\mathcal{E},\mathcal{F})$ is transient and $I(r) < \infty$ for any $r > 0$. If $r(t)$ is a positive and strictly increasing function on $(0,\infty)$ such that*

$$\int_{t_0}^\infty \frac{1}{I(r(s))}\sup_{y\in\mathcal{X}} p_s(x,y)\,\mathrm{d}s < \infty \quad \textit{for any } x\in\mathcal{X} \tag{2.3}$$

with some $t_0 > 0$, then

$$P_x\big(\rho(X_t)\ge r(t) \textit{ for all sufficiently large } t\big) = 1, \quad \textit{q.e. } x\in\mathcal{X}. \tag{2.4}$$

The function $r(t)$ in (2.4) is called a *lower rate function* for $\mathbf{M}$ with respect to ρ. On the other hand, a positive and strictly increasing function $\widetilde{r}(t)$ on $(0,\infty)$ is called a *lower rate function* for $\mathbf{M}$ if

$$P_x\big(d(x,X_t)\ge \widetilde{r}(t) \text{ for all sufficiently large } t\big) = 1, \quad \text{q.e. } x\in\mathcal{X}.$$

https://doi.org/10.4153/CJM-2015-014-x Published online by Cambridge University Press

We then see that, if $\rho(x) = d(o, x)$ for some $o \in \mathcal{X}$ and $r(t) \to \infty$ as $t \to \infty$ in (2.4), then for any small $\varepsilon > 0$, the function $(1-\varepsilon)r(t)$ is a lower rate function for **M**.

Remark 2.2 Let Assumption 1 hold. If

$$\int_1^\infty \sup_{y\in\mathcal{X}} p_s(x,y)\,\mathrm{d}s < \infty \quad \text{for any } x \in \mathcal{X},$$

then $Gf < \infty$ m-a.e. for any $f \in L^1(\mathcal{X}; m) \cap \mathcal{B}_b(\mathcal{X})$ strictly positive m-a.e. on $\mathcal{X}$, because

$$Gf = \int_0^\infty p_s f\,\mathrm{d}s, \quad m\text{-a.e. on } \mathcal{X}$$

and

$$\begin{aligned}
&\int_0^\infty p_s f(x)\,\mathrm{d}s \\
&= \int_0^1 \Big(\int_{\mathcal{X}} p_s(x,y) f(y)\,m(\mathrm{d}y) \Big)\,\mathrm{d}s + \int_1^\infty \Big(\int_{\mathcal{X}} p_s(x,y) f(y)\,m(\mathrm{d}y) \Big)\,\mathrm{d}s \\
&\le \|f\|_\infty + \|f\|_1 \int_1^\infty \sup_{y\in\mathcal{X}} p_s(x,y)\,\mathrm{d}s < \infty
\end{aligned}$$

for q.e. $x \in \mathcal{X}$. Therefore, $(\mathcal{E}, \mathcal{F})$ is transient as we mentioned in Subsection 2.1 (see also [8, Lemma 1.5.1]).

3 Proof of Theorem 2.1

In this section, we give a proof of Theorem 2.1. As mentioned before, our approach is similar to that of Grigor'yan [9].

Let K be a compact set in $\mathcal{X}$ and let $\sigma_K = \inf\{t > 0 \mid X_t \in K\}$ be the hitting time of **M** to K. If $(\mathcal{E}, \mathcal{F})$ is transient, then the function $p_K(x) := P_x(\sigma_K < \infty)$ is a quasi continuous modification of $e_K^{(0)} = U\nu_K$ ([8, Theorem 4.3.3]), whence $p_K = R\nu_K$ m-a.e. under Assumption 1

We first derive an upper bound of the probability $\psi_K(t,x)$ given by

$$\psi_K(t,x) = P_x(X_s \in K \text{ for some } s > t),\ x \in \mathcal{X} \setminus N,\ t \ge 0.$$

Lemma 3.1 *Let Assumption 1 hold and assume that $(\mathcal{E}, \mathcal{F})$ is transient. Then for any compact set K in $\mathcal{X}$,*

$$\psi_K(t,x) \le \mathrm{Cap}_{(0)}(K) \int_t^\infty \sup_{y\in\mathcal{X}} p_s(x,y)\,\mathrm{d}s, \quad t > 0,\ x \in \mathcal{X} \setminus N.$$

This lemma is a 0-order version of Bendikov and Saloff-Coste [2, Theorem 3.10], and our proof is similar to theirs.

Proof of Lemma 3.1 Let $\{\theta_t\}_{t\ge 0}$ be the shift operator of sample paths of **M**. Then by the Markov property,

$$\begin{aligned}
\psi_K(t,x) = P_x(\sigma_K \circ \theta_t < \infty) &= E_x[P_{X_t}(\sigma_K < \infty)] \\
&= \int_{\mathcal{X}} p_t(x,y) p_K(y)\,m(\mathrm{d}y).
\end{aligned} \tag{3.1}$$

https://doi.org/10.4153/CJM-2015-014-x Published online by Cambridge University Press

Since $p_K = R\nu_K$ m-a.e. and $\mathrm{supp}[\nu_K] \subset K$, the last expression in (3.1) is equal to

$$\int_{\mathcal{X}} p_t(x,y) R\nu_K(y)\, m(\mathrm{d}y) = \int_{\mathcal{X}} p_t(x,y)\Big(\int_K r(y,z)\,\nu_K(\mathrm{d}z)\Big)\, m(\mathrm{d}y)$$
$$= \int_K \Big(\int_{\mathcal{X}} p_t(x,y) r(y,z)\, m(\mathrm{d}y)\Big)\,\nu_K(\mathrm{d}z).$$

Then

$$\int_{\mathcal{X}} p_t(x,y) r(y,z)\, m(\mathrm{d}y) = \int_{\mathcal{X}} p_t(x,y)\Big(\int_0^\infty p_s(y,z)\,\mathrm{d}s\Big)\, m(\mathrm{d}y)$$
$$= \int_0^\infty \Big(\int_{\mathcal{X}} p_t(x,y) p_s(y,z)\, m(\mathrm{d}y)\Big)\,\mathrm{d}s. \tag{3.2}$$

By (2.2), the last expression in (3.2) is equal to

$$\int_0^\infty p_{t+s}(x,z)\,\mathrm{d}s = \int_t^\infty p_s(x,z)\,\mathrm{d}s.$$

Hence by (2.1),

$$\psi_K(t,x) = \int_K \Big(\int_t^\infty p_s(x,z)\,\mathrm{d}s\Big)\,\nu_K(\mathrm{d}z) = \int_t^\infty \Big(\int_K p_s(x,z)\nu_K(\mathrm{d}z)\Big)\,\mathrm{d}s$$
$$\le \nu_K(K)\int_t^\infty \sup_{z\in K} p_s(x,z)\,\mathrm{d}s = \mathrm{Cap}_{(0)}(K)\int_t^\infty \sup_{y\in\mathcal{X}} p_s(x,y)\,\mathrm{d}s. \qquad \blacksquare$$

We next obtain the capacitary upper bound as an application of the result by Ôkura [23].

Lemma 3.2 *Let Assumptions 2 and 3 hold. If $I(r) < \infty$ for any $r > 0$, then there exists $C > 0$ such that for any $r > 0$,*

$$\mathrm{Cap}(\overline{B}_\rho(r)) \le \frac{C}{I(r)}. \tag{3.3}$$

Proof Fix $\rho \in \mathcal{A}$. For $R > r > 0$, we define

$$\phi_{r,R}(x) = 0 \vee \Big(\frac{R-\rho(x)}{R-r}\Big) \wedge 1,$$
$$J_0(r,R) = \int_{B_\rho(r)} \Big(\int_{B_\rho(R)^c} J(x,\mathrm{d}y)\Big)\, m(\mathrm{d}x).$$

Note that $\phi_{r,R} \in \mathcal{F} \cap C_0(\mathcal{X})$. We will show that for any $c_0 > 1$,

$$3\mathcal{E}(\phi_{r,R},\phi_{r,R}) \le Cf(r)g(R) \quad \text{for } c_0 \le R/r \le c_0^2 \tag{3.4}$$

and

$$4J_0(r,R) \le Cf(r/c_0^2)g(c_0^2 R) \quad \text{for } R/r \ge c_0 \tag{3.5}$$

with some positive constant $C > 0$, which will be explicitly given below. By [23, Theorem 2.6], (3.4) and (3.5) imply that

$$\mathrm{Cap}(\overline{B}_\rho(r), B_\rho(R)) \le C\Big(\int_r^R \frac{h'(t)}{f(t)}\,\mathrm{d}t\Big)^{-1}$$

for $1 \le r < c_0 r \le R$. Here for a compact set K and a relatively compact open set G with $K \subset G$,

$$\mathrm{Cap}(K, G) = \inf\left\{ \mathcal{E}(u,u) \mid u \in \mathcal{F} \cap C_0(\mathcal{X}), u \ge 1 \text{ on } K, u = 0 \text{ on } G^c \right\}.$$

Noting that

$$\mathrm{Cap}_{(0)}(\overline{B}_\rho(r)) \le \mathrm{Cap}(\overline{B}_\rho(r), B_\rho(R)) \le C\Big(\int_r^R \frac{h'(t)}{f(t)}\, \mathrm{d}t \Big)^{-1},$$

we get (3.3) by letting $R \to \infty$.

In what follows, we show (3.4) and (3.5). Since

$$\Gamma(\phi_{r,R})(x) = \frac{1}{(R-r)^2} \Gamma(\rho)(x) \cdot \mathbf{1}_{\{r<\rho(x)<R\}}(x),$$

we have

$$\begin{aligned} \mathcal{E}^{(c)}(\phi_{r,R}, \phi_{r,R}) &= \frac{1}{2(R-r)^2} \int_{r<\rho(x)<R} \Gamma(\rho)(x)\, m(\mathrm{d}x) \\ &\le \frac{1}{2(R-r)^2} m(B_\rho(R)) \cdot w^{(c)}(R). \end{aligned} \tag{3.6}$$

On the other hand, since

$$|\phi_{r,R}(x) - \phi_{r,R}(y)| \le \frac{1}{R-r} \left\{ |\rho(x) - \rho(y)| \wedge (R-r) \right\},$$

we obtain for any $c > 1$,

$$\begin{aligned} &\iint_{\mathcal{X}\times\mathcal{X}\setminus \mathrm{diag}} \big(\phi_{r,R}(x) - \phi_{r,R}(y) \big)^2 J(x, \mathrm{d}y)\, m(\mathrm{d}x) \\ &= \iint_{B_\rho(cR)\times B_\rho(cR)\setminus \mathrm{diag}} \big(\phi_{r,R}(x) - \phi_{r,R}(y) \big)^2 J(x, \mathrm{d}y)\, m(\mathrm{d}x) \\ &\quad + 2 \iint_{B_\rho(cR)\times B_\rho(cR)^c} (\phi_{r,R}(x) - \phi_{r,R}(y))^2 J(x, \mathrm{d}y)\, m(\mathrm{d}x) \\ &\le \frac{3}{(R-r)^2} m\big(B_\rho(cR) \big) \cdot w^{(j)}(R). \end{aligned} \tag{3.7}$$

Assume first that $c_0 \le R/r \le c_0^2$ for some $c_0 > 1$. Then

$$\frac{1}{(R-r)^2} \le \Big(\frac{c_0}{c_0 - 1} \Big)^2 \cdot \frac{1}{R^2}. \tag{3.8}$$

If we choose $N \ge 1$ so that $cc_0^2/2^N < 1$, then

$$\begin{aligned} m(B_\rho(R)) \le m(B_\rho(cR)) &\le (c_V)^N m\big(B_\rho(cR/2^N) \big) \\ &\le (c_V)^N m\big(B_\rho(cc_0^2 r/2^N) \big) \le (c_V)^N m\big(B_\rho(r) \big), \end{aligned}$$

by Assumption 3. Hence the last expressions in (3.6) and (3.7) are less than

$$\frac{(c_V)^N}{2} \Big(\frac{c_0}{c_0-1} \Big)^2 \cdot m(B_\rho(r)) \cdot \frac{w^{(c)}(R)}{R^2}$$

and

$$3(c_V)^N \Big(\frac{c_0}{c_0-1} \Big)^2 \cdot m(B_\rho(r)) \cdot \frac{w^{(j)}(R)}{R^2},$$

respectively, which implies that

$$\mathcal{E}(\phi_{r,R},\phi_{r,R}) \le 3(c_V)^N\Big(\frac{c_0}{c_0-1}\Big)^2\cdot m(B_\rho(r))\cdot\frac{1}{R^2}\Big(w^{(c)}(R)+w^{(j)}(R)\Big).$$

Assume next that $R/r \ge c_0$ for some $c_0 > 1$. Then

(3.9)

$$\begin{aligned}
&J_0(r,R)\\
&\quad=\int_{B_\rho(r)}\Bigg(\int_{B_\rho(R)^c}\Big\{1\wedge\Big(\frac{\rho(y)-\rho(x)}{R-r}\Big)\Big\}^2 J(x,\mathrm{d}y)\Bigg)\,m(\mathrm{d}x)\\
&\quad\le\frac{1}{(R-r)^2}m(B_\rho(r))\cdot\operatorname*{ess\,sup}_{x\in B_\rho(r)}\int_{\mathcal{X}\setminus\{x\}}\Big\{(\rho(y)-\rho(x))^2\wedge(R-r)^2\Big\}\,J(x,\mathrm{d}y)\\
&\quad\le\Big(\frac{c_0}{c_0-1}\Big)^2 m(B_\rho(r))\cdot\frac{w^{(j)}(R)}{R^2}
\end{aligned}$$

by (3.8). We now take $M \ge 1$ so that $2^M \ge c_0^2$. Since

$$m(B_\rho(r)) \le (c_V)^M m(B_\rho(r/2^M)) \le (c_V)^M m(B_\rho(r/c_0^2))$$

by Assumption 3, the last expression in (3.9) is less than

$$\Big(\frac{c_0}{c_0-1}\Big)^2 c_0^4(c_V)^M\cdot m(B_\rho(r/c_0^2))\cdot\frac{1}{(c_0^2R)^2}w^{(j)}(c_0^2R).$$

As a result of the argument above, we arrive at (3.4) and (3.5) for any $c_0 > 1$, where

$$C=\Big(\frac{c_0}{c_0-1}\Big)^2\big(9(c_V)^N\vee 4c_0^4(c_V)^M\big).$$

Hence the proof is complete. ∎

Proof of Theorem 2.1 Fix $x \in \mathcal{X}$ and let

$$J(t)=\int_t^\infty \sup_{y\in\mathcal{X}} p_s(x,y)\,\mathrm{d}s.$$

Then by Lemmas 3.1 and 3.2,

$$\psi_{\overline{B}_\rho(r)}(t,x)\le \mathrm{Cap}_{(0)}(\overline{B}_\rho(r))J(t)\le C\frac{J(t)}{I(r)}.$$

Let $\{t_k\}_{k=0}^\infty$ be an increasing sequence such that

$$J(t_{k+1})=\frac{1}{2}J(t_k)$$

for any $k \ge 0$. Then

$$\begin{aligned}
J(t_{k-1}) &= 2J(t_k)=4J(t_k)-2J(t_k)=4\big(J(t_k)-J(t_{k+1})\big)\\
&=4\int_{t_k}^{t_{k+1}}\sup_{y\in\mathcal{X}}p_s(x,y)\,\mathrm{d}s,
\end{aligned}$$

https://doi.org/10.4153/CJM-2015-014-x Published online by Cambridge University Press

which implies that

$$\psi_{\overline{B_\rho(r(t_k))}}(t_{k-1},x) \le C\frac{J(t_{k-1})}{I(r(t_k))} = 4C\int_{t_k}^{t_{k+1}} \frac{1}{I(r(t_k))}\sup_{y\in\mathcal{X}} p_s(x,y)\,\mathrm{d}s$$
$$\le 4C\int_{t_k}^{t_{k+1}} \frac{1}{I(r(s))}\sup_{y\in\mathcal{X}} p_s(x,y)\,\mathrm{d}s$$

for any positive and strictly increasing function $r(t)$. Hence,

$$\sum_{k=1}^{\infty} \psi_{\overline{B_\rho(r(t_k))}}(t_{k-1},x) < \infty$$

by (2.3). In particular, since

$$\begin{aligned}
&P_x\big(\rho(X_t) \le r(t) \text{ for some } t \in [t_{k-1},t_k)\big)\\
&\quad\le P_x\big(\rho(X_t) \le r(t_k) \text{ for some } t \in [t_{k-1},t_k)\big)\\
&\quad\le P_x\big(\rho(X_t) \le r(t_k) \text{ for some } t \ge t_{k-1}\big)\\
&\quad= \psi_{\overline{B_\rho(r(t_k))}}(t_{k-1},x),
\end{aligned}$$

we have

$$\sum_{k=1}^{\infty} P_x\big(\rho(X_t) \le r(t) \text{ for some } t \in [t_{k-1},t_k)\big) < \infty$$

so that

$$P_x\big(\rho(X_t) \ge r(t) \text{ for all sufficiently large } t\big) = 1$$

by the Borel–Cantelli lemma. This completes the proof. ■

For applications of Theorem 2.1, we make the following assumption.

Assumption 4 In addition to Assumptions 1–3, the following hold.

(i) There exist $p > 0$ and $c_1 > 0$ such that for any $x \in \mathcal{X}$,

$$p_t(x,x) \le \frac{c_1}{f(t^p)} \quad \text{for all } t \ge 1.$$

(ii) There exist $\nu > 0$ and $c_2 > 0$ such that for any $r > 0$ and $R > r$,

$$\frac{f(R)}{f(r)} \ge c_2\Big(\frac{R}{r}\Big)^{\nu}.$$

(iii) There exists $c_3 > 1$ such that for any $R > 0$, $h(c_3R) \ge 2h(R)$.

Let $r(t)$ be a positive, strictly increasing function on $(0,\infty)$ such that $r(t)/t^p \to 0$ as $t \to \infty$. Then by Assumption 4(iii),

$$I(r(t)) \ge \int_{r(t)}^{c_3 r(t)} \frac{h'(u)}{f(u)}\,\mathrm{d}u \ge \frac{1}{f(c_3 r(t))}\big(h(c_3 r(t)) - h(r(t))\big) \ge \frac{h(r(t))}{f(c_3 r(t))}.$$

Hence, for all sufficiently large $t > 0$,

$$f(t^p)I(r(t)) \ge \frac{f(t^p)}{f(c_3 r(t))}h(r(t)) \ge c_2\Big(\frac{t^p}{c_3 r(t)}\Big)^{\nu} h(r(t)) = \frac{c_2}{c_3^{\nu}}\cdot\frac{t^{p\nu}}{r(t)^{\nu}}h(r(t))$$

by Assumption 4(ii). Since

$$\sup_{y\in\mathcal{X}} p_t(x,y) \le \sup_{y\in\mathcal{X}} p_t(y,y) \le \frac{c_1}{f(t^p)}$$

by (2.2) and Assumption 4(i), we have

$$\frac{1}{I(r(t))}\sup_{y\in\mathcal{X}} p_t(x,y) \le \frac{c_1}{f(t^p)I(r(t))} \le \frac{c_1c_3^\nu}{c_2}\cdot\frac{r(t)^\nu}{t^{p\nu}h(r(t))},$$

which implies the following corollary.

Corollary 3.3 *Let Assumption 4 hold. Assume that $(\mathcal{E},\mathcal{F})$ is transient and $I(r)<\infty$ for any $r>0$. If $r(t)$ is a positive and strictly increasing function on $(0,\infty)$ such that $r(t)/t^p \to 0$ as $t\to\infty$ and*

$$\int_{t_0}^\infty \frac{r(t)^\nu}{t^{p\nu}h(r(t))}\,\mathrm{d}t<\infty$$

for some $t_0>0$, then (2.4) holds.

4 Examples

In this section, we apply Theorem 2.1 and Corollary 3.3 to symmetric jump processes.

Example 4.1 For $x\in\mathcal{X}$ and $r>0$, let $B_x(r)=\{y\in\mathcal{X}\mid d(y,x)<r\}$. We assume that $B_x(r)$ is relatively compact for any $x\in\mathcal{X}$ and $r>0$, and that for some $\alpha>0$,

$$m(B_x(r))\asymp r^\alpha \quad \text{for any } x\in\mathcal{X}.$$

Let γ be a positive measurable function on $\mathcal{X}\times\mathcal{X}$ such that

$$\gamma_1\le\gamma(x,y)\le\gamma_2,\quad x,y\in\mathcal{X}$$

for some $\gamma_1,\gamma_2\in(0,2)$ with $\gamma_1<\gamma_2$. Let $J(x,y)$ be a symmetric and strictly positive function on $\mathcal{X}\times\mathcal{X}\setminus\mathrm{diag}$ such that

$$J(x,y)\asymp\frac{1}{d(x,y)^{\alpha+\gamma(x,y)}}.$$

We also assume that $C_0^{\mathrm{lip}}(\mathcal{X})\subset\mathcal{F}$ and

$$\mathcal{E}(u,u)=\iint_{\mathcal{X}\times\mathcal{X}\setminus\mathrm{diag}}(u(x)-u(y))^2J(x,y)m(\mathrm{d}x)m(\mathrm{d}y),\quad u\in\mathcal{F}\cap C_0(\mathcal{X}).$$

Here, $C_0^{\mathrm{lip}}(\mathcal{X})$ is the totality of Lipschitz continuous functions on $\mathcal{X}$ with compact support. Then for a fixed point $o\in\mathcal{X}$, the function $\rho(x):=d(o,x)$ belongs to $\mathcal{F}_{\mathrm{loc}}\cap C(\mathcal{X})$.

(i) Let

$$\gamma(x,y)=\begin{cases}\beta_1 & d(x,y)<1\\ \beta_2 & d(x,y)\ge1\end{cases}$$

for some $\beta_1, \beta_2 \in (0,2)$ and $\phi(s) = s^{\beta_1}\mathbf{1}_{\{s<1\}} + s^{\beta_2}\mathbf{1}_{\{s\geq 1\}}$. Then

$$J(x,y) \asymp \frac{1}{\phi(d(x,y))m(B_x(d(x,y)))}$$

for any $(x,y) \in \mathcal{X}\times\mathcal{X}\setminus \text{diag}$ and

$$\frac{\phi(R)}{\phi(r)} \geq \left(\frac{R}{r}\right)^{\beta_1\wedge\beta_2}$$

for any $0 < r < R < \infty$. Hence by [1, Theorem 3.1] and [5, Theorems 3.1 and 3.2], there exists a properly exceptional Borel set $N \subset \mathcal{X}$ such that

$$p_t(x,\mathrm{d}y) = p_t(x,y)m(\mathrm{d}y)$$

for some positive symmetric kernel $p_t(x,y)$ on $(0,\infty)\times(\mathcal{X}\setminus N)\times(\mathcal{X}\setminus N)$. Furthermore, there exist $c_1 > 0$ and $c_2 > 0$ such that

$$p_t(x,y) \leq \frac{c_2}{m(B(\phi^{-1}(c_1 t)))}$$

for any $t > 0$ and $x, y \in \mathcal{X}\setminus N$. Therefore, for some $c > 0$,

$$p_t(x,y) \leq \frac{c}{t^{\alpha/\beta_2}} \quad \text{for all } t \geq 1.$$

We now assume that $0 < \beta_2 < 2\wedge\alpha$ so that $(\mathcal{E},\mathcal{F})$ is transient by Remark 2.2. Since

$$\begin{aligned} w^{(j)}(t) &= \sup_{x\in\mathcal{X}} \int_{\mathcal{X}\setminus\{x\}} \left\{ (\rho(x)-\rho(y))^2 \wedge t^2 \right\} J(x,y)\, m(\mathrm{d}y) \\ &\leq \sup_{x\in\mathcal{X}} \int_{\mathcal{X}\setminus\{x\}} \left(d(x,y)^2 \wedge t^2 \right) J(x,y)\, m(\mathrm{d}y) \leq c t^{2-\beta_2} \end{aligned}$$

for some $c > 0$, we can take $h(t) = c' t^{\beta_2}$ for some $c' > 0$. Then by letting $\nu = \alpha$ and $p = 1/\beta_2$ in Corollary 3.3, we get

$$\int_1^\infty \frac{r(t)^\alpha}{t^{\alpha/\beta_2} h(r(t))}\,\mathrm{d}t \asymp \int_1^\infty \frac{r(t)^\alpha}{t^{\alpha/\beta_2} r(t)^{\beta_2}}\,\mathrm{d}t = \int_1^\infty \frac{r(t)^{\alpha-\beta_2}}{t^{\alpha/\beta_2}}\,\mathrm{d}t.$$

In particular, the last expression above is finite for $r(t) = c t^{1/\beta_2}/(\log t)^{\frac{1+\varepsilon}{\alpha-\beta_2}}$ with any $c > 0$ and $\varepsilon > 0$. We thus obtain

$$P_x\left(d(x,X_t) \geq t^{1/\beta_2}/(\log t)^{\frac{1+\varepsilon}{\alpha-\beta_2}} \text{ for all sufficiently large } t \right) = 1, \quad \text{q.e. } x \in \mathcal{X}.$$

This result is similar to that for the symmetric β_2-stable process on $\mathbb{R}^d$ (see Takeuchi [30]).

(ii) Assume that

$$\begin{aligned} \beta_1 \leq \gamma(x,y) \leq \beta_2 &\quad \text{for } d(x,y) < 1, \\ \gamma_1 \leq \gamma(x,y) \leq \gamma_2 &\quad \text{for } d(x,y) \geq 1 \end{aligned}$$

for some $\beta_1, \beta_2 \in (0,2)$ with $\beta_1 < \beta_2$ and $\gamma_1, \gamma_2 \in (0,2)$ with $\gamma_1 < \gamma_2$. Since

$$\begin{aligned} J(x,y) \geq \frac{c_1}{d(x,y)^{d+\beta_1}} &\quad \text{for } d(x,y) < 1, \\ J(x,y) \geq \frac{c_2}{d(x,y)^{d+\gamma_2}} &\quad \text{for } d(x,y) \geq 1, \end{aligned}$$

we have

$$\mathcal{E}(u,u) \geq c\left(\iint_{d(x,y)<1} \frac{(u(x)-u(y))^2}{d(x,y)^{\alpha+\beta_1}}\, m(\mathrm{d}x)m(\mathrm{d}y) + \iint_{d(x,y)\geq 1} \frac{(u(x)-u(y))^2}{d(x,y)^{\alpha+\gamma_2}}\, m(\mathrm{d}x)m(\mathrm{d}y)\right)$$

for some $c>0$. By [1, Theorem 3.1] and [5, Theorems 3.1 and 3.2] again, there exists a properly exceptional Borel set $N\subset\mathcal{X}$ such that $p_t(x,\mathrm{d}y)=p_t(x,y)m(\mathrm{d}y)$ for some positive symmetric kernel $p_t(x,y)$ on $(0,\infty)\times(\mathcal{X}\setminus N)\times(\mathcal{X}\setminus N)$ satisfying

$$p_t(x,y)\leq \frac{c}{t^{\alpha/\gamma_2}} \quad \text{for all } t\geq 1.$$

We assume that $0<\gamma_1<\gamma_2<2\wedge\alpha$ so that $(\mathcal{E},\mathcal{F})$ is transient by Remark 2.2. Since $w^{(j)}(t)\leq c't^{2-\gamma_1}$, we can take $h(t)=ct^{\gamma_1}$. Then by letting $v=\alpha$ and $p=1/\gamma_2$ in Corollary 3.3, we get

$$\int_1^\infty \frac{r(t)^\alpha}{t^{\alpha/\gamma_2}h(r(t))}\,\mathrm{d}t \asymp \int_1^\infty \frac{r(t)^\alpha}{t^{\alpha/\gamma_2}r(t)^{\gamma_1}}\,\mathrm{d}t = \int_1^\infty \frac{r(t)^{\alpha-\gamma_1}}{t^{\alpha/\gamma_2}}\,\mathrm{d}t.$$

The last expression is finite for $r(t)=ct^{\frac{1}{\gamma_2}\cdot\frac{\alpha-\gamma_2}{\alpha-\gamma_1}}/(\log t)^{\frac{1+\varepsilon}{\alpha-\gamma_1}}$ with any $c>0$ and $\varepsilon>0$, and thus

$$P_x\left(d(x,X_t)\geq t^{\frac{1}{\gamma_2}\cdot\frac{\alpha-\gamma_2}{\alpha-\gamma_1}}/(\log t)^{\frac{1+\varepsilon}{\alpha-\gamma_1}} \text{ for all sufficiently large } t\right)=1, \quad \text{q.e. } x\in\mathcal{X}.$$

Example 4.2 For each $i=1,2$, let $(\mathcal{X}^{(i)},d_i)$ be a locally compact separable metric space and m_i a positive Radon measure on $\mathcal{X}^{(i)}$ with full support. Set $\mathcal{X}=\mathcal{X}^{(1)}\times\mathcal{X}^{(2)}$ and $m=m_1\otimes m_2$. Let $(\mathcal{E}^{(i)},\mathcal{F}^{(i)})$ be a regular Dirichlet form on $L^2(\mathcal{X}^{(i)};m_i)$ and $\mathbf{M}^{(i)}=(\{X_t^{(i)}\}_{t\geq0},\{P_{x_i}\}_{x\in\mathcal{X}^{(i)}})$ an associated m_i-symmetric Hunt process on $\mathcal{X}^{(i)}$. Let $\mathbf{M}=(\{X_t\}_{t\geq0},\{P_x\}_{x\in\mathcal{X}})$ be the direct product of $\mathbf{M}^{(1)}$ and $\mathbf{M}^{(2)}$ defined by

$$X_t=(X_t^1,X_t^2), \quad P_x=P_{x_1}\otimes P_{x_2},\ x=(x_1,x_2)\in\mathcal{X}.$$

Then by [21, Theorem 1.4] and [22, Theorem 3.1], $\mathbf{M}$ is an m-symmetric Markov process on $\mathcal{X}$ and the associated Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L^2(\mathcal{X};m)$ is regular. Moreover, if $\mathcal{C}^{(i)}$ is a core for $(\mathcal{E}^{(i)},\mathcal{F}^{(i)})$, then so is $\mathcal{C}=\mathcal{C}^{(1)}\otimes\mathcal{C}^{(2)}$ for $(\mathcal{E},\mathcal{F})$ and

$$\mathcal{E}(u,v)=\int_{\mathcal{X}^{(1)}}\mathcal{E}^{(2)}(u(x_1,\cdot),v(x_1,\cdot))\,m_1(\mathrm{d}x_1) + \int_{\mathcal{X}^{(2)}}\mathcal{E}^{(1)}(u(\cdot,x_2),v(\cdot,x_2))\,m_2(\mathrm{d}x_2)$$

for $u,v\in\mathcal{C}$. Here $\mathcal{C}^{(1)}\otimes\mathcal{C}^{(2)}$ is the linear span of functions $u^{(1)}\otimes u^{(2)}(x,y):=u^{(1)}(x)u^{(2)}(y)$ for $u^{(i)}\in\mathcal{C}^{(i)}$.

In what follows, we assume that for each $i=1,2$, $C_0^{\mathrm{lip}}(\mathcal{X}^{(i)})\subset\mathcal{C}^{(i)}$ and

$$\mathcal{E}^{(i)}(u,u)=\iint_{\mathcal{X}^{(i)}\times\mathcal{X}^{(i)}\setminus\mathrm{diag}}(u(x)-u(y))^2J_i(x,y)m_i(\mathrm{d}x)m_i(\mathrm{d}y), \quad u\in\mathcal{F}^{(i)}\cap C_0(\mathcal{X}^{(i)})$$

for some positive measurable function $J_i(x,y)$ on $\mathcal{X}^{(i)} \times \mathcal{X}^{(i)} \setminus \mathrm{diag}$. Assume also that for some $\alpha_i > 0$ and $0 < \beta_i < 2$, we have $m_i(B_x^{(i)}(r)) \asymp r^{\alpha_i}$ for any $x \in \mathcal{X}^{(i)}$ and

$$J(x,y) \asymp \frac{1}{d_i(x,y)^{\alpha_i+\beta_i}}.$$

In a similar way to Example 4.1(i), there exists a properly exceptional Borel set $N^{(i)} \subset \mathcal{X}^{(i)}$ such that

$$p_t^{(i)}(x, \mathrm{d}y) = p_t^{(i)}(x,y) m_i(\mathrm{d}y)$$

for some positive symmetric kernel $p_t^{(i)}(x,y)$ on

$$(0,\infty) \times (\mathcal{X}^{(i)} \setminus N^{(i)}) \times (\mathcal{X}^{(i)} \setminus N^{(i)}).$$

Furthermore, there exists $c > 0$ such that for any $x, y \in \mathcal{X}^{(i)} \setminus N^{(i)}$,

$$p_t^{(i)}(x,y) \le \frac{c}{t^{\alpha_i/\beta_i}} \quad \text{for all } t \ge 1.$$

Therefore, if we denote by $p_t(x, \mathrm{d}y)$ the transition probability of $\mathbf{M}$, then

$$p_t(x,\mathrm{d}y) = p_t(x,y) m(\mathrm{d}y)$$

for $p_t(x,y) = p_t^{(1)}(x_1,y_1) p_t^{(2)}(x_2,y_2)$, and thus

$$p_t(x,y) \le \frac{c}{t^{\lambda}} \quad \text{for all } t \ge 1$$

for any $x = (x_1,x_2), y = (y_1,y_2) \in \mathcal{X} \setminus N$. Here,

$$\lambda = \alpha_1/\beta_1 + \alpha_2/\beta_2 \quad \text{and } N = \mathcal{X} \setminus \{(\mathcal{X}^{(1)} \setminus N^{(1)}) \times (\mathcal{X}^{(2)} \setminus N^{(2)})\}.$$

Note that N is of zero capacity with respect to $(\mathcal{E},\mathcal{F})$ ([22, Theorem 4.3 (3)]).

We now assume that $\beta_1 \ge \beta_2$ and $\lambda > 1$ so that $(\mathcal{E},\mathcal{F})$ is transient by Remark 2.2. We further assume that for each $i = 1,2$, the set $B_x^{(i)}(r) := \{y \in \mathcal{X}^{(i)} \mid d_i(x,y) < r\}$ is relatively compact for any $x \in \mathcal{X}^{(i)}$ and $r > 0$. Then by letting $d(x,y) = d_1(x_1,y_1) + d_2(x_2,y_2)$ for $x = (x_1,x_2)$, $y = (y_1,y_2) \in \mathcal{X}$, for a fixed point $o \in \mathcal{X}$, the function $\rho(x) := d(o,x)$ belongs to $\mathcal{F}_{\mathrm{loc}} \cap C(\mathcal{X})$ and

$$\begin{aligned} w^{(j)}(t) &\le \sup_{x\in\mathcal{X}} \Big\{ \int_{\mathcal{X}^{(1)}\setminus\{x_1\}} (d_1(x_1,y_1)^2 \wedge t^2)\, J_1(x_1,y_1) m_1(\mathrm{d}y_1) \Big\} \\ &\quad + \sup_{x\in\mathcal{X}} \Big\{ \int_{\mathcal{X}^{(2)}\setminus\{x_2\}} (d_2(x_2,y_2)^2 \wedge t^2)\, J_2(x_2,y_2)\, m_2(\mathrm{d}y_2) \Big\} \\ &\le c(t^{2-\beta_1} + t^{2-\beta_2}) \asymp t^{2-\beta_2}. \end{aligned}$$

This means that we can take $h(t) = ct^{\beta_2}$ for some $c > 0$. Since $m(B_\rho(r)) \asymp r^{\alpha_1+\alpha_2}$, we let $\nu = \alpha_1 + \alpha_2$ and $p = \lambda/(\alpha_1+\alpha_2)$ in Corollary 3.3 so that

$$\int_1^\infty \frac{r(t)^{\alpha_1+\alpha_2}}{t^\lambda h(r(t))}\,\mathrm{d}t \asymp \int_1^\infty \frac{r(t)^{\alpha_1+\alpha_2-\beta_2}}{t^\lambda}\,\mathrm{d}t.$$

For instance, if we set

$$r(t) = ct^{\frac{\lambda-1}{\alpha_1+\alpha_2-\beta_2}} / (\log t)^{\frac{1+\varepsilon}{\alpha_1+\alpha_2-\beta_2}}$$

for any $c > 0$, then the last expression above is finite for any $\varepsilon > 0$. Hence by Corollary 3.3,

$$P_x\big(d(x,(X_t^1,X_t^2)) \ge r(t) \text{ for all sufficiently large } t \big) = 1, \text{ q.e. } x \in \mathcal{X}.$$

Assume that $\mathbf{M}^{(i)}$ is a recurrent symmetric β_i-stable process on $\mathbb{R}^{d_i}$ for each $i = 1, 2$. Hendricks [11] showed that for a constant p, the function

$$s(t) = t^{\frac{1}{\beta_1}} / (\log t)^{\frac{1+p}{\beta_1(\lambda-1)}}$$

is a lower rate function for $\mathbf{M}$ if and only if $p > 0$, where $\lambda = d_1/\beta_1 + d_2/\beta_2$. Since

$$r(t) = s(t)^{\frac{(\lambda-1)\beta_1}{\alpha_1+\alpha_2-\beta_2}} \quad \text{and} \quad (\lambda - 1)\beta_1/(\alpha_1 + \alpha_2 - \beta_2) \le 1,$$

our result is not sharp for $\beta_1 > \beta_2$.

5 Time Changed Processes

In this section we discuss the escape rate of time changed processes. Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form on $L^2(\mathcal{X}; m)$ and let $\mathbf{M} = (\{X_t\}_{t\ge0}, \{P_x\}_{x\in\mathcal{X}}, \zeta)$ be an m-symmetric Hunt process on $\mathcal{X}$ generated by $(\mathcal{E}, \mathcal{F})$. Here, $\zeta := \inf\{t > 0 \mid X_t = \Delta\}$ is the *life time*. Let μ be a positive Radon measure on $\mathcal{X}$ charging no set of zero capacity and let A_t be the positive continuous additive functional with Revuz measure μ (see, *e.g.*, [3] or [8] for definitions). Let $\mathcal{Y}$ be the topological support of μ and let $\widetilde{\mathcal{Y}}$ be the quasi support of μ:

(a) $\widetilde{\mathcal{Y}}$ is a quasi closed set such that $\mu(\mathcal{X} \setminus \widetilde{\mathcal{Y}}) = 0$;
(b) If $\check{\mathcal{Y}}$ is a quasi closed set with $\mu(\mathcal{X} \setminus \check{\mathcal{Y}}) = 0$, then $\widetilde{\mathcal{Y}} \subset \check{\mathcal{Y}}$ q.e.

We denote by $\check{\mathbf{M}} = (\{\check{X}_t\}_{t\ge0}, \{P_x\}_{x\in\widetilde{\mathcal{Y}}})$ the time changed process of $\mathbf{M}$ with respect to μ:

$$\check{X}_t = X_{\tau_t}, \quad \tau_t = \inf\{s > 0 \mid A_s > t\}.$$

Then $\check{\mathbf{M}}$ is a μ-symmetric Markov process on $\widetilde{\mathcal{Y}}$ and the associated Dirichlet form $(\check{\mathcal{E}}, \check{\mathcal{F}})$ on $L^2(\mathcal{Y}; \mu)$ is regular (see [8, Theorem 6.2.1] or [3, Section 5.2]). In particular, if μ has full quasi support, then

$$\check{\mathcal{F}} = \mathcal{F}_e \cap L^2(\mathcal{X}; \mu), \quad \check{\mathcal{E}}(u, u) = \mathcal{E}(u, u), \quad u \in \check{\mathcal{F}}.$$

We now assume that $\mathbf{M}$ is *conservative*: $P_x(\zeta = \infty) = 1$ for q.e. $x \in \mathcal{X}$. If $f(t)$ and $g(t)$ are strictly increasing functions on $(0, \infty)$ such that

$$P_x(f(t) \le A_t \le g(t) \text{ for all sufficiently large } t) = 1, \quad \text{q.e. } x \in \mathcal{X},$$

then

$$P_x(g^{-1}(t) \le \tau_t \le f^{-1}(t) \text{ for all sufficiently large } t) = 1, \quad \text{q.e. } x \in \mathcal{X}.$$

Hence if $R(t)$ is an *upper rate function* for $\mathbf{M}$, that is,

$$P_x(d(x, X_t) \le R(t) \text{ for all sufficiently large } t) = 1, \quad \text{q.e. } x \in \mathcal{X},$$

then P_x-a.s.,

$$d(x, \check{X}_t) = d(x, X_{\tau_t}) \le R(\tau_t) \le R(f^{-1}(t))$$

for all sufficiently large $t > 0$. This means that $R(f^{-1}(t))$ is an upper rate function for $\check{\mathbf{M}}$. In the same way, if $r(t)$ is a lower rate function for $\mathbf{M}$, then so is $r(g^{-1}(t))$ for $\check{\mathbf{M}}$. A similar argument as above was used in [14] to obtain upper rate functions for Markov chains on weighted graphs.

https://doi.org/10.4153/CJM-2015-014-x Published online by Cambridge University Press

Example 5.1 Let $\{a_{ij}\}$ be a family of symmetric measurable functions on $\mathbb{R}^d$ such that

$$\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \asymp |\xi|^2 \quad \text{for any } x \in \mathbb{R}^d \text{ and } \xi \in \mathbb{R}^d.$$

We denote by $C_0^\infty(\mathbb{R}^d)$ the totality of smooth functions on $\mathbb{R}^d$ with compact support. If we define

$$\mathcal{E}(u,v) = \int_{\mathbb{R}^d} \sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial u}{\partial x_i}(x)\frac{\partial v}{\partial x_j}(x)\,\mathrm{d}x$$

for $u, v \in C_0^\infty(\mathbb{R}^d)$, then $(\mathcal{E}, C_0^\infty(\mathbb{R}^d))$ is closable on $L^2(\mathbb{R}^d)$ (see [8, p. 111]) and the closure $(\mathcal{E},\mathcal{F})$ is a regular Dirichlet form on $L^2(\mathbb{R}^d)$.

Let $\mathbf{M} = (\{X_t\}_{t\geq 0}, \{P_x\}_{x\in\mathbb{R}^d})$ be a symmetric diffusion process on $\mathbb{R}^d$ generated by $(\mathcal{E},\mathcal{F})$. Let h be a positive measurable function on $\mathbb{R}^d$ such that $h(x) \asymp 1/(1+|x|^2)^p$ for some $p > 0$ and $\mu(\mathrm{d}x) = h(x)\,\mathrm{d}x$. Then

$$A_t = \int_0^t h(X_s)\,\mathrm{d}s \asymp \int_0^t \frac{1}{(1+|X_s|^2)^p}\,\mathrm{d}s.$$

Assume that $0 < p \leq 1$. Let $\check{\mathbf{M}}$ be a time changed process of $\mathbf{M}$ with respect to μ. Since we see by [25, Example 3.4] that $R_1(t) := c\sqrt{t\log t}$ is an upper rate function for $\mathbf{M}$, there exists $c' > 0$ such that

$$A_t \geq c' \int_0^t \frac{1}{(1+R_1(s)^2)^p}\,\mathrm{d}s$$

for all sufficiently large $t > 0$. Hence if we define

$$f(t) = c' \int_0^t \frac{1}{(1+R_1(s)^2)^p}\,\mathrm{d}s,$$

then $R_1(f^{-1}(t))$ is an upper rate function for $\check{\mathbf{M}}$. For all sufficiently large $t > 0$, noting that

$$f(t) \asymp \begin{cases} t^{1-p}/(\log t)^p, & 0 < p < 1, \\ \log\log t, & p = 1, \end{cases}$$

we get

$$f^{-1}(t) \asymp \begin{cases} t^{\frac{1}{1-p}}(\log t)^{\frac{p}{1-p}}, & 0 < p < 1, \\ \exp(\exp(ct)), & p = 1, \end{cases}$$

and thus

$$R_1(f^{-1}(t)) \asymp \begin{cases} t^{\frac{1}{2(1-p)}}(\log t)^{\frac{1}{2(1-p)}}, & 0 < p < 1, \\ \exp(\exp(ct)/2)\cdot\exp(ct/2), & p = 1. \end{cases}$$

We next consider lower rate functions for $\check{\mathbf{M}}$. Assume that $d \geq 3$. Then $\mathbf{M}$ is transient and Corollary 3.3 implies that for any $\varepsilon > 0$, the function $r(t) = \sqrt{t}/(\log t)^{\frac{1+\varepsilon}{d-2}}$ is a lower rate function for $\mathbf{M}$. We note that if the coefficients a_{ij} are smooth, then Ichihara [15, Theorem E] obtained the same lower rate function and further showed the sharpness. Hence, for some $c > 0$,

$$A_t \leq c \int_0^t \frac{1}{(1+r(s)^2)^p}\,\mathrm{d}s$$

for all sufficiently large $t > 0$. This shows that $r(g^{-1}(t))$ is a lower rate function for **M**, where

$$g(t) = c\int_0^t \frac{1}{(1+r(s)^2)^p}\,\mathrm{d}s.$$

For all sufficiently large $t > 0$, since

$$g(t) \asymp \begin{cases} t^{1-p}(\log t)^{\frac{2p(1+\varepsilon)}{d-2}}, & 0 < p < 1, \\ c(\log t)^{\frac{d+2\varepsilon}{d-2}}, & p = 1, \end{cases}$$

we obtain

$$g^{-1}(t) \asymp \begin{cases} t^{\frac{1}{1-p}}/(\log t)^{\frac{2p}{1-p}\cdot\frac{1+\varepsilon}{d-2}}, & 0 < p < 1, \\ \exp\big(ct^{\frac{d-2}{d+2\varepsilon}}\big), & p = 1, \end{cases}$$

which implies that

$$r(g^{-1}(t)) \asymp \begin{cases} t^{\frac{1}{2(1-p)}}/(\log t)^{\frac{1}{1-p}\cdot\frac{1+\varepsilon}{d-2}}, & 0 < p < 1, \\ \exp\big(ct^{\frac{d-2}{d+2\varepsilon}}/2\big)/t^{\frac{1+\varepsilon}{d+2\varepsilon}}, & p = 1. \end{cases}$$

Example 5.2 Let $c(x,y)$ be a positive measurable function on $\mathbb{R}^d \times \mathbb{R}^d$ such that $c(x,y) \asymp 1$. If we define

$$\mathcal{E}(u,v) = \iint_{\mathbb{R}^d\times\mathbb{R}^d\setminus\mathrm{diag}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{d+\alpha}}c(x,y)\,\mathrm{d}x\mathrm{d}y$$

for $u,v \in C_0^\infty(\mathbb{R}^d)$, then $(\mathcal{E}, C_0^\infty(\mathbb{R}^d))$ is closable on $L^2(\mathbb{R}^d)$ (see [8, p. 111]), and the closure $(\mathcal{E},\mathcal{F})$ is a regular Dirichlet form on $L^2(\mathbb{R}^d)$.

Let $\mathbf{M} = (\{X_t\}_{t\geq 0}, \{P_x\}_{x\in\mathbb{R}^d})$ be a symmetric Hunt process on $\mathbb{R}^d$ generated by $(\mathcal{E},\mathcal{F})$. This process is called a symmetric stable-like process as introduced by Chen and Kumagai [4]. As in Example 5.1, let h be a positive measurable function on $\mathbb{R}^d$ such that $h(x) \asymp 1/(1+|x|^2)^p$ for some $p > 0$ and $\mu(\mathrm{d}x) = h(x)\,\mathrm{d}x$.

Let $\check{\mathbf{M}}$ be a time changed process of **M** with respect to μ. As mentioned in [27], $R(t) := t^{\frac{1}{\alpha}}(\log t)^{\frac{1+\varepsilon}{\alpha}}$ is an upper rate function for **M** if $\varepsilon > 0$. For the symmetric α-stable process, this upper rate function is obtained by Khintchine [16]. Then for some $c > 0$,

$$A_t \geq c\int_0^t \frac{1}{(1+R(s)^2)^p}\,\mathrm{d}s$$

for all sufficiently large $t > 0$. By letting

$$f(t) = c\int_0^t \frac{1}{(1+R(s)^2)^p}\,\mathrm{d}s,$$

$R(f^{-1}(t))$ is an upper rate function for $\check{\mathbf{M}}$ under the condition that $0 < p < \alpha/2$. We then get $f(t) \asymp t^{\frac{\alpha-2p}{\alpha}}/(\log t)^{\frac{2p(1+\varepsilon)}{\alpha}}$, and thus $f^{-1}(t) \asymp t^{\frac{\alpha}{\alpha-2p}}(\log t)^{\frac{2p(1+\varepsilon)}{\alpha-2p}}$. This implies that for all sufficiently large $t > 0$,

$$R(f^{-1}(t)) \asymp t^{\frac{1}{\alpha-2p}}(\log t)^{\frac{1+\varepsilon}{\alpha-2p}}.$$

We next consider lower rate functions for $\check{\mathbf{M}}$. If we assume that $d > \alpha$, then **M** is transient and Corollary 3.3 implies that for any $\varepsilon > 0$, the function $r(t) =$

https://doi.org/10.4153/CJM-2015-014-x Published online by Cambridge University Press

$t^{\frac{1}{\alpha}}/(\log t)^{\frac{1+\varepsilon}{d-\alpha}}$ is a lower rate function for $\mathbf{M}$ (see Takeuchi [30] for symmetric stable processes). This yields that for some $c > 0$,

$$A_t \le c \int_0^t \frac{1}{(1+r(s)^2)^p}\,\mathrm{d}s$$

for all sufficiently large $t > 0$. Thus, if we define

$$g(t) = c \int_0^t \frac{1}{(1+r(s)^2)^p}\,\mathrm{d}s,$$

then $r(g^{-1}(t))$ is a lower rate function for $\check{\mathbf{M}}$. For all sufficiently large $t > 0$, noting that

$$g(t) \asymp \begin{cases} t^{\frac{\alpha-2p}{\alpha}}(\log t)^{\frac{2p(1+\varepsilon)}{d-\alpha}}, & 0 < p < \alpha/2, \\ (\log t)^{\frac{d+\alpha\varepsilon}{d-\alpha}}, & p = \alpha/2, \end{cases}$$

we have

$$g^{-1}(t) \asymp \begin{cases} t^{\frac{\alpha}{\alpha-2p}}/(\log t)^{\frac{2p\alpha}{\alpha-2p}\cdot\frac{1+\varepsilon}{d-\alpha}}, & 0 < p < \alpha/2, \\ \exp\big(ct^{\frac{d-\alpha}{d+\alpha\varepsilon}}\big), & p = 1, \end{cases}$$

which implies that

$$r(g^{-1}(t)) \asymp \begin{cases} t^{\frac{1}{\alpha-2p}}/(\log t)^{\frac{\alpha}{\alpha-2p}\cdot\frac{1+\varepsilon}{d-\alpha}}, & 0 < p < \alpha/2, \\ \exp\big(c't^{\frac{d-\alpha}{d+\alpha\varepsilon}}\big)/t^{\frac{1+\varepsilon}{d+\alpha\varepsilon}}, & p = \alpha/2. \end{cases}$$

This lower rate function is compatible with that for $\alpha = 2$ (see Example 5.1).

Acknowledgments The author would like to thank Professor Kaneharu Tsuchida for his careful reading of the manuscript and valuable comments. He is indebted to Professor Jian Wang for his interest in this work and drawing the author's attention to Remark 2.2. Thanks are also due to the referee for valuable suggestions on the draft.

References

[1] M. Barlow, R. Bass, Z.-Q. Chen, and M. Kassmann, *Non-local Dirichlet forms and symmetric jump processes,*. Trans. Amer. Math. Soc. **361**(2009), 1963–1999. http://dx.doi.org/10.1090/S0002-9947-08-04544-3

[2] A. Bendikov and L. Saloff-Coste, *On the regularity of sample paths of sub-elliptic diffusions on manifolds.* Osaka J. Math. **42**(2005), 677–722.

[3] Z.-Q. Chen and M. Fukushima, *Symmetric Markov processes, time change, and boundary theory.* London Mathematical Society Monographs Series, 35, Princeton University Press, Princeton, NJ, 2012.

[4] Z.-Q. Chen and T. Kumagai, *Heat kernel estimates for stable-like processes on d-sets.* Stochastic Process. Appl. **108**(2003), 27–62. http://dx.doi.org/10.1016/S0304-4149(03)00105-4

[5] ———, *Heat kernel estimates for jump processes of mixed types on metric measure spaces.* Probab. Theory Related Fields **140**(2008), 277–317. http://dx.doi.org/10.1007/s00440-007-0070-5

[6] A. Dvoretzky and P. Erdös, *Some problems on random walk in space.* Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, 1950, University of California Press, Berkeley and Los Angeles, 1951, pp. 353–367.

[7] M. Folz, *Volume growth and stochastic completeness of graphs.* Trans. Amer. Math. Soc. **366**(2014), 2089–2119. http://dx.doi.org/10.1090/S0002-9947-2013-05930-2

[8] M. Fukushima, Y. Oshima, and M. Takeda, *Dirichlet forms and symmetric Markov processes*. de Gruyter Studies in Mathematics, 19, Walter de Gruyter, Berlin, 2011.

[9] A. Grigor'yan, *Escape rate of Brownian motion on Riemannian manifolds.* Appl. Anal. 71(1999), 63–89.

[10] A. Grigor'yan, X. Huang, and J. Masamune, *On stochastic completeness of jump processes.* Math. Z. **27**(2012), 1211–1239. http://dx.doi.org/10.1007/s00209-011-0911-x
[11] W. J. Hendricks, *Lower envelopes near zero and infinity for processes with stable components.* Z. Wahrscheinlichkeitstheorie verw. Geb. **16**(1970), 261–278. http://dx.doi.org/10.1007/BF00535132
[12] X. Huang, *Escape rate of Markov chains on infinite graphs.* J. Theoret. Probab. **27**(2014), 634–682. http://dx.doi.org/10.1007/s10959-012-0456-x
[13] ———, *A note on the volume growth criterion for stochastic completeness of weighted graphs.* Potential Anal. **40**(2014), 117–142. http://dx.doi.org/10.1007/s11118-013-9342-0
[14] X. Huang and Y. Shiozawa, *Upper escape rate of Markov chains on weighted graphs.* Stochastic Process. Appl. **124**(2014), 317–347. http://dx.doi.org/10.1016/j.spa.2013.08.004
[15] K. Ichihara, *Some global properties of symmetric diffusion processes.* Publ. Res. Inst. Math. Sci. **14**(1978), 441–486. http://dx.doi.org/10.2977/prims/1195189073
[16] A. Khintchine, *Zwei Sätze über stochastische Prozesse mit stabilen Verteilungen.* Rec. Math. [Mat. Sbornik] N.S., **3**(1938), 577–584.
[17] D. Khoshnevisan, *Escape rates for Lévy processes.* Studia Sci. Math. Hungar. **33**(1997), 177–183.
[18] J. Masamune and T. Uemura, *Conservation property of symmetric jump processes.* Ann. Inst. Henri Poincaré Probab. Stat. **47**(2011), no. 3, 650–662. http://dx.doi.org/10.1214/09-AIHP368
[19] J. Masamune, T. Uemura, and J. Wang, *On the conservativeness and the recurrence of symmetric jump-diffusions.* J. Funct. Anal. **263**(2012), 3984–4008. http://dx.doi.org/10.1016/j.jfa.2012.09.014
[20] G. Metafune and C. Spina, *Heat kernel estimates for some elliptic operators with unbounded diffusion coefficients.* Discrete Contin. Dyn. Syst. **32**(2012), 2285–2299. http://dx.doi.org/10.3934/dcds.2012.32.2285
[21] H. Ôkura, *Recurrence criteria for skew products of symmetric Markov processes.* Forum Math. **1**(1989), 331–357.
[22] ———, *A new approach to the skew product of symmetric Markov processes.* Mem. Fac. Engrg. Design Kyoto Inst. Tech. Ser. Sci. Tech. **46**(1997), 1–12.
[23] ———, *Capacitary upper estimates for symmetric Dirichlet forms.* Potential Anal. **19**(2003), 211–235. http://dx.doi.org/10.1023/A:1024037212801
[24] H. Ôkura and T. Uemura, *On the recurrence of symmetric jump processes.* Forum Math., to appear.
[25] S. Ouyang, *Volume growth, comparison theorem and escape rate of diffusion process.* arxiv:1310.3996v1
[26] Y. Shiozawa, *Conservation property of symmetric jump-diffusion processes.* Forum Math. **27**(2015), 519–548.
[27] ———, *Escape rate of symmetric jump-diffusion processes.* Trans. Amer. Math. Soc., to appear.
[28] Y. Shiozawa and T. Uemura, *Explosion of jump-type symmetric Dirichlet forms on* $\mathbb{R}^d$. J. Theoret. Probab. **27**(2014), 404–432. http://dx.doi.org/10.1007/s10959-012-0424-5
[29] K.-T. Sturm, *Sharp estimates for capacities and applications to symmetric diffusions.* Probab. Theory Related Fields **103**(1995), 73–89. http://dx.doi.org/10.1007/BF01199032
[30] J. Takeuchi, *On the sample paths of the symmetric stable processes in spaces.* J. Math. Soc. Japan **16**(1964), 109–127. http://dx.doi.org/10.2969/jmsj/01620109

Graduate School of Natural Science and Technology, Department of Environmental and Mathematical Sciences, Okayama University, Okayama 700-8530, Japan
e-mail: shiozawa@ems.okayama-u.ac.jp