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ON GROUPS AND FIELDS DEFINABLE IN 1-H-MINIMAL FIELDS

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Abstract We show that an infinite group G definable in a 1-h-minimal field admits a strictly K-differentiable structure with respect to which G is a (weak) Lie group, and we show that definable local subgroups sharing the same Lie algebra have the same germ at the identity. We conclude that infinite fields definable in K are definably isomorphic to finite extensions of K and that 1-dimensional groups definable in K are finite-by-abelian-by-finite. Along the way, we develop the basic theory of definable weak K-manifolds and definable morphisms between them.

1. Introduction

Various Henselian valued fields are amenable to model theoretic study. Those include the p-adic numbers (more generally, p-adically closed fields) and (non-trivially) valued real closed and algebraically closed fields, as well as various expansions thereof (e.g., by restricted analytic functions). Recently, a new axiomatic framework for tame valued fields (of characteristic 0) was introduced. This framework, known as Hensel-minimality,[1] was suggested in [4] and [5] as a valued field analogue of o-minimality. The notion of 1-h-minimality is both broad and powerful. Known examples include, among others, all pure Henselian valued fields of characteristic 0 as well as their expansions by

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[1] In [4] and [5], various notions of Hensel-minimality – n-h-minimality – for $n \in \mathbb{N} \cup \{\omega\}$ were introduced. For the sake of clarity of exposition, we will only discuss 1-h-minimality.

restricted analytic functions. Known tameness consequences of 1-h-minimality include a well-behaved dimension theory and strong regularity of definable functions (e.g., a generic Taylor approximation theorem for definable functions).

In the present paper, we initiate a study of groups definable in 1-h-minimal fields. Using the above-mentioned tameness and regularity conditions provided by 1-h-minimality and inspired by similar studies in the o-minimal setting (initiated in [11]) and in p-adically closed fields ([12]), our first theorem (Proposition 6.4, stated here in a slightly weaker form) is as follows:

Theorem 1. *Let K be a 1-h-minimal field and G be an infinite group definable in K. Then G admits a definable weak $\mathcal{C}^k$ (any k) manifold structure with respect to which G has the structure of a strictly differentiable weak $\mathcal{C}^k$-Lie group. More precisely, the forgetful functor from definable strictly differentiable weak Lie groups to definable groups is an equivalence of categories. If algebraic closure coincides with definable closure in K, then a definable weak Lie group is a definable Lie group.*

Above, by a definable weak Lie group (over K), we mean a Lie group whose underlying K-manifold structure may not have a definable (so, in particular, finite) atlas but can be covered by (the domains of) finitely many compatible étale maps. We do not know whether this is a necessary requirement for the correctness of the statement, or an artifact of the proof: we follow Pillay's argument in the o-minimal and p-adic contexts ([11], [12]), but the fact that, in the present setting, finite covers are not generically trivial requires that we work with weakly definable manifolds, in the above sense. To pursue this argument, we have to extend the study of definable functions beyond what was done in [4] (and its sequel). Specifically, instead of working with continuously differentiable functions (as is the case in the o-minimal setting), we are working with strictly differentiable functions, and for those we prove an inverse function theorem, allowing us to deduce an implicit function theorem for definable functions as well as other standard consequences of these theorems. This allows us to develop the basic theory of the category of (weak) manifolds definable in this setting. Among others, we prove a definable version of Sard's Lemma.

We do not know whether strict differentiability follows in the 1-h-minimal context from continuous differentiability (as is the case in real analysis), but it can be easily inferred from a multi-variable Taylor approximation theorem for definable functions available in this context.

Having established that definable groups are Lie, our next theorem establishes the natural Lie correspondence (asserting that the germ of a definable group morphism at the identity is determined by its derivative at that point). For applications, it is convenient to state the result for local groups (Corollary 6.11):

Theorem 2. *Let K be a 1-h-minimal field, U and V definable strictly differentiable local Lie groups and $g, f : U \to V$ definable strictly differentiable local Lie group morphisms. If we denote $Z = \{x \in U : g(x) = f(x)\}$, then $\dim_e Z = \dim(\ker(f'(e) - g'(e)))$, where e denotes the identity of the local group U.*

We then prove two applications. First, we show – adapting techniques from the o-minimal context – that every infinite field definable in a 1-h-minimal field K is definably

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isomorphic to a finite extension of K, Proposition 7.3. This generalises an analogous result for real closed valued fields ([2]) and p-adically closed fields ([12]). It will be interesting to know whether these results can be extended to *interpretable* fields (in the spirit of [6] or [9, §6] under suitable additional assumptions on the RV-sort.

Our next application is a proof that definable 1-dimensional groups are finite-by-abelian-by-finite, Corollary 8.10. This generalises analogous results in the o-minimal context ([11]), in p-adically closed fields ([12]) and combines with [1] to give a complete classification of 1-dimensional groups definable in ACVF_0.

The present paper is a first step toward the study of groups definable in 1-h-minimal fields. It seems that more standard results on Lie groups over complete local fields can be extended to this context. Thus, for example, it can be shown that any definable local group contains a definable open subgroup. As the proof is long and involves new techniques, we postpone it to a subsequent paper.

1.1. Structure of the paper

In Section 2, we review the basics of 1-h-minimality and dimension theory in geometric structures. In Section 3, we prove a multi-variable Taylor approximation theorem for 1-h-minimal fields, and we formulate some strong regularity conditions (implied, generically, by Taylor's theorem) that will be needed in later parts of the paper. These results are probably known to the experts, and we include them mostly for the sake of completeness and clarity of exposition (as some of them do not seem to exist in writing).

In Section 4, we prove the inverse function theorem and related theorems on the local structure of immersions, submersions and constant rank functions. Though some proofs are similar to those of analogous statements in real analysis (and, more generally, in the o-minimal context), this is not true throughout. Specifically, 1-h-minimality is invoked in a crucial way in the proof that a function with vanishing derivative is locally constant, which, in turn, is used in our proof of the Lie correspondence for definable groups.

In Section 5, we introduce several versions of definable manifolds in 1-h-minimal fields, and we develop their basic theory.

Using the results of the first sections, our study of definable groups starts in Section 6. We first show that definable groups can be endowed with an essentially unique strictly differentiable weak Lie group structure and that the germ of definable group morphisms are determined by their derivative at the identity. We then define the (definable) Lie algebra associated with a definable Lie group, and show that it satisfies the familiar properties of Lie algebras. This is done using a local computation, after characterising the Lie bracket as the second order part of the commutator function near the identity.

Section 7 is dedicated to the classification of fields definable in 1-h-minimal fields, and in Section 8, we prove our results on definable one-dimensional groups.

2. Preliminaries

In this section, we describe some background definitions, notation and basic relevant results, used in later sections. Most of the terminology below is either standard or taken from [4]. Throughout, K will denote a non-trivially valued field. We will not distinguish notationally between the structure and its universe. Formally, we allow K to be a multi-sorted structure (with all sorts coming from K^{eq}), but by a definable set we mean, unless explicitly stated otherwise, a subset of K^n definable with parameters. All tuples are finite, and we write (as is common in model theory) $a \in K$ for $a \in K^n$, where $n = \text{length}(a)$. We apply the same convention to variables.

To stress the analogy of the current setting with the Real numbers, we use multiplicative notation for the valuation. Thus, the valued group is denoted $(\Gamma, \cdot)$ and the valuation $|\cdot| : K \to \Gamma_0 = \Gamma \cup \{0\}$, and if $x \in K^n$, we set $|x| := \max_{1 \leq k \leq n} |x_k|$.

An open ball of (valuative) radius $r \in \Gamma$ in K^n is a set of the form $B = \{x \in K^n : |x - a| < r\}$ for $a \in K^n$. The balls endow K with a field topology (the valuation topology). Up until Section 5, all topological notions mentioned in the text will refer solely to this topology and the product topology it induces on K^n.

We denote $\mathcal{O} := \{x : |x| \leq 1\}$, the valuation ring, $\mathcal{M} := \{x \in \mathcal{O} : |x| < 1\}$, the valuation ideal, and $k := \mathcal{O}/\mathcal{M}$, the residue field. We also denote $RV = K^\times/(1+\mathcal{M})$. More generally, whenever $s \in \Gamma$ and $s \leq 1$, we denote $\mathcal{M}_s = \{x \in K : |x| < s\}$, and $RV_s = K^\times/(1+\mathcal{M}_s)$. If K has mixed characteristic $(0,p)$, we denote $RV_{p,n} = RV_{|p|^n}$ and $RV_{p,\bullet} = \bigcup_n RV_{p,n}$.

It is convenient, when discussing approximation theorems, to adopt the big-O notation from real analysis. For the sake of clarity, we recall this notation in the valued field setting:

Definition 2.1. (1) If $f : U \to K^m$ and $g : U \to \Gamma_0$ are functions defined in an open neighborhood of 0 in K^n, then $f(x) = O(g(x))$ means that there are $r, M > 0$ in Γ, such that if $|x| < r$, then $|f(x)| \leq M g(x)$. We also denote $f_1(x) = f_2(x) + O(g(x))$ if $f_1(x) - f_2(x) = O(g(x))$.

(2) If $g : U \to K^r$, and $s \in \mathbb{N}$, then $O(g(x)^s) = O(|g(x)|^s)$.

(3) If $f : Y \times U \to K^m$ is a function where U is an open neighborhood of 0 in K^n, and if $g : U \to \Gamma_0$, then $f(y,x) = O_y(g(x))$ means that for every $y \in Y$, there are $r_y, M_y > 0$, such that if $|x| < r_y$, then $|f(y,x)| \leq M_y g(x)$.

As mentioned in the introduction, in the present paper, we are working with the notion of strict differentiability that we now recall:

Definition 2.2. Let $U \subset K^n$ be an open subset and $f : U \to K^m$ be a map. Then f is strictly differentiable at $a \in U$ if there is a linear map $A : K^n \to K^m$ such that for every $\epsilon > 0$, there exists $\delta > 0$ satisfying $|f(x) - f(y) - A(x - y)| \leq \epsilon |x - y|$ for every x, y such that $|x - a| < \delta$ and $|y - a| < \delta$.

f is strictly differentiable in U if it is strictly differentiable at every point of U.

In the situation of the definition, the linear map A is uniquely determined and denoted $f'(a)$. If f is strictly differentiable in an open U, then it is continuously differentiable.

Definition 2.3. Let $U \subset K^n$ and $V \subset K^n$ be open subsets. Then $f : U \to V$ is a strict diffeomorphism if it is strictly differentiable and bijective and if its inverse is strictly differentiable.

As we will see, a strict diffeomorphism is just a strictly differentiable diffeomorphism.

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Given an open ball $B \subseteq K^n$ of radius r, a subset Y of K^n, and an element $s \in \Gamma$ with $s \leq 1$, we say that B is s-away from Y if $B' \cap Y = \emptyset$ for B' the open ball of radius $s^{-1}r$ containing B. Note the closely related definition in [4] of a ball $B \subset K$ being s-next to a finite set $Y \subset K$. In that terminology, a ball B is s-next to Y if it is maximal among the balls s-away from Y.

Note that every point not in the closure of Y is contained in a ball s-away from Y. This is because if B is an open ball of radius r disjoint from Y, then every open ball of radius sr contained in B is s-away from Y.

Following [4], we say that a finite set $Y \subset K$ prepares the set $X \subset K$ if every ball B disjoint from Y is either disjoint from X or contained in X. More generally, if $s \in \Gamma$ is such that $s \leq 1$, then Y s-prepares X if every open ball B s-away from Y is either contained in X or disjoint from X.

If K is a valued field of mixed characteristic $(0,p)$, given an integer $m \in \mathbb{N}$, an open ball, $B \subseteq K^n$ and a set $Y \subseteq K^n$, we say that B is m-away from Y if it is $|p|^m$-away from Y. Similarly, if $s \in \Gamma$ and $s \leq 1$, then B is m-s-away from Y if it is $|p|^m s$-away from Y. Given a finite $Y \subset K$ and $X \subset K$, we say that Y m-prepares (resp. m-s-prepares) the set X if Y $|p|^m$-prepares X (resp. Y $|p|^m s$-prepares X).

Next, we recall the definitions of 1-h-minimality defined in the equi-characteristic 0 ([4]) and in the mixed characteristic ([5]) settings:

Definition 2.4. Let K be an $\aleph_0$-saturated non-trivially valued field of characteristic 0, which is a structure in a language expanding the language of valued fields.

(1) If K has residue characteristic 0, then K is 1-h-minimal, if for any $s \leq 1$ in Γ any $A \subseteq K$, $A' \in RV_s$ (a singleton) and every $(A \cup RV \cup A')$-definable set $X \subset K$, there is an A-definable finite set $Y \subset K$ s-preparing X.

(2) If K has mixed characteristic $(0,p)$, then K is 1-h-minimal, if for any $s \leq 1$ in Γ any $A \subseteq K$, $A' \in RV_s$ (a singleton) and every $(A \cup RV_{p,\bullet} \cup A')$-definable set $X \subset K$, there is $m \in \mathbb{N}$ and an A-definable finite set $Y \subset K$ which m-s-prepares X.

In the sequel, when appealing directly to the definition, we will only need the case $s = 1$ (so A' does not appear). The parameter s does appear implicitly, though, when applying properties of 1-h-minimality such as generic continuity of definable functions (see [4, Proposition 5.1.1]).

Below we will need to study properties of 'one-to-finite definable functions' (definable correspondences, in the terminology of [15]). It turns out that statements regarding such objects can sometimes be reduced to statements on definable functions in expansions of the language by algebraic Skolem functions (i.e., Skolem functions for definable finite sets). For this, the following will be convenient (see [4, Proposition 4.3.3] and [5, Proposition 3.2.2]):

Fact 2.5. Suppose K is a 1-h-minimal valued field. Then there exists a language $\mathcal{L}' \supseteq \mathcal{L}$, an elementary extension K' of K and an $\aleph_0$-saturated $\mathcal{L}'$-structure on K' extending the $\mathcal{L}$-structure of K', such that K' is 1-h-minimal as an $\mathcal{L}'$-structure, and such that $\mathrm{acl}_{\mathcal{L}'}(A) = \mathrm{dcl}_{\mathcal{L}'}(A)$ for all $A \subseteq K'$.

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Above and throughout, algebraic and definable closures are always assumed to be taken in the K sort. In the sequel, we will refer to the property appearing in the conclusion of Fact 2.5 simply as ‘$\mathrm{acl} = \mathrm{dcl}$’.

Remark 2.6. Given an $\mathcal{L}$-definable set S, statements concerning topological or geometric properties of S are often expressible by first-order $\mathcal{L}$-formulas. As the topology on K is definable in the valued field language, and the dimension of definable sets in 1-h-minimal structures is determined by the topology (see Proposition 2.11), the truth values of the hypothesis and conclusion of such statements (for our fixed $\mathcal{L}$-definable set S) are the same in K and in any elementary extension $K \prec K'$, as well as in any 1-h-minimal expansion of the latter. Therefore, by Fact 2.5, in the proof of such statements (for a fixed definable S), there is no harm assuming $\mathrm{acl} = \mathrm{dcl}$.

2.1. Geometric structures

We collect a few basic facts about geometric structures. Starting with Proposition 2.11, we apply them to the context of 1-h-mininal fields. Occasionally, we may state the hypothesis in the context of geometric or pregeometric structures when the proof is not simpler in the 1-h-minimal context, but the main application in this paper will always be to 1-h-minimal structures. Geometric structures were introduced in [8, §2]. Let us recall the definition: An $\aleph_0$-saturated structure, M is *pregeometric* if $\mathrm{acl}(\cdot)$ is a matroid; that is, it satisfies the exchange property:

$$\text{if } a \in \mathrm{acl}(Ab) \setminus \mathrm{acl}(A), \text{ then } b \in \mathrm{acl}(Aa) \text{ for singletons } a,b \in M.$$

In this situation, the matroid gives a notion of dimension, $\dim(a/b)$, the dimension of a tuple a over a tuple b, as the smallest length of a sub-tuple a' of a such that $a \in \mathrm{acl}(a'b)$, and the dimension of a b-definable set X as the maximum of the dimensions $\dim(a/b)$ with $a \in X$ (this does not depend on b). As is customary, we set $\dim(\emptyset) = -\infty$. We recall the basic properties of dimension (see [8, §2] for all references). This dimension satisfies the additivity property

$$\dim(ab/c) = \dim(a/bc) + \dim(b/c)$$

that we will invoke without further reference. We call a and b algebraically independent over c if $\dim(a/bc) = \dim(a/c)$. Note that by additivity of dimension, this is a symmetric relation. Note also that additivity implies that if b,c are inter-algebraic over a, meaning $b \in \mathrm{acl}(ac)$ and $c \in \mathrm{acl}(ab)$, then $\dim(b/a) = \dim(c/a)$ (in particular, this holds when c is the image of b under an a-definable bijection). If M is a pregeometric structure and $f : X \to Y$ is a surjective definable function with fibers of constant dimension k, then $\dim(X) = \dim(Y) + k$. This is a consequence of the additivity formula.

Given an a-definable set X, a generic element of X over a is an element $b \in X$ such that $\dim(b/a) = \dim(X)$. Generic elements can always be found in the model by using $\aleph_0$-saturation. We call $Y \subset X$ large if $\dim(X \setminus Y) < \dim(X)$. This is equivalent to Y containing every generic point of X.

A pregeometric structure M is called geometric if it eliminates the quantifier $\exists^\infty$. If M is geometric, then dimension is definable in definable families. Namely, for $\{X_a\}_{a\in S}$, a definable family, the set $\{a\in S : \dim(X_a)=k\}$ is definable.

The following simple fact is a translation of the definition of a pregeometry to a property of definable sets. Note as an aside that this reformulation implies that the property of being a pregeometry is preserved under reducts. That is, if M is an $\aleph_0$-saturated pregeometric $\mathcal{L}'$-structure, and $\mathcal{L}\subset\mathcal{L}'$, then M is also a pregeometric $\mathcal{L}$-structure. For the sake of completeness, we give the proof:

Fact 2.7. Suppose M is an $\aleph_0$-saturated structure. Then M is pregeometric if and only if for every definable $X\subset M\times M$, if the projection, $\pi_1 : X\to M$, into the first factor is finite-to-one, then the set $Y=\{c\in M : \pi_2^{-1}(c)\cap X \text{ is infinite}\}$ is finite, where π_2 is the projection into the second factor.

Proof. Suppose M is pregeometric and suppose $X\subset M\times M$ is A-definable such that $\pi_1^{-1}(x)\cap X$ is finite for all $x\in M$. Suppose also that $Y=\{y\in M : \pi_2^{-1}(y)\cap X \text{ is infinite}\}$ is infinite. By compactness and saturation, we can choose $b\in Y$ such that $\dim(b/A)=1$. Similarly, we can find $a\in\pi_2^{-1}(b)\cap X$ such that $\dim(a/Ab)=1$. We conclude that $\dim(ab/A)=2$, and so $\dim(X)\geq 2$. This contradicts the fact that $\pi_1^{-1}(x)\cap X$ is finite for all $x\in M$.

For the converse, suppose A is a finite subset of M and $a,b\in M$ are singletons such that $a\in\operatorname{acl}(Ab)\setminus\operatorname{acl}(A)$. Then there is an A-definable set $X\subset M\times M$ such that $(b,a)\in X$ and $\pi_1^{-1}(b)\cap X$ is finite, say of cardinality k. If we take $Z=\{c\in M : \pi_1^{-1}(c)\cap X \text{ has cardinality } k\}$, then we may replace X by $X\cap Z\times M$, and we may assume that $\pi_1^{-1}(c)\cap X$ is either empty or of constant finite cardinality for all $c\in M$. In this case, by the hypothesis, we conclude that $Y=\{y\in M : \pi_2^{-1}(y)\cap M \text{ is infinite}\}$ is finite. Note that Y is A-invariant and definable, so it is A-definable. We conclude that $a\notin Y$ because $a\notin\operatorname{acl}(A)$, and so $b\in\operatorname{acl}(Aa)$ as required. □

The next characterisation of the acl-dimension should be well known:

Fact 2.8. Suppose M is an $\aleph_0$-saturated structure, which eliminates the $\exists^\infty$ quantifier. Suppose there is a function, $X\mapsto d(X)$, from the nonempty definable subsets of (cartesian powers of) M into $\mathbb{N}$ satisfying the following:

(1) If $X\subset M^n\times M$ is such that the first coordinate projection $\pi_1 : X\to M^n$ is finite to one, then $d(X)=d(\pi_1(X))$.

(2) If $X\subset M^n\times M$ and $\pi_1 : X\to M^n$ is a projection, all of whose fibres are either empty or infinite, then $d(X)=d(\pi_1(X))+1$.

(3) If $\pi : M^n\to M^n$ is a coordinate permutation, then $d(X)=d(\pi(X))$.

(4) $d(X\cup Y)=\max\{d(X),d(Y)\}$.

(5) $d(M)=1$

(6) $d(X)=0$ if and only if X is finite.

Then M is a geometric structure and d coincides with its *acl*-dimension.

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Proof. It suffices to show that M is pregeometric. We use Fact 2.7. Let $X \subset M \times M$ be such that $\pi_1^{-1}(x) \cap X$ is finite for all $x \in M$. Take $Y = \{y \in M : \pi_2^{-1}(y) \cap X \text{ is infinite}\}$. Because M eliminates the $\exists^\infty$ quantifier, we have that Y is definable. If Y is infinite, we conclude that $d(X) \geq d(X \cap \pi_2^{-1}(Y)) = d(Y) + 1 = 2$, the first inequality by item (4), the second equality by items (3) and (2), and the third by item (6). However, $d(X) = d(\pi_1(X)) \leq d(M) = 1$, the first equality by item (1), the second inequality by item (4) and the third by item (5). This is a contradiction and finishes the proof.

To see that $d(X) = \dim(X)$ for $X \subset M^n$, we may proceed by induction on n. The base case $n = 1$ follows from item (4), (5) and (6). So suppose that $X \subset M^n \times M$. Denote $Y = \{x \in M^n : \pi_1^{-1}(x) \cap X \text{ is infinite}\}$. By hypothesis, Y is a definable set. Denote $X_1 = \pi_1^{-1}(Y) \cap X$ and $X_2 = X \setminus X_1$. Then by items (1), (2) and (4), we conclude that $d(X) = \max\{d(X_1), d(X_2)\} = \max\{d(Y)+1, d(\pi_1(X) \setminus Y)\}$. For the same reason, we have the formula $\dim(X) = \max\{\dim(Y)+1, \dim(\pi_1(X) \setminus Y)\}$, so $d(X) = \dim(X)$, as required. □

The next fact is also standard:

Fact 2.9. Suppose M is a geometric structure. Suppose $X \subset M^n$ is a-definable. Then there is a partition of X into a finite number of a-definable sets $X = X_1 \cup \cdots \cup X_n$, such that for each member of the partition X_k, there is a coordinate projection $\pi : X_k \to M^r$ which is finite to one and has image of dimension r.

Remark 2.10. For this statement, we need to allow the identity $\mathrm{id} : M^n \to M^n$ as well as the constant function $M^n \to M^0$ as coordinate projections. For the above, recall that M^0 is a set consisting of one element.

Proof. By induction on the dimension of the ambient space n. Consider the projection onto the first $n-1$ coordinates $\pi_1 : M^n \to M^{n-1}$. Then the set $Y \subset M^{n-1}$ of y such that the fibers $X_y = \pi_1^{-1}(y) \cap X$ are infinite is definable. So partitioning X, we may assume all the nonempty fibers of X over M^{n-1} are finite, or all are infinite. If all the fibers of $X \to K^{n-1}$ are finite then we finish by induction.

If all the nonempty fibers are infinite, then by the induction, there is a partition $Y = \bigcup_i Y_i$, and for each Y_i, there is a coordinate projection $\tau : Y_i \to K^r$ with finite fibers and $r = \dim(Y_i)$. Denote $\pi_2 : M^n \to M$ the projection onto the last coordinate. Then setting $X_i = X \cap \pi_1^{-1}(Y_i)$, the projection $\pi(x) = (\tau(\pi_1(x)), \pi_2(x))$ has the desired properties. □

The next proposition is key. It asserts that 1-h-minimal fields are geometric, and it connects (combined with the previous fact) topology and dimension in such structures:

Proposition 2.11. *Suppose K is a 1-h-minimal valued field. Then*

(1) *K is a geometric structure.*

(2) *Every definable $X \subset K^n$ satisfies $\dim(X) = n$ if and only if X has nonempty interior, and $\dim(X) < n$ if and only if X is nowhere dense.*

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(3) *For a definable* $X \subset K^n$*, we have* $\dim(X) = \max_{x \in X} \dim_x(X)$*, where we denote* $\dim_x(X)$*, the local dimension of* X *at* x*, defined as*

$$\dim_x(X) = \min\{\dim(B \cap X) : x \in B \textit{ is an open ball}\}.$$

Proof. This is essentially items (1)–(5) of [4, Proposition 5.3.4] in residue characteristic 0 and contained in [5, Proposition 3.1.1] in mixed characteristic.

For example, assume K has residue characteristic 0. That K is geometric is proved in the course of the proof of [4, Proposition 5.3.4]. We can also derive it from Fact 2.8 and [4, Proposition 5.3.4].

The topological characterisation asserting that $\dim(X) = n$ if and only if X has nonempty interior is a particular case of item (1) in [4, Proposition 5.3.4]. That $\dim(X) < n$ if and only if X is nowhere dense follows from this. Indeed, if $\dim(X) = n$, then X has nonempty interior and so it is somewhere dense. If $\dim(X) < n$ and $U \subset K^n$ is nonempty open, then $\dim(U \setminus X) = n$, and so $U \setminus X$ has nonempty interior. This implies X is nowhere dense.

That dimension is the maximum of the local dimensions is item (5) of Proposition 5.3.4 of [4] □

Proposition 2.12. *Suppose* K *is a 1-h-minimal field. Suppose* $f : U \to K^m$ *is a definable function. Then there is a definable open dense subset* $U' \subset U$ *such that* $f : U' \to K^m$ *is continuous.*

Proof. This is essentially a particular case of [4, Proposition 5.1.1] in residue characteristic 0 and contained in [5, Proposition 3.1.1] in mixed characteristic.

Indeed, because the intersection of open dense sets is open and dense, we reduce to the case $m = 1$. From those propositions, one gets that the set Z of points where f is continuous is dense in U. As Z is somewhere dense, we conclude using item (2) of Proposition 2.11 that $\dim(Z) = n$ and so Z has nonempty interior. If $V \subset U$ is a nonempty open definable subset, then, as $Z \cap V$ is the set of points at which $f|_V$ is continuous, by what we just proved, $Z \cap V$ has nonempty interior. We conclude that the set of points at which f is continuous has a dense interior in U, as desired. □

Next, we describe a topology for $Y^{[s]}$, the set of subsets of Y of cardinality s, for Y a Hausdorff topological space, and s a positive integer. We prove a slightly more general statement that will be applied when X is $Y^s \setminus \Delta$, the set of tuples of Y^s with distinct coordinates and the symmetric group, S_s, on s elements acting on Y^s by coordinate permutation, in which case the orbit space is identified with $Y^{[s]}$.

Fact 2.13. Suppose X is a Hausdorff topological space and G is a finite group acting on X by homeomorphisms, such that every $x \in X$ has a trivial stabiliser in G. Then X/G equipped with the quotient topology is Hausdorff and the map $p : X \to X/G$ is a closed finite covering map. In fact, for every $x \in X$, there is an open set $x \in U \subset X$ such that $\{gU : g \in G\}$ are pairwise disjoint, $p^{-1}p(U) = \bigcup_g gU$ and $p|_{gU}$ is a homeomorphism onto $p(U)$.

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Proof. We know that p is open since $p^{-1}p(U) = \bigcup_{g\in G} gU$ is open for U open. Consider the orbit $\{gx\}_{g\in G}$ of x. By assumption, if $g \neq h$, then $gx \neq hx$. Let V be an open set in X containing $\{gx\}_{g\in G}$. Now, because X is Hausdorff, there are U_g open neighborhoods of gx, contained in V, such that $U_g \cap U_h = \emptyset$ for $g \neq h$. If we take $U = \bigcap_{g\in G} g^{-1}U_g$, then $gU \subset U_g$ and so $\{gU : g \in G\}$ are pairwise disjoint. We conclude that p is closed and restricted to gU is a homeomorphism. That X/G is Hausdorff now follows from this. Indeed, if $p(x) \neq p(y)$, then there are open sets V_1 and V_2 of X, which are disjoint and such that $p^{-1}p(x) \subset V_1$ and $p^{-1}p(y) \subset V_2$. Because p is closed, there are open sets $p(x) \in U_1$ and $p(y) \in U_2$ in X/G such that $p^{-1}(U_i) \subset V_i$. We conclude that U_1 and U_2 are disjoint. □

With respect to this topology, we get the following:

Proposition 2.14. *Let K be a 1-h-minimal valued field. Suppose $U \subset K^n$ is open and $f : U \to (K^r)^{[s]}$ is definable. Then there is an open dense definable set $U' \subset U$ such that f is continuous in U'.*

Proof. This statement is equivalent to saying that the interior of the set of points on which f is continuous is dense. As this property is expressible by a first order formula, we may assume $\mathrm{acl} = \mathrm{dcl}$; see Fact 2.5 and the remark following it.

In that case, we have a definable section $\sigma : (K^r)^{[s]} \to K^{rs}$, and if $V \subseteq U$ is open dense such that σf is continuous, as provided by Proposition 2.12, then f is continuous in V. □

Proposition 2.15. *Suppose $X \subset K^n$ is b-definable. Then there is finite partition of X into b-definable sets, such that for each element Y of the partition, there is a coordinate projection $\pi : Y \to U$ onto an open set $U \subset K^m$, such that the fibers of π all have the same cardinality equal to s, and the associated map $f : U \to (K^{n-m})^{[s]}$ is continuous.*

Remark 2.16. As in Remark 2.10, we need to allow the two cases $m = 0$ and $m = n$. The set K^0 consists of a single point and has a unique topology.

Proof. This is a consequence of dimension theory and the previous observation. In more detail, we proceed by induction on the dimension of X.

First, recall that X has a finite partition into b-definable sets such that for each set X' in the partition, there is a coordinate projection $\pi : X' \to K^r$ with finite fibers and $r = \dim(X')$; see Fact 2.9.

So now assume $\pi : X \to K^r$ is a coordinate projection with finite fibers and $r = \dim(X)$, and denote $\pi' : X \to K^{n-r}$ the projection into the other coordinates. There is an integer s which bounds the cardinality of the fibers of π. If we denote Y_k the set of elements $a \in K^r$ such that $X_a = \pi'(\pi^{-1}(a))$ has cardinality k, then we get $Y_0 \cup \cdots \cup Y_s = K^r$. Now let $V_j \subset Y_j$ be open dense in the interior of Y_j and such that the map $V_j \to (K^{n-r})^{[j]}$ given by $a \mapsto X_a$ is continuous; see Proposition 2.14. Then the set $\{x \in X : \pi(x) \in Y_j \setminus V_j, 1 \leq j \leq s\}$ is of lower dimension than X, by item 2 of Proposition 2.11, and so we may apply the induction hypothesis on it. □

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Recall that a subset $Y \subset X$ of a topological space X is locally closed if it is the intersection of an open set and a closed set. This is equivalent to Y being relatively open in its closure. It is also equivalent to, for every point $y \in Y$, the existence of a neighborhood V of y, such that $Y \cap V$ is relatively closed in V.

Proposition 2.17. *Suppose K is 1-h-minimal and $X \subset K^n$ an a-definable set. Then X is a finite union of a-definable locally closed subsets of K^n.*

Proof. This is a consequence of Proposition 2.15. Namely, there is a partition of X into a finite union of a-definable subsets for each of which there is a coordinate projection with finite fibers onto an open set U, so we may assume X is of this form. We may further assume that the fibers have constant cardinality k and the associated mapping $U \to (K^r)^{[k]}$ is continuous. Then X is closed in $U \times K^r$ and so locally closed. □

We finish by reviewing a more difficult property of dimension. We will only use this in Proposition 5.20, Proposition 6.9 and Corollary 6.14, which are not used in the main theorems.

Proposition 2.18. *Suppose K is a 1-h-minimal field and $X \subset K^n$. Then* $\dim(\mathrm{cl}(X) \setminus X) < \dim(X)$.

This is item 6 of [4, Proposition 5.3.4] for the residue characteristic 0, and it is contained in Proposition 3.1.1 of [5] in the mixed characteristic case.

3. Taylor approximations

In this section, we show that, in the 1-h-minimal setting, the generic one variable Taylor approximation theorem ([5, Theorem 3.1.2]) implies a multi-variable version of the theorem. In equi-characteristic 0, this is [4, Theorem 5.6.1]. Although the proof in mixed characteristic is essentially similar, we give the details for the sake of completeness and in view of the importance of this result in the sequel.

We then proceed to introducing some regularity conditions for definable functions (implied in the present context by Taylor's approximation theorem) necessary for computations related to the Lie algebra of definable groups.

First, we recall the multi-index notation. If $i = (i_1, \dots, i_n) \in \mathbb{N}^n$, we denote $|i| = i_1 + \cdots + i_n$ and $i! = i_1! \cdots i_n!$. For $x = (x_1, \cdots, x_n) \in K^n$, we denote $x^i = x_1^{i_1} \cdots x_n^{i_n}$. Also, if $f : U \to K$ is a function defined in an open set of K^n, we denote $f^{(i)}(x) = (\frac{\partial^{i_1}}{\partial x_1^{i_1}} \cdots \frac{\partial^{i_n}}{\partial x_n^{i_n}} f)(x)$ whenever it exists. Note that we are not assuming equality of mixed derivatives, but see Corollary 3.6.

Proposition 3.1. *Let K be a 1-h-minimal field of residue characteristic* 0*. Suppose $f : U \to K$ is an a-definable function with $U \subset K^n$ open and let $r \in \mathbb{N}$. Then there is an a-definable set C, of dimension strictly smaller than n, such that for any open ball $B \subseteq U$ disjoint from C, the derivative $f^{(i)}$ exists in B for every i with $|i| \leq r$ and has constant valuation in B. Moreover,*

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$$\left| f(x) - \sum_{\{i:|i|<r\}} \frac{1}{i!} f^{(i)}(x_0)(x-x_0)^i \right| \le \max_{\{i:|i|=r\}} \left| \frac{1}{i!} f^{(i)}(x_0)(x-x_0)^i \right|$$

for every $x, x_0 \in B$.

This is [4, Theorem 5.6.1]. Our first order of business is to adapt this result to positive residue characteristic.

The following fact is proved by a standard compactness argument and is often applied implicitly. We add this argument for convenience.

Fact 3.2. Let M be an $\aleph_0$-saturated structure and $\{\Phi^l(\bar{D})\}_{l\in I}$ be a family of properties of definable sets $\bar{D} = (D_1, \dots, D_n)$ in M, indexed by a directed set I. Let b be a tuple in M and S a b-definable set. Assume that

(1) For all l, the property Φ^l is definable in definable families. That is, if for $i = 1, \dots, n$, we have $\{D_{i,a}\}_{a\in T}$ b-definable families, then the set $\{a \in T : \Phi^l(\bar{D}_a) \text{ holds}\}$ is b-definable.

(2) Φ^l implies $\Phi^{l'}$ for all $l \le l'$.

(3) For every $a \in S$, and for $i = 1, \dots, n$, there are ba-definable sets $D_{i,a}$, satisfying Φ^{l_a} for some $l_a \in I$.

Then for $i = 1, \dots, n$, there are $\{D_{i,a}\}_{a\in S}$ b-definable families of sets, and a fixed $l \in I$, such that $\Phi^l(\bar{D}_a)$ holds for every $a \in S$.

Remark 3.3. Formally, Φ^l is a subset of

$$\{(D_1, \dots, D_n) : D_i \text{ is a definable set}\},$$

and we say $\Phi^l(D_1, \dots, D_n)$ holds if the tuple $(D_1, \cdots, D_n)$ belongs to Φ^l.

Note also that the tuple $(D_1, \dots, D_n)$ can be replaced with $D_1 \times \cdots \times D_n$, so there is no loss of generality in taking Φ^l of the form $\Phi^l(D)$.

Proof. Note that by the previous remark, it suffices to prove the result for $n = 1$, as we shall presently proceed to do. Let $a \in S$. By hypothesis, there is a b-definable family $\{D^a_{a'}\}_{a'\in S^{0,a}}$ and an element $l^a \in I$, such that D^a_a satisfies Φ^{l^a}. Consider S^a to be the set of $a' \in S$ such that $a' \in S^{0,a}$ and such that $\Phi^{l^a}(D^a_{a'})$ holds. By hypothesis, this is a b-definable set contained in S and containing a.

We conclude that $S = \bigcup_{a\in S} S^a$ is a cover of S by b-definable sets, and so by compactness and saturation, there is a finite sub-cover, say $S = S^1 \cup \cdots \cup S^k$ for $S^r = S^{a_r}$. Indeed, if there was no finite sub-cover, then the partial type expressing $x \in S$ and $x \notin S^a$ for all $a \in S$ is a consistent b-type, and so a realization in M would contradict $S = \bigcup_{a\in S} S^a$.

Then D_a defined as $D^{a_r}_a$ if $a \in S^r \setminus \bigcup_{r'<r} S^{r'}$ satisfies that $\{D_a\}_{a\in S}$ forms a b-definable family. If we take l such that $l \ge l^{a_1}, \dots, l^{a_k}$, then we get that $\Phi^l(D_a)$ holds for every $a \in S$, as required. □

Notation 3.4. If $D \subset E \times F$, and $a \in E$, we often denote $D_a = \{b \in F : (a,b) \in D\}$. If $b \in F$, we denote, when no ambiguity can occur, $D_b = \{a \in E : (a,b) \in D\}$. If $f : D \to C$

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is a function, we let $f_a : D_a \to C$ denote the function $f_a(b) = f(a,b)$, and similarly, f_b for $b \in F$.

The positive residue characteristic versions of the multi-variable Taylor Theorem is as follows:

Proposition 3.5. *Let K be a 1-h-minimal field of positive residue characteristic, let $f : U \to K$ be an a-definable function with $U \subset K^n$ open, and let $r \in \mathbb{N}$. Then there is an integer m, and a set C, which is closed, a-definable and with $\dim(C) < n$, such that for every open ball, $B \subseteq U$ m-away from C, $f^{(i)}$ exists in B for every i with $|i| \le r$, and $f^{(i)}$ has constant valuation in B. Moreover,*

$$\left| f(x) - \sum_{\{i:|i|<r\}} \frac{1}{i!} f^{(i)}(x_0)(x-x_0)^i \right| \le \max_{\{i:|i|=r\}} \left| \frac{1}{i!} f^{(i)}(x_0)(x-x_0)^i \right|$$

for every $x, x_0 \in B$.

Proof. We proceed by induction on n, the case $n = 1$ being [5, Theorem 3.1.2]. Assume the result for n and let $f : U \to K$ be an a-definable function with $U \subset K^n \times K$ open, i a multi-index with $|i| \le r$. Then for every $x \in K^n$, there is a finite ax-definable set $C_x \subset K$ and an integer m_x such that

$$|f_x(y) - \sum_{s<r} \frac{1}{s!} f_x^{(s)}(y_0)(y-y_0)^s| \le |\frac{1}{r!} f_x^{(r)}(y_0)(y-y_0)^r| \tag{1}$$

for every y and y_0 in an open ball m_x-away from C_x, and such that $|f_x^{(s)}(y)|$ exists and is constant in any such open ball. By a standard compactness argument (see Fact 3.2), we may assume that the C_x are uniformly definable and that there is some $m \in \mathbb{N}$ such that $m_x = m$ for all x. Define $C = \bigcup_x (\{x\} \times C_x)$. By induction, for each $y \in K$, we can approximate the functions $g_{s,y}(x) = f_x^{(s)}(y)$ defined on $V_y = \mathrm{Int}(U_y \setminus C_y)$ up to order $r - s$. By a similar application of Fact 3.2, we obtain a natural number m' and an a-definable family $\{D_y\}_{y \in K^n}$ of subsets $D_y \subseteq V_y$ with $\dim(D_y) < n$ such that $g_{s,y}^{(i)}$ exists and has constant valuation on any ball m'-away from D_y in V_y, for every multi-index i, with $|i| \le r - s$. Moreover,

$$|g_{s,y}(x) - \sum_{\{i:|i|<r-s\}} \frac{1}{i!} g_{s,y}^{(i)}(x_0)(x-x_0)^i| \le \max_{\{i:|i|=r-s\}} |\frac{1}{i!} g_{s,y}^{(i)}(x_0)(x-x_0)^i|. \tag{2}$$

Replacing m and m' by their maximum, we may assume $m = m'$. Define $D := \bigcup_y D_y \times \{y\}$. By additivity of dimension, $\dim(C) \le n$ and $\dim(D) \le n$. Now take $E = C \cup D \cup \bigcup_y (U_y \setminus V_y) \times \{y\}$. Similar dimension considerations show that $\dim(E) < n + 1$.

Note that for $(x,y) \in U \setminus E$, we have that, for i and s such that $|i| + s \le r$, $f^{(i,s)}(x,y)$ and $g_{s,y}^{(i)}(x)$ exist and are equal.

Now, for $x \in K^n$, define $W_x = \mathrm{Int}(U_x \setminus E_x)$, and for the functions $h_{x,s,i} : y \mapsto f^{(i,s)}(x,y)$ with $s + |i| \le r$ defined on W_x, we find a finite set $F_x \subset W_x$ such that $\{F_x\}_x$ is an a-definable family, and there is an integer m', such that in every ball in W_x m'-away from F_x, $h_{x,s,i}$ has constant valuation. We may assume that $m' = m$ as before. Let G be the closure of $E \cup \bigcup_x (\{x\} \times F_x) \cup \bigcup_x (\{x\} \times (U_x \setminus W_x))$. Note that $\dim(G) < n+1$.

Take $B_1 \times B_2$ a ball in U, m-away from G. Then for every $x \in B_1$, we get that B_2 is m-away from both C_x and F_x and $B_2 \subseteq W_x$. Similarly, for every $y \in B_2$, $B_1 \subseteq V_y$ is m-away from D_y.

We conclude that for every $(x,y) \in B_1 \times B_2$, $f^{(i,s)}(x,y)$ exists and has constant valuation, for every index (i,s) such that $|(i,s)| \leq r$. Indeed, we have for every $(x,y),(x',y') \in B_1 \times B_2$ that

$$|f^{(i,s)}(x',y')| = |g^{(i)}_{s,y'}(x')| = |g^{(i)}_{s,y'}(x)| = |h_{x,s,i}(y')| = |h_{x,s,i}(y)| = |f^{(i,s)}(x,y)|,$$

as the second equality follows from the condition on $D_{y'}$, and the fourth from those on F_x and W_x.

Now, if (x,y) and (x_0,y_0) are in $B_1 \times B_2$, then equations 1 and 2 hold, and for the error term of 1, we have $|f^{(r)}_x(y_0)| = |f^{(0,r)}(x,y_0)| = |f^{(0,r)}(x_0,y_0)|$. Denote

$$M = \max\left\{ \frac{1}{i!s!} |f^{(i,s)}(x_0,y_0)(x-x_0)^i (y-y_0)^s| : |(i,s)| = r \right\}.$$

Then Equation 1 yields $|f(x,y) - \sum_{s<r} \frac{1}{s!} f^{(0,s)}(x,y_0)(y-y_0)^s| \leq M$. Also, from Equation 2, we have that

$$\left| \frac{1}{s!} f^{(0,s)}(x,y_0)(y-y_0)^s - \sum_{\{i:|i|<r-s\}} \frac{1}{i!s!} f^{(i,s)}(x_0,y_0)(x-x_0)^i (y-y_0)^s \right| \leq M.$$

Taking the sum over s smaller than r and using the ultrametric inequality, we obtain

$$\left| \sum_s \frac{1}{s!} f^{(0,s)}(x,y_0)(y-y_0)^s - \sum_{\{(i,s):|i|+s<r\}} \frac{1}{i!s!} f^{(i,s)}(x_0,y_0)(x-x_0)^i (y-y_0)^s \right| \leq M.$$

Summing this with Equation 1 and using the ultrametric inequality once more, we conclude. □

As a consequence of the previous theorem, we obtain that partial derivatives of definable functions commute generically.

Corollary 3.6. *Suppose $f : U \to K$ is a definable function for some open $U \subset K \times K$. Then there exists a dense open $U' \subset U$ such that for every $(x,y) \in U'$,*

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} f(x,y)\right) = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} f(x,y)\right),$$

and, in particular, the terms of the above equation exist in U'.

Moreover, if $f : U \to K$ is such that the partial derivatives $\frac{\partial}{\partial x}\frac{\partial}{\partial y} f(x,y)$, $\frac{\partial}{\partial y}\frac{\partial}{\partial x} f(x,y)$ exist and are continuous in U, then they are equal.

Proof. Take a 1-dimensional closed $C \subset K \times K$ and m an integer as provided by the Taylor approximation property for errors of order 3. We may also assume that $\pi(C)$ and

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m satisfy the same Taylor approximation property for the function $f\pi$, where π is the coordinate permutation $(x,y) \mapsto (y,x)$.

Then for $(x,y),(x_0,y_0) \in B_1 \times B_2$ in a ball m-away from C, we obtain (see Definition 2.1 for the big-O notation) that

$$f(x,y) = f(x_0,y_0) + (x-x_0)^2 \frac{1}{2} \frac{\partial^2}{\partial x^2} f(x_0,y_0) + (y-y_0)^2 \frac{1}{2} \frac{\partial^2}{\partial y^2} f(x_0,y_0) + (x-x_0)(y-y_0) \frac{\partial}{\partial x} \frac{\partial}{\partial y} f(x_0,y_0) + O((x-x_0,y-y_0)^3).$$

Similarly,

$$f\pi(y,x) = f(x,y) = f(x_0,y_0) + (x-x_0)^2 \frac{1}{2} \frac{\partial^2}{\partial x^2} f(x_0,y_0) + (y-y_0)^2 \frac{1}{2} \frac{\partial^2}{\partial y^2} f(x_0,y_0) + (x-x_0)(y-y_0) \frac{\partial}{\partial y} \frac{\partial}{\partial x} f(x_0,y_0) + O((x-x_0,y-y_0)^3).$$

Taking the difference, we obtain

$$(x-x_0)(y-y_0) \frac{\partial}{\partial y} \frac{\partial}{\partial x} f(x_0,y_0) - (x-x_0)(y-y_0) \frac{\partial}{\partial x} \frac{\partial}{\partial y} f(x_0,y_0) = O((x-x_0,y-y_0)^3).$$

Taking $h = (x-x_0) = (y-y_0)$ small, we get $h^2(\frac{\partial}{\partial y} \frac{\partial}{\partial x} f(x_0,y_0) - \frac{\partial}{\partial x} \frac{\partial}{\partial y} f(x_0,y_0)) = O(h^3)$, so $\frac{\partial}{\partial y} \frac{\partial}{\partial x} f(x_0,y_0) - \frac{\partial}{\partial x} \frac{\partial}{\partial y} f(x_0,y_0) = O(h)$. This is only possible when the left-hand side is 0, as desired.

The second statement follows from the assumed continuity, as two continuous functions into a Hausdorff space agreeing on a dense subset are equal. □

The following notation is intended to provide a cleaner expression for the Taylor approximation of multivariate functions.

Definition 3.7. Let m,n be positive integers, and $r \in \mathbb{N}$. Let $J = J(r,n) = \{j \in \mathbb{N}^n : |j| = r\}$. Let $a = (a_j)_{j \in J}$ be such that $a_j \in K^m$ for all $j \in J$. Then, for $x \in K^n$, we define $ax^r = \sum_{j \in J} a_j x^j$, where $x^j := \prod_{i=1}^n x_i^{j(i)}$. Note that $x \mapsto ax^r$ is a function $K^n \to K^m$.

As an example, consider, in the above notation, the case $r = 1$. In this case, $J = \{e_1,\dots,e_n\}$, and for $j \in J$, we have $x^j = x_j$ (where $x = (x_1,\dots,x_n)$), so for $a = (a_j)_{j \in J}$ with $a_j \in K^m$, we get that $ax = A \cdot x$, where A is the matrix whose j-th column is a_j.

In the same spirit, we have the following:

Definition 3.8. Let m,n,k be positive integers, and $r,s \in \mathbb{N}$. Let $J_1 = J(r,n)$ and $J_2 = J(s,m)$ be as in Definition 3.7. Let $a = (a_{j,j'})_{j \in J_1, j' \in J_2}$ be such that $a_{j,j'} \in K^k$. Then, for $x \in K^n$ and $y \in K^m$, we define $ax^r y^s = \sum_{j \in J_1, j' \in J_2} a_{j,j'} x^j y^{j'}$. Note that $(x,y) \mapsto ax^r y^s$ is a function $K^n \times K^m \to K^k$.

As an example, note that the functions $(x,y) \mapsto axy$ are exactly the bilinear functions $K^n \times K^m \to K^k$.

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Using the above notation, we introduce the following:

Definition 3.9. Let $U \subset K^k$ be open, $f : U \to K^m$ a function and $a \in U$. We say that f is P_n at a if it is approximable by polynomials of degree n near a in the following sense: there are constants $b_0, \cdots, b_n$ such that $f(a+x) = \sum_{r \le n} b_r x^r + O(x^{n+1})$.

We say f is P_n in U if it is P_n at every point of U.

In view of the example after Definition 3.7, it follows immediately from the definition that a P_1 function is differentiable, and for the coefficient b_1 in the definition, we may take $f'(a)$ (or, more precisely, $b_1^t = f'(a)$). Note also that a P_n function is also P_m for every $m \le n$.

Lemma 3.10. *If f is P_n at a, then for every number $i \le n$, the coefficients b_i in Definition 3.9 are determined by f.*

Proof. The problem readily reduces to the case of f a polynomial restricted to some open neighborhood, U, of the origin. That is, we have to show that if $\sum_{i \le n} b_i x^i = O(x^{n+1})$ in any open $U \subseteq K^r$, then $b_i = 0$ for all i. For x_0 fixed, let $x = tx_0$ and consider the single variable polynomial $P(tx_0) = \sum_{i \le n} (b_i x_0^i) t^i = O(t^{n+1})$. If we knew the result for $r = 1$, this would give that $b_i x_0^i = 0$. Since $x_0 \in U$ was arbitrary and U contains a cartesian product of r infinite sets, this implies $b_i = 0$ for all i. So we are reduced to proving the result for $r = 1$.

In this case, if i is the smallest with $b_i \neq 0$, we get $x^i = O(x^{i+1})$, which is a contradiction. □

The next technical definition is only used to obtain the Lie algebra of a definable Lie group; see Proposition 6.19. For the latter, Lemma 3.16 below is key.

We mention that this notion of T_n was selected to be first order expressible, stronger than P_n, and satisfying Propositions 3.12,3.13 and, crucially, Lemma 3.16. Readers willing to accept these results may treat the definition as a black box and skip to the next section.

Definition 3.11. Let $U \subset K^k$ be open, $f : U \to K^m$ a function and $a \in U$. We say f is T_n at a if there is $\gamma \in \Gamma$ such that for every x, x' with $|x-a|, |x'-a| < \gamma$, we have $f(x) = \sum_{r \le n} c_r(x')(x-x')^r + O(x-x')^{n+1}$ for a function c_r that is P_{n-r} at a.

We say f is T_n in U if it is T_n at every point of U.

Note that in the previous definition, the constant implicit in the notation $O(x-x')^{n+1}$ (see Definition 2.1) does not depend on x', so this definition requires some uniformity with respect to the center x' which is not implied by simply assuming f is P_n at every point of a ball around a.

Note also that if f is T_n at $c = (b,a)$, then, in particular, $f(z, a+x) = f(z) + f_1(z)x + \cdots + f_n(z)x^n + O(x^{n+1})$ for functions f_k that are P_{n-k} at b (and a constant in $O(x^{n+1})$ uniform in z). This follows from the definition by taking $x = (z, a+x)$ and $x' = (z,a)$.

Sums and products of P_n (resp. T_n) functions are P_n (resp. T_n), and a vector function is P_n (resp. T_n) if and only if its coordinate functions are P_n (resp. T_n).

Note also that a T_n function is T_m for every $m < n$, and a T_1 function at a is strictly differentiable at a (with $f'(a) = c_1(a)$).

We could also require the stronger condition, say ST_n, defined similarly to T_n, but requiring inductively the functions c_r be ST_{n-r} for $r = 1,\dots,n$ (the base case $ST_0 = T_0$). A possible advantage of ST_n is that it can be shown that ST_n functions are n-times strictly differentiable. We do not know whether this holds of T_n functions as well. Since all we need is the easy observation that T_1 functions are strictly differentiable, and T_2 is precisely what is needed to achieve the conclusion of Lemma 3.16, we opted to keep the simpler, though possibly less natural definition.

Proposition 3.12. *If g is P_n at a and f is P_n at $g(a)$ then the composition, $f \circ g$ is P_n at a.*

If g is T_n at a and f is T_n at $g(a)$, then $f \circ g$ is T_n at a.

Proof. For the first statement, we write

$$\begin{aligned}
&f(g(a+x)) = \\
&f(g(a)+g_1(a)x+\ldots+g_n(a)x^n+O(x^{n+1})) = \\
&f(g(a))+f_1(g(a))h(a,x)+f_2(g(a))h(a,x)^2+\ldots+O(h(a,x)^{n+1}) = \\
&fg(a)+b_1(a)x+\cdots+b_n(a)x^n+O(x^{n+1})+O(h(a,x)^{n+1}),
\end{aligned}$$

where

(1) $h(a,x) = g(a+x)-g(a) = g_1(a)x+\cdots+g_n(a)x^n+O(x^{n+1})$

(2) The second inequality is the application of the assumption that f is P_n at $g(a)$.

(3) The coefficients b_i arise by expanding the expression

$$f_k(a)h(a,x)^k = f_k(a)(g_1(a)x+\cdots+g_n(a)x^n+O(x^{n+1}))^k.$$

To conclude, we note that, as in the proof of Lemma 3.10, $h(a,x) = O(x)$, and so $O(h(a,x)^{n+1}) = O(x^{n+1})$.

The proof of the second statement is, essentially, similar:

$$\begin{aligned}
&f(g(x)) = \\
&f(g(x')+g_1(x')(x-x')+\ldots+g_n(x')(x-x')^n+O(x-x')^{n+1}) = \\
&f(g(x'))+f_1(g(x'))h(x,x')+f_2(g(x'))h(x,x')^2+\ldots+f_n(g(x'))h(x,x')^n+O(h(x,x')^{n+1}) = \\
&f(g(x'))+b_1(x')(x-x')+\cdots+b_n(x')(x-x')^n+O(x-x')^{n+1},
\end{aligned}$$

where $h(x,x') = g(x)-g(x') = g_1(x')(x-x')+\cdots+g_n(x')(x-x')^n+O(x-x')^{n+1} = O(x-x')$, and the coordinates of the coefficients $b_k(x')$ are sums and products of the coordinates of the coefficients of $f_i(g(x'))$ and $g_j(x')$ with $i,j \le k$. By what we have just proved, those are P_{n-i} functions. The constant appearing on $h(x,x') = O(x-x')$ does not depend on x' because the g_i are continuous at a. We conclude that the b_k are P_{n-k} at a, as claimed. □

Proposition 3.13. *Let K be 1-h-minimal and $f : U \to K^m$ as definable function. Then there exists $U' \subset U$ definable open and dense, such that f is T_n at every point of U'.*

In particular, for every $f : U \to K^m$, there is a definable open dense subset $U' \subset U$ such that f is strictly differentiable in U'.

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This follows from Taylor's approximation theorem (Proposition 3.1 in residue characteristic 0 and Proposition 3.5 in positive residue characteristic). The second statement follows because a T_1 function is strictly differentiable.

In the next section, we show that a strictly differentiable map with invertible derivative, definable in a 1-h-minimal valued field, is a local homeomorphism. Here, we show that the local inverse is strictly differentiable. We then proceed to showing that the properties P_n and T_n are also preserved in this local inverse, though this latter fact is not used for the proof of our main results.

Proposition 3.14. *Suppose $f: U \to V$ is a bijection where $U \subset K^n$ and $V \subset K^n$ are open. Suppose f satisfies $|f(x) - f(y) - (x-y)| < |x-y|$ for $x, y \in U$ distinct. Assume f is differentiable at a. Then $f'(a)$ is invertible, f^{-1} is differentiable at $b = f(a)$ and $(f^{-1})'(b) = f'(f^{-1}(b))^{-1}$.*

If f is strictly differentiable at a, then f^{-1} is strictly differentiable at b.

Proof. Note that the hypothesis implies $|f(x) - f(x')| = |x - x'|$. This implies that $f'(a)$ is invertible. Indeed, assume otherwise and take x close to a such that $f'(a)(x-a) = 0$ to get $|f(x) - f(a)| < |x-a|$, a contradiction.

Assume that f is strictly differentiable at a. Take $\epsilon > 0$ in Γ. Then there is $0 < r \in \Gamma$ such that if $|x-a|, |x'-a| < r$, then $|f(x) - f(x') - f'(a)(x-x')| \leq \epsilon|x-x'|$. If we denote $y = f(x)$ and $y' = f(x')$, then we have $|y - y'| = |x - x'|$, so multiplying the above inequality by $f'(a)^{-1}$, we obtain

$$\begin{aligned}
&|f^{-1}(y) - f^{-1}(y') - f'(a)^{-1}(y-y')| = \\
&|f'(a)^{-1}(f'(a)(x-x') - (f(x) - f(x'))| \leq \\
&|f'(a)^{-1}||f(x) - f(x') - f'(a)(x-x')| \leq \\
&\epsilon|f'(a)^{-1}||x-x'| = \epsilon|f'(a)^{-1}||y-y'|,
\end{aligned}$$

where, for a linear map, A, represented by the matrix $(a_{ij})_{i,j}$, we denote $|A| = \max_{i,j}|a_{ij}|$ and use the ultra-metric inequality to get $|Ax| \leq |A||x|$, which we apply to obtain the first inequality in the above computation.

So we conclude that $|f^{-1}(y) - f^{-1}(y') - f'(a)^{-1}(y-y')| \leq \epsilon|f'(a)|^{-1}|y-y'|$ for any y, y' such that $|y-b| = |x-a| < r$ and $|y'-b| = |x'-a| < r$. We have thus shown that f^{-1} is strictly differentiable at b and $(f^{-1})'(b) = f'(f^{-1}(b))^{-1}$.

To show that f^{-1} is differentiable if f is, substitute $x' = a$ in the above argument. □

Proposition 3.15. *Suppose $f: U \to V$ is a bijection where $U \subset K^n$ and $V \subset K^n$ are open. Suppose f satisfies $|f(x) - f(y) - (x-y)| < |x-y|$ for $x, y \in U$ distinct. Then if f is P_n (resp. T_n) at b, f^{-1} is P_n (resp. T_n) at $f(b)$.*

Proof. Denote $a = f(b)$. Note that the hypothesis implies then $|f(x) - f(y)| = |x-y|$ for all distinct $x, y \in U$, so the inverse map f^{-1} is continuous and in fact satisfies $|f^{-1}(x) - f^{-1}(y)| = |x-y|$ for distinct $x, y \in V$. In particular, f^{-1} is T_0 in V.

Now, assume that f is P_n at b, with $n \geq 1$. In particular, by Proposition 3.14, it is differentiable and $f'(b)$ is invertible.

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Apply the fact that f is P_n (and see also the discussion following the definition) to get

$$f(y)-f(b)=f'(b)(y-b)+f_2(b)(y-b)^2+\cdots+f_n(b)(y-b)^n+O(y-b)^{n+1}.$$

Rearranging, we get

$$y-b=f'(b)^{-1}(f(y)-f(b))-f'(b)^{-1}f_2(b)(y-b)^2-\cdots-f'(b)^{-1}f_n(b)(y-b)^n+O(y-b)^{n+1}.$$

Putting $y=f^{-1}(x)$, and remembering $x-a=O(y-b)$, we conclude

$$f^{-1}(x)-f^{-1}(a)=f'(f^{-1}(a))^{-1}(x-a)+\sum_{2\le i\le n}c_i(a)(f^{-1}(x)-f^{-1}(a))^i+O(x-a)^{n+1}, \tag{$\Diamond$}$$

for some constants $c_i(a)$.

Next, we proceed to showing (by induction on $k\le n$) that f^{-1} is P_k. As P_0 follows from the equality $|f^{-1}(x)-f^{-1}(y)|=|x-y|$, we assume that $k\ge 1$. So suppose f^{-1} is P_{k-1}. Using this, we can write $f^{-1}(x)-f^{-1}(a)=\sum_{1\le j<k}b_j(a)(x-a)^j+O(x-a)^k$ and apply a direct computation to obtain that

$$c_i(a)(f^{-1}(x)-f^{-1}(a))^i=\sum_{i\le j\le k}d_{ij}(a)(x-a)^j+O(x-a)^{k+1}$$

for some constants d_{ij}. Note that as $i\ge 2$, we obtain the improved error $O(x-a)^{k+1}$. Substituting this in ($\diamond$), it follows that f^{-1} is P^k, as required.

Now suppose f is T_n at b. The proof in this case is similar:

$$f^{-1}(x)-f^{-1}(x')=f'(f^{-1}(x'))^{-1}(x-x')+\sum_{2\le i\le n}c_i(f^{-1}(x'))(f^{-1}(x)-f^{-1}(x'))^i+O(x-x')^{n+1},$$

so, as above, if $f^{-1}(x)$ is T_{k-1}, we obtain

$$f^{-1}(x)-f^{-1}(x')=f'(f^{-1}(x'))^{-1}(x-x')+\sum_{2\le i\le k}d_i(x')(x-x')^i+O(x-x')^{k+1}.$$

Here, note that f' is P_{n-1} at b and so $f'(f^{-1}(x'))^{-1}$ is P_{k-1} at a. Also, following the above argument, we see that the coordinates of $d_i(x')$ are sums and products of functions of the form $b_{i'}(x')$ with $1\le i'<i$, for $b_{i'}(x')$ a $P_{k-1-i'}$ function at a (by the induction hypothesis that f^{-1} is T_{k-1}), and functions of the form $c_{i'}(f^{-1}(x'))$ for $c_{i'}(y')$ a $P_{k-i'}$ function at b, $i'\le i$ (by the assumption that f is T_n). So $d_i(x')$ is P_{k-i} at a. □

The next lemma will be important in our study of the differential structure of definable groups. For the statement, recall that O_x means that the constant implicit in the notation depends on x; see Definition 2.1. In the statement and proof, we are also using the notation axy introduced in Definition 3.8.

Lemma 3.16. *Let $f:U\times V\to K^r$ be a definable function, where $U\subset K^n$ and $V\subset K^m$ are open sets around 0. Suppose $f(x,y)$ is T_2 at $(0,0)$, and $f(x,y)=O(x,y)^3$. If $axy+f(x,y)=O_x(y^2)$, then $a=0$.*

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Proof. By the definition of T_2 (with $x = x'$, $y' = 0$), we get $f(x,y) = f_0(x) + f_1(x)y + f_2(x)y^2 + O(y^3)$ for f_2 a P_0 function at 0, f_1 a P_1 function at 0 and f_0 a P_2 function at 0. Fixing x and expanding the Taylor polynomial of $f(x,y)$, the uniqueness of Taylor coefficients (Lemma 3.10) gives, using our assumption, $axy + f_0(x) + f_1(x)y = 0$.

Expanding f_0, f_1, f_2 around 0 and keeping in mind $f(x,y) = O(x,y)^3$, we get $f_0(x) = O(x^3)$ and $f_1(x) = O(x^2)$. Indeed, we have $f_0(x) = b_0 + b_1x + b_2x^2 + O(x^3)$, $f_1(x) = c_0 + c_1x + O(x^2)$ and $f_2 = d_0 + O(x)$, so $f(x,y) = b_0 + b_1x + b_2x^2 + c_0y + c_1xy + d_0y^2 + O(x,y)^3 = O(x,y)^3$. So from the uniqueness of the Taylor coefficients, we get $b_0 = b_1 = b_2 = c_0 = c_1 = 0$.

Now from $axy = O(x,y)^3$, we get $a = 0$, by the uniqueness of Taylor coefficients again. □

4. Strictly differentiable definable maps

In this section, we prove an inverse function theorem for definable strictly differentiable maps in a 1-h-minimal valued field. This is done by adapting a standard argument from real analysis using Banach's fixed point theorem. In the present section, we use definable spherical completeness to obtain a definable version of Banach's fixed point theorem, implying, almost formally, the desired inverse function theorem. From the inverse function theorem, we deduce results on the local structure of immersions and submersions in the usual way. We then proceed to proving a generic version of the theorem on the local structure of functions of constant rank (Proposition 4.11). This last result is obtained only generically. The reason is that definable functions whose partial derivative with respect to a variable x is 0 on an open set. U need not be locally constant in x in U, as shown in Example 4.7 below. For that reason, we give a different argument for a weaker result (see Proposition 4.8) and the discussion preceding it.

Throughout the rest of this section, we fix an $\aleph_0$-saturated 1-h-minimal valued field K.

We start with a fixed point theorem, mentioned in [4, Remark 2.7.3]. We first note that a version of definable spherical completeness of 1-h-minimal fields ([4, Lemma 2.7.1]) holds in positive residue characteristic:

Lemma 4.1. *Suppose K has positive residue characteristic p. Suppose $\{B_i\}_i$ is a definable chain of open balls or a definable chain of closed balls. Suppose, further, that for every i, there is j such that $rad(B_j) \leq |p| rad(B_i)$. Then $\bigcap_i B_i \neq \emptyset$.*

Proof. The proof is similar to spherical completeness in residue characteristic 0; see Lemma 2.7.1 of [4]. It is enough consider the 1-dimensional case since the higher dimensional case follows by considering the coordinate projections. Note also that our assumption implies that the chain $\{B_i\}$ has no minimal element (as such an element would have valuative radius 0).

The closed case follows from the open case as follows: given a definable chain $\{B_i\}_{i \in I}$ of closed balls, for each i, let r_i be the valuative radius of B_i and let B_i' be the unique open ball $B \subseteq B_i$ of valuative radius r_i with the additional property that $B \supseteq B_j$ for all $j < i$. Obviously, $\bigcap_i B_i = \bigcap_i B_i'$ (unless the chain B_i has a minimal element, in which case there is nothing to prove).

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Note that, in the above notation, the map $B_i \mapsto r_i$ is injective, so there is no harm assuming that $\{B_i\}$ is indexed by a subset of Γ. Thus, our chain $\{B_i\}$ has index set interpretable in RV, so by [5, Proposition 2.3.2], there is a finite set C m-preparing the chain $\{B_i\}$ for some $m \in \mathbb{N}$. We claim that $C \cap B_i \neq \emptyset$ for all $i \in I$. This would finish the proof since C is finite. Assume, therefore, that this is not the case, and let $i_0 \in I$ be such that $B_{i_0} \cap C = \emptyset$. By assumption, we can find $i < i_0$ such that $r_i < |p^m| r_{i_0}$. Then B_i is a ball m-away from C, and since our chain has no minimal element, any ball $B \subsetneq B_i$ that is an element of our chain is not m-prepared by C, a contradiction. □

Note that by [3, Exanple 1.5], infinitely ramified 1-h-minimal fields of positive residue characteristic need not be definably spherically model complete. Thus, the extra condition in the assumption of the above lemma is not superfluous.

Proposition 4.2. *Let $B_r = \{x \in K^n : |x| \leq r\}$. Suppose $f : B_r \to B_r$ is a definable function. Assume that for distinct $x,y \in B_r$, we have*

(1) $|f(x) - f(y)| < |x - y|$ *if the residue characteristic is* 0.

(2) $|f(x) - f(y)| \leq |p||x - y|$ *if the residue characteristic is* $p > 0$.

Then f has a unique fixed point in B_r.

Proof. Uniqueness is immediate from the hypothesis. For existence, take the family of balls of the form $B(a)_{|f(a)-a|}$. It is a definable chain of balls indexed by $a \in B_r$. Indeed, if $a,b \in B_r$ are distinct and the balls are disjoint, then $|f(a) - f(b)| = |a - b|$, as the distance of points in disjoint balls does not change. Note that in positive residue characteristic, one has the additional hypothesis in Lemma 4.1 because $|f(f(a)) - f(a)| \leq |p||f(a) - a|$, by assumption 2 on f.

By the appropriate version of definable spherical completeness of 1-h-minimal fields (Lemma 2.7.1 of [4] for residue characteristic 0, Lemma 4.1 otherwise), we obtain a point x in the intersection of all balls. Then x is a fixed point of f. Indeed, if we assume otherwise, then for $y = f(x)$, we have $|f(y) - y| < |f(x) - x|$ by the hypothesis. However, if a is arbitrary then, as $x \in B(a)_{|f(a)-a|}$, one has $|f(x) - f(a)| \leq |x - a| \leq |f(a) - a|$ and so $|f(x) - x| \leq |f(a) - a|$. This is a contradiction and finishes the proof. □

Just as in real analysis, this fixed point theorem implies an inverse function theorem.

Proposition 4.3. *Suppose $f : U \to K^n$ is a definable function from an open set $U \subset K^n$ satisfying the following 'bilipschitz condition': for every distinct $x,y \in U$,*

(1) $|f(x) - f(y) - (x - y)| < |x - y|$ *if the residue characteristic is* 0.

(2) $|f(x) - f(y) - (x - y) \leq |p||x - y|$ *if the residue characteristic is* $p > 0$.

Then $f(U)$ is open and f is a homeomorphism from U to $f(U)$. If f is (strictly) differentiable, then f^{-1} is (strictly) differentiable.

Proof. Injectivity of the map follows directly from the hypothesis. The same assumptions also imply that if $x,y \in U$ are distinct, then $|f(x) - f(y)| = |x - y|$, implying continuity of the inverse.

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The main difficulty is showing that $f(U)$ is open. Translating, we may assume $0 \in U$ and $f(0) = 0$. We have to find an open neighborhood of 0 in $f(U)$. Take $r > 0$ such that $0 \in B_r \subset U$. Then $B_r \subset f(U)$. Indeed, if $|a| \leq r$, then, by the same reasoning as above, the function $g(x) = x + a - f(x)$ satisfies $g(B_r) \subset B_r$. By the assumptions on f, this implies that g satisfies the hypothesis of Proposition 4.2. So $g(x_0) = x_0$ for some x_0, namely, $a = f(x_0)$, as claimed.

Differentiability (and strict differentiability) of f^{-1} now follow from 3.14. □

We can finally formulate and prove the inverse function theorem for 1-h-minimal fields:

Proposition 4.4. *Suppose $f : U \to K^n$ is a definable function from an open set $U \subset K^n$. Suppose f is strictly differentiable at a and $f'(a)$ is invertible. Then there is an open set $V \subseteq U$ around a such that $f(V)$ is open and $f : V \to f(V)$ is a bijection whose inverse is strictly differentiable at $f(a)$.*

Proof. By the definition of strict differentiability, the function $f'(a)^{-1}f$ satisfies the hypothesis of the previous proposition in a small open ball around a. The conclusion follows. □

We do not know whether a definable function, $f : U \to K$, with continuous partial derivatives, but such that f is not strictly differentiable in U, could exist.[2] Clearly, sums, products and compositions of strictly differentiable functions are strictly differentiable, and so are locally analytic functions. Moreover, strict differentiability is first order definable and therefore extends to elementary extensions. Also, by the generic Taylor approximation theorem, in the 1-h-minimal context, any definable function in an open subset of K^n is strictly differentiable in a dense open subset. See Proposition 3.13.

Our next goal is to study definable functions of constant rank. We first note that without the assumption of definability, a strictly differentiable function whose derivative vanishes identically need not be locally constant:

Example 4.5. Consider a function $f : \mathcal{O} \to \mathcal{O}$ that is locally constant in $B \setminus \{0\}$ but near 0, it grows like x^2. Such a function f will be strictly differentiable, with $f' \equiv 0$, but f is not locally constant at 0.

Roughly, a function as in the above example involves an infinite number of choices, so it is not definable. In contrast, we have the following:

Proposition 4.6. *Let $f : U \to K^m$ be a function definable in an open set $U \subset K^n$. Assume f is continuous. Assume f is differentiable with derivative 0 on an open dense subset of U. Then f is locally constant with finite image.*

Proof. We proceed by induction on n, the dimension of the domain. We may assume $m = 1$.

[2]In real analysis, it is well known that a function $f : U \to \mathbb{R}$ is $\mathcal{C}^1$ in U if and only if it is strictly differentiable there.

First, assume $n = 1$. By the valuative Jacobian property in [4, Corollary 3.1.6] and [5, Corollary 3.1.3], there is a finite set $C \subset U$ such that in $U \setminus C$ the function f is locally constant. This implies that the fibers of $f|_{U\setminus C}$ are of dimension 1, and so the image of f is finite. As f is continuous, it is locally constant; the fibers form a finite partition of closed and so open sets on which f is constant.

Now assume the proposition is valid for n, and suppose $U \subset K^n \times K$. We denote $\pi_1 : K^n \times K \to K^n$ the projection onto the first factor and $\pi_2 : K^n \times K \to K$ the projection onto the second factor.

Let $V \subset U$ be an open dense subset such that f is differentiable at V with derivative 0. Denote $C = U \setminus V$. Let $T = \{x \in K^n \mid \dim(\pi_1^{-1}(x) \cap C) = 1\}$. Then $\dim(T) < n$. We conclude that there is an open dense set $W \subset K^n$ such that $\dim(\pi_1^{-1}(x) \cap C) = 0$ for all $x \in W$. Similarly, there is an open dense set $P \subset K$ such that $\dim(\pi_2^{-1}(x) \cap C) < n$ for every $x \in P$. Shrinking V to $V \cap \pi_1^{-1}(W) \cap \pi_2^{-1}(P)$, we may assume that if $(x,y) \in V$, then V_x is an open dense subset of U_x and V_y is an open dense subset of U_y. By the induction hypothesis and the $n = 1$ case, we conclude that f_x and f_y are locally constant. This implies that the fiber $f^{-1}f(x,y)$ has dimension $n+1$. Indeed, if B is an open neighborhood of y on which f_x is constant and for each $y' \in B$ we take $R_{y'} \subset U_x$ an open neighborhood of x on which $f_{y'}$ is constant, then f is constant on $\bigcup_{y' \in B} R_{y'} \times \{y'\}$. By dimension considerations, we conclude that the image $f(V)$ is finite. As f is continuous, $f^{-1}f(V)$ is closed in U, and as V is dense in U, we conclude $f(U) = f(V)$ is finite. As f is continuous, we conclude f is locally constant as before. □

Below, we let $D_y f(x,y)$ be the differential of the function f_x, given by $f_x(y) = f(x,y)$; we call this the derivative of f with respect to y (where y can be a tuple of variables).

Given the previous proposition, we may expect that a definable strictly differentiable function $f : U \to K^m$ with open domain $U \subset K^r \times K^s$ and satisfying that $D_y f(x,y) = 0$ is locally of the form $f(x,y) = g(x)$. Unfortunately, this is not true.

Example 4.7. Take $f : \mathcal{O} \times \mathcal{O} \to \mathcal{O}$ defined by $f(x,y) = 0$ if $|y| > |x|$ and $f(x,y) = x^2$ if $|y| \leq |x|$. Then f is strictly differentiable, $f(x,\cdot)$ is locally constant, but f is not of the form $g(x)$ near $(0,0)$.

It is due to this pathology that the conclusion of Proposition 4.11 below only holds generically.

Proposition 4.8. *Suppose $U \subset K^n$, $V \subset K^r$ are open and $f : U \times V \to K^m$ is a definable function such that f is continuous and $D_y f = 0$ on a dense open subset of $\mathrm{dom}(f)$. Then there exists an open dense set $U' \subset U$ such that $f|_{U' \times V}$ is locally of the form $g(x)$.*

Proof. The set D of points $x \in U$ such that for every point of $\{x\} \times V$ f is locally of the form $g(x)$ is definable. More precisely, $x \in D$ exactly when for all $y \in V$, there exists an open ball $B \ni (x,y)$, such that for all $(x',y'),(x',y'') \in B$, we have $f(x',y') = f(x',y'')$. Thus, the statement that D has dense interior in U is a first order expressible property, so we may assume that $\mathrm{acl} = \mathrm{dcl}$; see Fact 2.5 and the subsequent remark.

In the course of the proof, we may replace U by a dense open subset a finite number of times. Fix $W \subset U \times V$, a dense open set where f is differentiable and its derivative with

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respect to y is 0. Shrinking U, we may assume $W_x \subset V$ is dense, open for all $x \in U$. By Proposition 4.6, we know that f_x is locally constant with finite image for every $x \in U$ (recall that $f_x(y) := f(x,y)$). The sets $\mathrm{Im}(f_x)$ form a definable family of finite sets indexed by $x \in U$, so there is a uniform bound, n, on their cardinalities. Denoting $A_k = \{x \in U : |\mathrm{Im}(f_x)| = k\}$, we have $U = A_1 \cup \cdots \cup A_n$, so the union of the interiors of the A_k form a dense open subset of U. So we may assume that $|\mathrm{Im}(f_x)| = k$ for all x and some fixed k. Since we assumed that acl = dcl, there are definable functions $r_1, \ldots, r_k : U \to K^m$ such that $\{r_1(x), \ldots, r_k(x)\} = \mathrm{Im}(f_x)$. By generic continuity of definable functions (Proposition 2.12), we may assume that r_i are all continuous. Then the sets $B_i = \{(x,y) : f(x,y) = r_i(x)\}$ form a finite partition of $U \times V$ into closed, and so open subsets. □

The next two results, describing the local structure of definable maps of full rank, are standard applications of the inverse function theorem:

Proposition 4.9. *Suppose $U \subset K^k$ is a definable open set and $f : U \to K^k \times K^r$ is a definable, strictly differentiable map. Suppose that for some $a \in U$, the derivative $f'(a)$ has full rank. Then there is a ball $a \in B \subset U$, a ball $B_2 \ni 0$, a definable open set $V \subset K^k \times K^r$ and a definable strict diffeomorphism $\varphi : V \to B \times B_2$ such that $f(B) \subset V$ and the composition $\varphi f : B \to B \times B_2$ is the inclusion $b \mapsto (b,0)$.*

Proof. After a coordinate permutation in the target, we may assume the principal $k \times k$ minor of $f'(a)$ is invertible. Consider the function $g : U \times K^r \to K^k \times K^r$ defined as $g(x,y) = f(x) + (0,y)$. Then g is strictly differentiable and has invertible derivative at $(a,0)$ so by the inverse function theorem, Proposition 4.4, we can find a ball B around a and a ball B_2 around 0, and open set $f(a) \in V$ such that g restrict to a strict diffeomorphism $g : B \times B_2 \to V$. If $i : B \to B \times B_2$ is the inclusion $i(b) = (b,0)$, then we get that $gi = f$, so we conclude the statement is valid with $\varphi = g^{-1}$. □

Proposition 4.10. *Suppose $U \subset K^k \times K^r$ is a definable open set and $f : U \to K^k$ is a definable strictly differentiable map. Let $a \in U$. Suppose $f'(a)$ has full rank. Then there exists a definable open set $a \in U' \subset U$, a ball $f(a) \in B$, a ball $B_2 \subseteq K^r$ and a definable strict diffeomorphism $\varphi : B \times B_2 \to U'$, such that $f(U') \subset B$ and the composition $f\varphi : B \times B_2 \to B$ is the projection $(b,c) \mapsto b$.*

Proof. After applying a coordinate permutation to U, we may assume that the principal $k \times k$ minor of $f'(a)$ is invertible.

Consider the function $g : U \to K^k \times K^r$ defined as $g(x,y) = (f(x,y),y)$. Then g is strictly differentiable with invertible differential, so by the inverse function theorem, Proposition 4.4, there is an open set $a \in U' \subset U$ such that $g(U')$ is open and $g : U' \to g(U')$ is a strict diffeomorphism.

Shrinking U', we may assume $g(U') = B \times B_2$ is a product of two balls. Then if $p : B \times B_2 \to B$ is the projection $p(b,c) = b$, we get that $pg = f$, and so the statement is valid with $\varphi = g^{-1}$. □

We can finally prove our result on the local structure of definable functions of constant rank:

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Proposition 4.11. *Let $U \subset K^k \times K^r$ and $V \subset K^k \times K^s$ be open definable sets and let $f : U \to V$ be a definable strictly differentiable map such that for all $a \in U$, the rank of $f'(a)$ is constant equal to k. Then there exist $U' \subset U$ and $V' \subset V$ definable open sets, such that $f(U') \subset V'$, and there are definable strict diffeomorphisms $\varphi_1 : B_1 \times B_2 \to U'$ and $\varphi_2 : V' \to B_1 \times B_3$, such that the composition $\varphi_2 f \varphi_1 : B_1 \times B_2 \to B_1 \times B_3$ is the map $(a,b) \mapsto (a,0)$.*

Proof. Take a point $(b,c) \in U$. After a coordinate permutation in U and V, we may assume $f'(b,c)$ has its first $k \times k$ minor invertible. Then by the theorem on submersions, Proposition 4.10, applied to the composition of $f : U \to K^k \times K^s$ with the projection $K^k \times K^s \to K^k$ onto the first factor, we may assume (absorbing the diffeomorphism φ_1 provided by that proposition) that U is of the form $B_1 \times B_2$ and f is of the form $f(x,y) = (x,g(x,y))$. As f' has constant rank equal to k, we conclude that $D_y g = 0$. By Proposition 4.8, we may assume $g(x,y)$ is of the form $g(x,y) = g(x)$ (after passing, if needed, to smaller open sub-balls of B_1 and B_2, not necessarily containing (b,c)). Now the function $h : B_1 \to K^k \times K^s$ defined by $h(x) = (x,g(x))$ is a definable strictly differentiable immersion, so by the theorem on immersions 4.9, we may, after shrinking B_1 and composing with a definable diffeomorphism φ_2 in the target, assume that h is of the form $h(x) = (x,0)$. This finishes the proof. □

5. Strictly differentiable definable manifolds

In this section, we introduce several variants of definable manifolds in a 1-h-minimal field. Those are manifolds covered by a finite number of definable charts, with compatibility functions of various kinds.

Throughout, we keep the convention that K is an $\aleph_0$-saturated 1-h-minimal field. In case acl_K is not the same as dcl_K, it is better to take 'étale domains' instead of open subsets of K^n as the local model of the manifold. This is because the cell decomposition, as provided by Proposition 2.15, decomposes a definable set into a finite number of pieces, each of which is only a finite cover of an open set, instead of an open set. We describe this notion formally below:

Definition 5.1. Let $S \subset K^m$. A definable function $f : S \to K^n$ is (topologically) étale if it is a local homeomorphism. In other words, for every $x \in S$, there is a ball $B \ni x$ such that $f(B \cap S)$ is open and the inverse map $f(B \cap S) \to B \cap S \to K^m$ is continuous.

Informally, we think of étale maps as similar to open immersions and will denote such maps accordingly (e.g., $i : U \to K^n$). We now proceed to describing the differential structure of étale maps (or, rather, étale domains):

Definition 5.2. Suppose $i : U \to K^n$ and $j : V \to K^m$ are étale maps. A definable function $f : U \to V$ is strictly differentiable at $x \in U$ if there are balls $x \in B$ and $f(x) \in B'$ such that $i : B \cap U \to i(B \cap U)$, $j : B' \cap V \to j(B' \cap V)$ are homeomorphisms onto open sets, such that $f(B \cap U) \subset B' \cap V$, and the map $i(B \cap U) \xrightarrow{i^{-1}} B \cap U \xrightarrow{f} B' \cap V \xrightarrow{j} j(B' \cap V)$ is

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strictly differentiable at $i(x)$. In this case, the derivative $f'(x)$ is defined as the derivative of $i(B \cap U) \to j(B' \cap V)$.

The function $f: U \to V$ is T_k at x if the composition $i(B \cap U) \to j(B' \cap V)$ is T_k at $i(x)$.

Note that with this definition, the given inclusion $U \subset K^r$ is not necessarily strictly differentiable because the local inverses $i(U \cap B) \to K^r$ of the map $i: U \to K^n$ are only topological embeddings, so not necessarily strictly differentiable.

For the rest of this section, let $\mathcal{P}$ stand for any one of the following adjectives: topological, strictly differentiable or T_n.

Definition 5.3. A definable weak $\mathcal{P}$-n-manifold is a definable set, M, equipped with a finite number of definable injections, $\varphi_i: U_i \to M$, and each U_i comes equipped with an étale map $r_i: U_i \to K^n$. We require further that the sets $U_{ij} := \varphi_i^{-1}(\varphi_j(U_j))$ are open in U_i and that the transition maps $U_{ij} \to U_{ji}$, $\varphi_j^{-1}\varphi_i$ are $\mathcal{P}$-maps. We call the (U_i, φ_i) appearing in this definition the charts of M.

We further define the following:

(1) A definable weak $\mathcal{P}$-manifold is a weak $\mathcal{P}$-n-manifold for some n.

(2) A definable weak $\mathcal{P}$-manifold is equipped with a topology making the structure maps, φ_i, open immersions.

(3) A morphism of definable weak $\mathcal{P}$-manifolds is a definable function $f: M \to N$, such that for any charts $\varphi_i: U_i \to M$ and $\tau_j: V_j \to N$, the set $W_{ij} = \varphi_i f^{-1} \tau_j(V_j)$ is open in U_i and the map $W_{ij} \to V_j$ given by $x \mapsto \tau_j^{-1} f \varphi_i(x)$ is a $\mathcal{P}$-map.

(4) A definable $\mathcal{P}$-n-manifold is a definable weak $\mathcal{P}$-n-manifold, where the U_i are open subsets of K^n (and the maps $U_i \to K^n$ are inclusions).

(5) A morphism of definable $\mathcal{P}$-manifolds is a morphism of weak definable $\mathcal{P}$-manifolds.

The last two definitions introduce the categories of definable (weak) $\mathcal{P}$-n-manifolds, hence also the notion of isomorphism.

It is common to identify two manifold structures on the same set if the identity map on the underlying set is an isomorphism. We prefer not to do this, and simply treat them as isomorphic objects in the category. The resulting category is, of course, equivalent.

Definable weak manifolds are, immediately from the definition, (abstract) manifolds over K. As such, definable differentiable weak manifolds inherit the classical differential structure. For the sake of completeness, we recall the relevant definitions:

Definition 5.4. If M is a definable strictly differentiable weak manifold and $x \in M$, then the tangent space of M at x, $T_x(M)$ is $(\bigsqcup_i T_i)/E$, where the disjoint union is over the charts (U_i, φ_i) around x, $T_i = K^n$ (which we informally think of as the tangent space of $\varphi_i^{-1}(x)$ in U_i), and the equivalence relation E results from identifying $v \in T_i$ with $(\varphi_j^{-1}\varphi_i)'(\varphi_i^{-1}(x))(v) \in T_j$.

For a strictly differentiable definable morphism $f: M \to N$ of definable strictly differentiable weak manifolds, we have a map of K-vector spaces $f'(x): T_x(M) \to T_{f(x)}(N)$ given by the differential of the map appearing in Definition 5.3(3) above.

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As usual, once we have a chart around a point in a weak strictly differentiable manifold, we get an identification of $T_x(M)$ with K^n, but distinct charts may give distinct isomorphisms.

The map $(M,x) \mapsto T_x(M)$ from pointed definable weak $\mathcal{P}$-n-manifolds to K-vector spaces is a functor or, more precisely, it is the map on objects of a functor (with the obvious definition of the category of pointed weak $\mathcal{P}$-n-manifolds). In particular, two isomorphic objects (M,x) and (M',x') have isomorphic tangent spaces $T_x(M)$ and $T_{x'}(M')$.

Definition 5.5. A definable (weak) $\mathcal{P}$-Lie group is a group object in the category of definable (weak) $\mathcal{P}$-manifolds.

Lemma 5.6. *Suppose $i : U \to K^n$ and $j : V \to K^m$ are étale and $f : U \to V$ is a definable map. Then f is continuous in an open dense subset of U.*

Also, f is strictly differentiable, and T_k in an open dense subset of U.

Proof. For the statement about continuity, note that V has the subspace topology (of $V \subset K^r$), so we may assume $V = K^r$. If we denote U' the interior (relative to U) of the set of points of U where f is continuous, then in every ball B where i is a homeomorphism $i : B \cap U \to i(B \cap U)$, we get that $B \cap U'$ is dense in $B \cap U$, by generic continuity of definable functions. We conclude that U' is dense as required.

For strict differentiability and T_k, by the above, we may assume that f is continuous. Let U' be the interior of the set of all points where $f : U \to V$ is strictly differentiable and T_k. This is a definable open set. By generic differentiability and generic T_k property for functions defined on open sets, for every point $x \in U$, there is an open ball $B \ni x$, such that $B \cap U'$ is dense in $U \cap B$. Thus, we conclude that U' is dense in U. □

Note that the previous lemma implies that a (weak) definable topological manifold M contains an open dense subset $U \subset M$, which admits a structure of a (weak) definable T_n manifold extending the given (weak) definable topological manifold structure. As a consequence of Proposition 5.7 below, this structure on U is unique up to restriction to a dense open subset and isomorphism. More precisely, if τ_1 and τ_2 are two structures of T_n-manifold such that the identity map $i : (U,\tau_1) \to (U,\tau_2)$ is an isomorphism of weak definable topological manifolds, there exists $V \subset U$ open and dense such that the restriction $i : (V,\tau_1) \to (V,\tau_2)$ is an isomorphism of weak definable T_n-manifolds. For that reason, several of the statements below hold (essentially unaltered) for definable weak manifolds (without further assumptions on differentiability or T_n). For the sake of clarity of the exposition, we keep these assumptions.

Proposition 5.7. *If $f : M \to N$ is a definable function and M, N are definable weak $\mathcal{P}$-manifolds, then f is a $\mathcal{P}$-map in an open dense set of M.*

Proof. Considering the charts in M, we may assume $M = U \to K^n$ is étale. Now if (V_i,τ_i) are charts for N, then $f^{-1}\tau_i(V_i)$ cover U, and so the union of their interiors is open dense in U. So we may assume $N = V \to K^m$ is étale. This case is Lemma 5.6. □

Recall that the local dimension of a definable set X is defined as

$$\dim_x X = \min\{\dim(B \cap X) : x \in B \text{ is a definable open neighborhood of } x \text{ in } M\}.$$

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Note, in particular, that, in our definition of weak manifolds, the U_i are étale domains, so that for every $u \in U_i$ there is a definable local homeomorphism of U_i at u with an open subset of K^n. In particular, the local dimension of each U_i, and therefore also of M, is n at every point.

The next lemma is standard:

Lemma 5.8. *Suppose M is a definable topological weak manifold. Let $X \subset M$ be a definable subset. Then* $\dim(X) = \max_{x\in X} \dim_x(X)$.

If G is a definable weak topological group and H is a subgroup, then the dimension of H is the local dimension of H at any point.

Proof. If $M = U_1 \cup \cdots \cup U_n$ is a covering by open sets and $\varphi_i : U_i \to V_i$ is a homeomorphism onto a set V_i, with an étale map $V_i \to K^n$, then $\dim(X) = \max_i(\dim(\varphi_i(X \cap U_i)))$, and the local dimension of X at $x \in X \cap U_i$ is the local dimension of $\varphi_i(X \cap U_i)$ at $\varphi_i(x)$, so we reduce to the case $M = V$ is étale over K^n. In fact, the result is true whenever $M \subset K^m$ with the subspace topology, as then the local dimension of $X \subset M$ at a point x equals the local dimension of X at x in K^m, and so the result follows from Proposition 2.11(3).

If G is a definable weak topological group and H is a subgroup, then the local dimension of H at any point $h \in H$ is constant independent of h. Indeed, the left translation $L_h : G \to G$ is a definable homeomorphism that sends e to h and satisfies $L_h(H) = H$, so $\dim_e(H) = \dim_h(H)$. □

Proposition 5.9. *Suppose $T \subset K^m$ is such that there is a coordinate projection $\pi : T \to U$ onto an open subset $U \subset K^n$ and such that the fibres of π are finite of constant cardinality, s. Assume that the associated map $f : U \to (K^{m-n})^{[s]}$ is continuous. Then $T \to K^n$ is étale.*

Proof. Let $x \in T$. Replacing U by a smaller neighborhood around $\pi(x)$, we may assume, using Fact 2.13, that f lifts to a continuous function $g : U \to (K^{m-n})^s$, $g = (g_1, \cdots, g_s)$. In this case, one gets that T is homeomorphic to $\bigsqcup_{i=1}^{s} U$ over U, via the map $(a,i) \mapsto (a, g_i(a))$. □

Lemma 5.10. *Suppose $M = \bigcup_{i=1}^{r} \varphi_i(U_i)$, where $\varphi_i : U_i \to M$ are injective definable functions, such that the U_i are definable (weak) $\mathcal{P}$-n-manifolds. Suppose further that for all i,j, the sets $U_{ij} := \varphi_i^{-1}(\varphi_j(U_j))$ are open in U_i, and the transition maps $U_{ij} \to U_{ji}$ given by $x \mapsto \varphi_j^{-1}\varphi_i(x)$ are $\mathcal{P}$-maps. Then M has a unique (up to isomorphism) structure of a definable (weak) $\mathcal{P}$-n-manifold such that $\varphi_i : U_i \to M$ is an open immersion.*

The proof is straightforward and omitted.

Proposition 5.11. *Suppose M is a definable weak topological manifold. Then $X \subset M$ is large (i.e., $\dim(M \setminus X) < \dim(M)$) if and only if the interior of X in M is dense in M.*

Proof. Because the dimension of $M \setminus X$ is the maximum of the local dimension at its points by Lemma 5.8, we conclude that both conditions are local, and so we may assume $M = U \subset K^n$ is open. Here, the result follows from dimension theory. □

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Proposition 5.12. *Suppose M is a weak definable topological manifold and $X \subset M$ is definable. Then X is a finite union of locally closed definable subsets of M.*

Proof. There is an immediate reduction to the case where $M = U \to K^n$ is étale. In this case, U has the subspace topology $U \subset K^s$ for some s. So it is enough to prove this for $X \subset K^s$. This is a consequence of Proposition 2.17. □

In case $\mathrm{acl} = \mathrm{dcl}$, a weak manifold is generically a manifold:

Proposition 5.13. *Suppose* $\mathrm{acl} = \mathrm{dcl}$. *If M is a definable weak $\mathcal{P}$-manifold, then there is a definable open dense subset $U \subset M$ which is a definable $\mathcal{P}$-manifold.*

Proof. There is an immediate reduction to the case in which $i : M = U \to K^n$ is étale. Let r be a uniform bound on the cardinality of the fibers of i. Denote $X_s := \{x \in K^n : \#(i^{-1}(x)) = s\}$. Letting $U_s \subset X_k$ be the interior of X_s, we see that $\bigcup_{s \leq r} U_s$ is open, dense in K^n. Replacing U with $i^{-1}(U_s)$, we may assume that the nonempty fibers of i have constant cardinality. From the assumption that $\mathrm{acl} = \mathrm{dcl}$, we conclude that the map $i(U) \to (K^r)^{[s]}$ lifts to a definable map $i(U) \to (K^r)^s$. There is an open dense subset $V' \subset i(U)$ such that $V' \to i(U) \to (K^r)^s$ is a $\mathcal{P}$-map (see Proposition 5.6), and we conclude that $i^{-1}(V') \cong \bigsqcup_{i=1}^{s} V'$ over V', which is clearly a $\mathcal{P}$-manifold. □

It seems possible that in this situation, a weak manifold is already a manifold, but as this is not needed for the sequel, we have not looked into it.

The next couple of results are not used in the main theorems but may be of independent interest. We recall some standard definitions:

Definition 5.14. Suppose M and N are definable strictly differentiable weak manifolds, and $f : M \to N$ a definable strictly differentiable function.

Then f is called an immersion if the derivative $f'(x)$ is injective at all points $x \in M$.

f is called an embedding if f is an immersion and a homeomorphism onto its image.

f is called a submersion if the derivative $f'(x)$ is surjective for all $x \in M$.

These notions have the expected properties.

Proposition 5.15. *Suppose $f : M \to N$ is a strictly differentiable definable map of definable strictly differentiable weak manifolds. If f is an immersion, then M satisfies the following universal property: For every definable strictly differentiable weak manifold P, and $g : P \to M$, the function g is strictly differentiable and definable if and only if fg is strictly differentiable and g is definable and continuous.*

If f is an embedding and $g : P \to M$ is a function, then g is a strictly differentiable definable map if and only if fg is a strictly differentiable definable map.

Proposition 5.16. *If $f : M \to N$ is a surjective submersion, then a map $g : N \to K$ is a strictly differentiable definable function if and only if the composition gf is strictly differentiable and definable.*

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These two properties are a consequence of the theorems on the local structure of immersions and submersions, Propositions 4.9 and 4.10. We leave the details for the interested reader to fill.

Suppose M is a definable strictly differentiable weak manifold. If $M \to N$ is a surjective map of sets, it determines at most one structure of a definable strictly differentiable weak manifold on N in such a way that $M \to N$ is a submersion. Also, an injective map $N \to M$ determines at most one structure of a strictly differentiable weak manifold on N in such a way that $N \to M$ is an embedding. The subsets $N \subset M$ admitting such a structure are called submanifolds of M. We also get that if N is a definable topological space, and $N \to M$ is a definable and continuous function, then there is at most one structure of a strictly differentiable definable manifold on N extending the given topology (and the definable structure), and for which $N \to M$ is an immersion. In other words, the strictly differentiable weak manifold structure that makes $N \to M$ an embedding is determined by the set N, and the strictly differentiable weak manifold structure that makes $N \to M$ an immersion is determined by the topological space, and definable set N.

Proposition 5.17. *Suppose M,N are definable strictly differentiable weak manifolds, and let $f : M \to N$ be an injective definable map. Then there is a definable open dense subset $U \subset M$ such that $f|_U$ is an immersion.*

Proof. By Proposition 5.7 we may assume f is strictly differentiable. We have to show that the interior of the set $\{x \in M : f'(x) \text{ is injective}\}$ is dense in M. If this is not the case, we can find an open nonempty subset of M such that f is not an immersion at any point.

So suppose M is an open subset of K^n, N is an open subset of K^m and f is not an immersion at any point. For dimension reasons, $n \leq m$. If we define X_k to be the set of points x of M such that $f'(x)$ is of rank k, then $X_0 \cup \cdots \cup X_{n-1} = M$, and so if U_r is the interior of X_r we have that $U_1 \cup \cdots \cup U_{n-1}$ is open dense in M. So we may assume that f is of constant rank. This contradicts the result in Proposition 4.11 since the map $(x,y) \mapsto (x,0)$ is not injective. □

The following facts are standard and are probably known:

Fact 5.18. Suppose X is a Hausdorff space and $X \to Y$ is a surjective local homeomorphism with fibers of constant cardinality s. Then the map $t : Y \to X^{[s]}$ given by the fibers of p is continuous.

Proof. Let $\pi : X^s \setminus \Delta \to X^{[s]}$ be the canonical projection. Take $y \in Y$ and $\{x_1,\ldots,x_s\} = p^{-1}(y) = t(y)$. A basic open neighborhood of $t(y)$ is of the form $\pi(U_1 \times \cdots \times U_s)$ for $U_k \ni x_k$ open and U_k pairwise disjoint. Shrinking U_k, we may assume $p|_{U_k}$ is a homeomorphism onto an open set. If $V = \bigcap_k p(U_k)$, then $t^{-1}(V) \subset \pi(U_1 \times \cdots \times U_s)$. □

Fact 5.19. Let X,Y,Z be topological spaces, $p : X \to Z$, $q : Y \to Z$ be surjective continuous functions and $f : X \to Y$ be a continuous bijection such that $qf = p$.

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Assume that X and Y are Hausdorff spaces and $p: X \to Z$ has finite fibers of constant cardinality, s. If the map $t: Z \to X^{[s]}$ given by $z \mapsto p^{-1}(z)$ is continuous, then f is a homeomorphism.

Proof. Since f is continuous and bijective, we only need to show that it is open, which is a local property. Fix some $x \in X$ and $z = p(x)$. By Fact 2.13, the map $Z \to X^{[s]}$ lifts, locally near z, to a continuous map $(l_1,\dots,l_s): Z \to X^s$. Shrinking Z to this neighborhood, and reducing X and Y accordingly, we may assume that X is homeomorphic to $\bigsqcup_{i\le s} Z$ over Z, via the homeomorphism $F_l : \bigsqcup_{i\le s} Z \to X$, given by $(i,z) \mapsto l_i(z)$. To see that this is a homeomorphism, note that the image of the i-th-cofactor via F_l is the set $X_i = \{x \in X \mid x = l_i p(x)\}$, which is closed in X (because X is Hausdorff). Since there are only finitely many X_i (and they are pairwise disjoint), they are also open. Finally, the inverse of F_l restricted to X_i coincides with p which is continuous.

Similarly, we have a homeomorphism $F_{fl} : \bigsqcup_{i\le s} Z \to Y$, which is compatible with f in the sense that $fF_l = F_{fl}$. We conclude that f is a homeomorphism, as required. □

Proposition 5.20. *Let M,N be strictly differentiable weak manifolds, and $f : M \to N$ be an injective definable function. Then there is a definable dense open $U \subset M$ such that $f|_U$ is an embedding.*

Proof. By Proposition 5.17, we may assume that f is an immersion. If $V_1,\dots,V_n$ is a finite open cover of N and the statement is valid for $f : f^{-1}V_i \to V_i$, then it is also valid for f. So we may assume $N = V \to K^m$ is étale. From the definition, we have that $V \subset K^d$ has the subspace topology. Since $f : M \to V$ is an immersion, to verify that it is an embedding it suffices to show that it is a topological embedding into K^d. So we may assume $N = K^m$.

Now consider $U_1 \cup \dots \cup U_n = M$ a finite open cover of M. Assume, first, that $f|_{U_i}$ is an embedding. Define $U_i' = \mathrm{Int}(U_i \setminus \bigcup_{j<i} U_i)$, and $U_i'' = U_i' \setminus \bigcup_{j\ne i} f^{-1}\mathrm{cl}(f(U_j'))$. Note that $\bigcup_i U_i' \subset M$ is an open dense set. Also, note that $f(U_i' \setminus U_i'') = \bigcup_{j\ne i} f(U_i') \cap \mathrm{cl}(f(U_j')) \subset \bigcup_{j\ne i} \mathrm{cl}(f(U_j')) \setminus f(U_j')$. So we conclude that $\dim(U_i' \setminus U_i'') < \dim(M)$, by Proposition 2.18. Thus, replacing U_i with U_i'', we may assume $\mathrm{cl}(f(U_i)) \cap f(U_j) = \emptyset$ for distinct i,j. In this case, one verifies that f is a topological embedding.

We are thus reduced to the case where $M = U \to K^n$ is étale.

Consider for each $I \subset \{1,\dots,m\}$ of size n, the set A_I of $x \in U$ such that the I-th-minor of $f'(x)$ is invertible. As $U = \bigcup_I A_I$, we conclude that $U' = \bigcup_I \mathrm{Int}(A_I)$ is open dense in U, so by the reduction in the previous paragraph, we may assume that the composition of $f : U \to K^m$ with the projection onto the first n coordinates $p : K^m \to K^n$ is an étale immersion. If s is a uniform bound for the size of the fibers of U over K^n, then we can take A_k the set of $x \in K^m$ such that the fiber $(pf)^{-1}(x)$ has k elements and consider $U' = \bigcup_{k\le s} \mathrm{Int}(A_k)$. So we may assume that if $V = pf(U)$, the fibers of $pf : U \to V$ have the same size s.

In this case, the function $V \to U^{[s]}$, given by $x \mapsto (pf)^{-1}(x)$, is continuous, and the function f is a topological homeomorphism $U \to f(U)$; see facts 5.18 and 5.19. □

We can now prove a 1-h-minimal version of Sard's Lemma (compare with [16, Theorem 2.7] for an analogous result in the o-minimal setting). Namely, given a definable strictly

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differentiable morphism, f, of definable weak manifolds, call a value x of f regular when f is a submersion at every point of $f^{-1}(x)$. The statement is, then, that the set of singular values is small:

Proposition 5.21. *Suppose M and N are definable strictly differentiable weak manifolds. If $f : M \to N$ is a strictly differentiable map, then there exists an open dense subset $U \subset N$ such that $f : f^{-1}(U) \to U$ is a submersion.*

Proof. We have to see that the image via f of the set of points $x \in M$ such that $f'(x)$ is not surjective, and is nowhere dense in N. This property is expressible by a first order formula, so we may assume that acl = dcl; see Fact 2.5 and the subsequent remark.

Let $m = \dim(N)$. Let $X \subset M$ be a definable set such that for all $x \in X$, $f'(x)$ is not surjective. We have to see that $\dim f(X) < m$. We do this by induction on the dimension of X. The base case, when X is finite, is trivial.

The dimension of $f(X)$ is the maximum of the local dimensions at points; see Proposition 5.8. So we may assume $N \subset K^m$ is open. Covering M by a finite number of charts, we may assume $M \to K^n$ is étale, say $M \subset K^r$.

Then by Proposition 2.15, there exists a finite partition of X into definable sets such that if X' is an element of the partition, there exists a coordinate projection $p : K^r \to K^l$, which restricted to X' is a surjection $X' \to U$ onto an open subset $U \subset K^l$ with finite fibers of constant cardinality. If we prove that $\dim f(X') < m$ for every element X' of the partition, then also $\dim f(X) < m$. So we may assume there is a coordinate projection $p : X \to U$ onto an open subset $U \subset K^l$, such that $p^{-1}(u)$ has t elements for all $u \in U$. From the assumption acl = dcl, we get that there are definable sections $s_1, \dots, s_t : U \to X$, such that $\{s_1(u), \cdots, s_t(u)\} = p^{-1}(u)$, for all $u \in U$. As $X = \bigcup_{i \leq t} s_i(U)$, we may assume that $p : X \to U$ is a bijection with inverse $s : U \to X$. The map $s : U \to M$ becomes strictly differentiable in an open dense $V \subset U$; see Proposition 5.7. As $s(U \setminus V)$ has smaller dimension than X, we may assume that s is strictly differentiable. Now note that s has image in X, and so the composition $U \to M \to N$ has derivative which is not surjective at any point of U. So we have reduced to the case in which $M = U \subset K^n$ is open and $X = U$.

If we consider $A_k \subset U$, the set defined by $A_k = \{x \in U : f'(x) \text{ has rank } k\}$, then $\bigcup_{k<m} A_k = U$, and so $\bigcup_{k<m} \operatorname{Int}(A_k)$ is open and dense in U. We conclude by the induction hypothesis that the image of $U \setminus \bigcup_{k<m} \operatorname{Int}(A_k)$ is nowhere dense in N, and so we may assume that $f'(x)$ has constant rank k in U, for a $k < m$.

Consider the set $Y = \{x \in U : \dim f^{-1} f(x) \geq \dim(U) - k\}$. Then Y is definable because dimension is definable in definable families. Also, Y has dense interior by the constant rank theorem, Proposition 4.11. So once more by the induction hypothesis, we may assume $f^{-1} f(x)$ has dimension at least $\dim(U) - k$ for all $x \in U$. Then the dimension of $f(U)$ is at most k, by the additivity of dimension. □

If $M \to N$ is a map of strictly differentiable weak manifolds, and $y \in N$ is a regular value, then one can show $f^{-1}(y) \subset M$ is a strictly differentiable weak submanifold.

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6. Definable Lie groups

In this section, we show that every definable group is a definable weak Lie group and that the germ of a definable weak Lie group morphism is determined by its derivative at the identity.

The proof of the following lemma was communicated to us by Martin Hils.

Lemma 6.1. *Suppose G is a group a-definable in a pregeometric theory, and that $X,Y \subset G$ are nonempty a-definable sets of dimension smaller than* $\dim(G)$*. If $g \in G$ is such that* $\dim(gX \cap Y) = \dim(X)$*, then* $\dim(g/a) \leq \dim(Y)$*.*

In particular, there exists $g \in G$ such that $\dim(gX \cap Y) < \dim(X)$*.*

Proof. Denote $d = \dim(X)$ and $d' = \dim(Y)$. Suppose $\dim(gX \cap Y) = d$. Note that $d \leq d'$. Let $h' \in gX \cap Y$ be such that $\dim(h'/ag) = d$. Let $h = g^{-1}h'$.

As $h \in X$, we have $d \geq \dim(h/a) \geq \dim(h/ag) = \dim(h'/ag) = d$. The first inequality is because $h \in X$, the third one because h and h' are inter-definable over ag, and the fourth one by choice of h'. We conclude that h and g are algebraically independent over a.

Then we obtain that $d' \geq \dim(h'/a) \geq \dim(h'/ah) = \dim(g/ah) = \dim(g/a)$. The first inequality because $h' \in Y$, the third equality because h' and g are inter-definable over ah, and the fourth equality because h and g are algebraically independent over a.

For the second statement, note that if $g \in G$ is such that $\dim(g/a) = \dim(G)$, or more generally $\dim(g/a) > d'$, then by what we have just shown, $\dim(gX \cap Y) < \dim(Y)$. □

The next lemma generalises Lemma 2.4 of [11] for o-minimal theories. Pillay's proof can be seen to generalise, with some effort, to geometric theories. We give a different proof:

Lemma 6.2. *Suppose G is a group definable in a pregeometric theory and $X \subset G$ is large (i.e.,* $\dim(G \setminus X) < \dim(G)$*). Then a finite number of translates of X cover G.*

Proof. Suppose we have $g_0, \cdots, g_n \in G$ such that $\dim(G \setminus (\bigcup_k g_k X)) = m$. By Lemma 6.1 applied to $G \setminus X$ and $G \setminus (\bigcup_k g_k X)$, we get that there is $g_{n+1} \in G$ such that $\dim(G \setminus \bigcup_{k \leq (n+1)} g_k X) < m$, which – within at most m iterations – finishes the proof. □

Lemma 6.3. *Suppose G is a definable group in a pregeometric theory and $V \subset G$ is large. Then every $g \in G$ is a product of two elements in V.*

Proof. The proof of [11, Lemma 2.1] works: if we take $h \in G$ generic over g, then $h^{-1}g$ is also generic over g, and so $h, h^{-1}g \in V$ and their product is g. □

Proposition 6.4. *A definable group can be given the structure of a definable strictly differentiable weak T_k-Lie group. The forgetful functor from definable strictly differentiable weak Lie groups to definable groups is an equivalence of categories.*

If $\mathrm{acl} = \mathrm{dcl}$*, the forgetful functor from definable strictly differentiable T_k-Lie groups to definable groups is an equivalence of categories.*

Proof. That the forgetful functor is full follows from Proposition 5.7. Indeed, suppose G and H are strictly differentiable or T_k-Lie groups, and let $f : G \to H$ be a definable group morphism. Then by Proposition 5.7, there is an open dense $U \subset G$ such that $f : U \to H$

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is strictly differentiable or T_k. If $g_0 \in U$ is arbitrary, and $g \in G$, consider the formula $f = L_{f(g)f(g_0)^{-1}} f L_{g_0 g^{-1}}$, where we are denoting L_h the left translate by h. Now, $L_{g_0 g^{-1}}$ is a strict diffeomorphism or T_k-isomorphism which sends g to g_0, and $L_{f(g)f(g_0)^{-1}}$ is a strict diffeomorphism or T_k-isomorphism. We conclude that f being strictly differentiable or T_k at g_0 implies that f is strictly differentiable of T_k at g.

To see that the forgetful functor is essentially surjective, one follows the proof of [11, Proposition 2.5]. Namely, let G be of dimension n. Decompose G as in Proposition 2.15, and let $V_0 \subset G$ be the union of the n-dimensional pieces $U_0, \cdots, U_r$. Give V_0 the structure of a weak strictly differentiable manifold with charts the inclusions $U_i \to V_0$; see Proposition 5.9.

Note as an aside that this gives V_0 a topology which might be different from the subspace topology of $V_0 \subset G \subset K^s$. In what follows topological notions on V_0 refer to the manifold topology, not this subspace topology. These two topologies coincide in an open dense subset of V_0, by Proposition 5.20, but this is not needed for this proof.

Note that $V_0^{-1} \subset G$ is large in G, as the inverse function is a definable bijection, sending the large subset V_0 onto V_0^{-1}. As the intersection of two large sets is large, we conclude that $V_0 \cap V_0^{-1}$ is large in G. A fortiori, $V_0 \cap V_0^{-1}$ is large in V_0 and so it contains an open dense subset of V_0; see Proposition 5.11. Let $V_1 \subset V_0 \cap V_0^{-1}$ be open dense in V_0 such that the inverse function on V_1 (and into V_0) is strictly differentiable and T_k; see Proposition 5.7.

Similarly, we have that $V_0 \times V_0 \cap m^{-1}(V_0)$ is large in $G \times G$. Indeed, $m^{-1}(V_0)$ is the inverse image of the large subset $G \times V_0$ of $G \times G$ under the definable bijection $(\mathrm{Id}, m) : G \times G \to G \times G$. In the same way as before, we find $Y_0 \subset V_0 \times V_0 \cap m^{-1}(V_0)$ open and dense in $V_0 \times V_0$ such that the multiplication map $Y_0 \to V_0$ is strictly differentiable and T_k.

Now we take

$$V_1' = \{g \in V_1 : (h,g),(h^{-1},hg) \in Y_0 \text{ for all } h \text{ generic over } g\}.$$

Note that V_1' is definable because $g \in V_1'$ is equivalent to $\dim(G \setminus X_g) < n$ for $X_g = \{h \in G : (h,g),(h^{-1},hg) \in Y_0\}$, and dimension is definable in definable families in geometric theories. Note also that V_1' is large in G because if $g \in G$ is generic and $h \in G$ is generic over g, then (h,g) is generic in $G \times G$, and (h^{-1},hg), being the image of a definable bijection at (h,g), is also generic in $G \times G$, so they belong to Y_0 because Y_0 is large in $G \times G$.

Now take V_2 the interior of V_1' in V_0 and $V = V_2 \cap V_2^{-1}$. Then V_2 is large in V_0 by Proposition 5.11, and so it is also large in G. So we conclude that V is an open dense subset of V_0.

Define also $Y = \{(g,h) : g,h,gh \in V, (g,h) \in Y_0\}$. Then Y is open dense in $V_0 \times V_0$. This is because Y is large in $G \times G$, with arguments as above, and it is open in Y_0 because multiplication is continuous in Y_0.

Then we have shown the following:

(1) V is large in G.

(2) Y is a dense open subset of $V \times V$, and multiplication $Y \to V$ is strictly differentiable and T_k.

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(3) Inversion is a strictly differentiable T_k-map from V onto V.

(4) If $g \in V$ and $h \in G$ is generic in G over g, then $(h,g),(h^{-1},hg) \in Y$.

For the last item, note that $h,hg,h^{-1} \in G$ are generic, and so they belong to V. Also, because $g \in V_1'$, one has that $(h,g),(h^{-1},hg) \in Y_0$.

From this, one gets the following:

(a) For every $g,h \in G$, the set $Z = \{x \in V : gxh \in V\}$ is open, and $Z \to V$ given by $x \mapsto gxh$ is strictly differentiable and T_k.

(b) For every $g,h \in G$, the set $W = \{(x,y) \in V \times V : gxhy \in V\}$ is open in $V \times V$, and the map $W \to V$ given by $(x,y) \mapsto gxhy$ is strictly differentiable and T_k.

Indeed, for (a), assume $x_0 \in Z$. Take h_1 generic over h and k generic over g,x,h,h_1. Take $h_2 = h_1^{-1}h$. Note that $h_1,h_2 \in V$. Now one writes $f(x) = gxh$ as a composition of strictly differentiable and T_k functions defined on an open neighborhood of x_0 in the following way. Consider the set $Z_1 = \{x \in V : (kg,x) \in Y,(kgx,h_1) \in Y,(kgxh_1,h_2) \in Y,(k^{-1},kgxh) \in Y\}$. Then by item 2, we have that Z_1 is open, and the map $x \mapsto gxh = k^{-1}(((kgx)h_1)h_2)$ is a composition of strictly differentiable and T_k functions. Also, $x_0 \in Z_1$ by item 4.

Similarly, for (b), given $(x_0,y_0) \in W$, the set

$$W_1 = \{(x,y) \in V : (kg,x),(kgx,h_1),(kgxh_1,h_2),(kgxh,y),(k^{-1},kgxhy) \in Y\}$$

is open by item (3) and contains (x_0,y_0) by item (4). And in W_1, the required map is a composition of strictly differentiable and T_k functions.

By (1) above and Lemma 6.2, a finite number of translates, $g_0V,\dots,g_nV$, cover G. Consider the maps $\varphi_i : V \to G$ given by $\varphi_i(x) = g_ix$. It is straightforward to verify, using (a), (b) and (3) above, that these charts endow G with a (unique) structure of a strictly differentiable or T_k manifold, as in Lemma 5.10, and with this structure, G is a Lie group.

For example, to see that the transition maps are strictly differentiable or T_k, we have to see that the sets $V \cap g_i^{-1}g_jV$ are open, and the maps $\varphi_{i,j} : V \cap g_i^{-1}g_jV \to V \cap g_j^{-1}g_iV$ given by $x \mapsto g_j^{-1}g_ix$ are strictly differentiable or T_k. This is a particular case of (a). Similarly, (b) translates into the multiplication being strictly differentiable or T_k and (a) and (3) translate into the inversion being strictly differentiable and T_k.

When $\mathrm{acl} = \mathrm{dcl}$, an appropriate version of cell decomposition in Proposition 2.15 gives the result by repeating the above proof. Alternatively, we can see it directly from the result we have just proved and Proposition 5.13 and Lemma 6.2 (and the appropriate version of the Lemma 5.10). Indeed, if G is a definable group in a 1-h-minimal field with $\mathrm{acl} = \mathrm{dcl}$, then G has the structure of a weak strictly differentiable or T_k-Lie group. By Proposition 5.13, there is an open dense $U \subset G$ such that U is a strictly differentiable or T_k-manifold. By Proposition 5.11 and Lemma 6.2, a finite number of translates of U cover G, $g_1U \cup \cdots \cup g_nU = G$. Then the functions $\varphi_i : U \to G$ given by $x \mapsto g_ix$ form a gluing data for G which makes it a strictly differentiable or T_k-Lie group. □

As the previous result implies that every definable group G admits a structure of a definable weak Lie group which is unique up to a unique isomorphism, whenever we mention a property of the weak Lie group structure, we understand it with respect to this structure.

Definition 6.5. A definable strictly differentiable local Lie group is given by a definable open set containing a distinguished point $e \in U \subset K^n$, a definable open subset $e \in U_1 \subset U$, and definable strictly differentiable maps $U_1 \times U_1 \to U$ denoted as $(a,b) \mapsto a \cdot b$ and $U_1 \to U$ denoted as $a \mapsto a^{-1}$, such that there exists $e \in U_2 \subset U_1$ definable open such that

- $a \cdot e = e \cdot a = a$ for $a \in U_2$.
- If $a,b,c \in U_2$, then $a \cdot b \in U_1, b \cdot c \in U_1$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
- If $a \in U_2$, then $a^{-1} \in U_1$ and $a \cdot a^{-1} = a^{-1} \cdot a = e$.

Given two definable strictly differentiable local Lie groups, U and V, a definable strictly differentiable local Lie group morphism is given by a definable strictly differentiable map $f : U' \to V_1$ for a $e \in U' \subset U_1$ open, with U_1 and V_1 as in the above definition, and such that $f(e) = e$, $f(a \cdot b) = f(a) \cdot f(b)$ and $f(a^{-1}) = f(a)^{-1}$ for $a \in U'$. Also, two such maps f_1 and f_2 are identified as morphisms if they have the same germ around 0 – in other words, if there is a definable open neighborhood of the identity $W \subset \mathrm{dom}(f_1) \cap \mathrm{dom}(f_2)$ such that $f_1|_W = f_2|_W$.

It is common to only consider local groups where $e = 0$, and translating, we see that every local group is isomorphic to one with this condition. In this case, we denote the distinguished element by e whenever we emphasise its role as a local group identity.

We will usually identify a local group with its germ at e. In those terms, the prototypical example of a local Lie group is the germ around the identity of a Lie group.

The following fact is a well-known application of the chain rule. We give the short proof for completeness:

Fact 6.6. Suppose U is a local definable strictly differentiable Lie group. Then the multiplication map $m : U_1 \times U_1 \to U$ has derivative $m'(0)(u,v) = u + v$. The inverse $i : U_1 \to U$ has derivative $i'(0)(x) = -x$. Raising to power n, $p_n : U_n \to U$ has derivative $p_n'(0)(x) = nx$.

Proof. The formula for $m'(0)$ follows formally from the equations $m(x,0) = x, m(0,y) = y$.

In detail, $m'(0)(x,y) = ax + by$ for some matrices a and b. If we denote $s(x) = (x,0)$, we have that $ms = 1$, so applying the chain rule, we conclude that $m'(0)s'(0) = 1$, and as $s'(0) = (1,0)$, we conclude that $a = 1$. Similarly, we have $b = 1$.

From this, the formula for $i'(0)$ follows from $m(x,i(x)) = 0$ and the chain rule.

The formula for p_n follows inductively from the chain rule and $p_n(x) = m(p_{n-1}(x),x)$. □

We give some results on subgroups and quotient groups. These are not needed for the main applications.

Proposition 6.7. *Suppose $f : G \to H$ is a surjective definable group morphism. Then f is a submersion.*

Proof. This is a consequence of Sard's Lemma, Proposition 5.21. □

Fact 6.8. Suppose X is a topological space and $Y \subset X$ is a finite union of locally closed subsets of X. Then every open nonempty subset of X contains an open nonempty subset which is either disjoint from Y or contained in Y.

Proof. If Y_1, Y_2 are sets satisfying the claim, then so do $Y_1 \cup Y_2$, $Y_1 \cap Y_2$ and $Y_1 \setminus Y_2$. Since the claim is trivially true if Y is open, the conclusion follows. □

Proposition 6.9. *Suppose G is a definable group and $H \subset G$ is a definable subgroup. Then H is closed in G.*

Proof. Recall that H is a finite union of locally closed subsets of G; see, for instance, Proposition 5.12. So by applying Fact 6.8 to $H \subset \bar{H}$, we conclude that H has nonempty relative interior in $\bar{H}$. As H is a subgroup, so is $\hat{H}$ (this is true in any topological group), and we conclude by translation that H is open in $\bar{H}$. An open subgroup is the complement of some of its translates, so it is also closed. We conclude that $H = \bar{H}$ is closed. □

Proposition 6.10. *Suppose $H \subset G$ is a subgroup of G. Then with the structure of weak definable strictly differentiable manifolds on G and H, the inclusion $i : H \to G$ is a closed embedding.*

Proof. By Proposition 5.20, there is an open dense set $U \subset H$ such that $i|_U$ is an embedding. Replacing U by $U' = U \setminus i^{-1}(\mathrm{cl}(i(H) \setminus i(U)))$ if necessary, and keeping in mind Proposition 2.18 to show U' is large in H, we may assume $i(U)$ is open in $i(H)$. By translation, we conclude that i is an immersion. Also for an open set $V \subset H$, we have that $i(V) = i(\bigcup_{h \in H} hU \cap V) = \bigcup_h hi(U \cap h^{-1}V)$ is open in $i(H)$. Since i is injective, the conclusion follows. □

As a consequence of the theorem on constant rank functions, Proposition 4.11, we have the following result:

Corollary 6.11. *Suppose U and V are definable strictly differentiable local Lie groups, and let $g, f : U \to V$ be definable strictly differentiable local Lie group morphisms. If we denote $Z = \{x \in U : g(x) = f(x)\}$, then $\dim_e Z = \dim(\ker(f'(e) - g'(e)))$.*

In particular, if G and H are definable strictly differentiable weak Lie groups and g, f are definable strictly differentiable Lie group morphisms, then $\dim\{x : f(x) = g(x)\} = \dim(\ker(f'(e) - g'(e)))$.

Proof. The second result follows from the first because of Lemma 5.8.

To keep the proof readable, we only verify the first statement in the case of weak Lie groups, and in this case, we denote the groups G and H instead of U and V. The proof for local Lie groups is similar. By translating in G, we see that the map $f \cdot g^{-1} : G \to H$ has, at any point of G, derivatives of constant rank equal to $\dim(G) - k$, for $k = \dim(\ker(f'(e) - g'(e)))$. Indeed, if $u \in G$, then $(f \cdot g^{-1})L_u = L_{f(u)} R_{g(u)^{-1}} (f \cdot g^{-1})$, where L_u, R_u denote the left and right translations by u, respectively. By the chain rule, we get

$$(f \cdot g^{-1})'(u) L'_u(e) = (L_{f(u)} R_{g(u)^{-1}})'(f(u) \cdot g(u)^{-1})(f \cdot g^{-1})'(e).$$

As L_u and $L_{f(u)} R_{g(u)}$ are definable strict diffeomorphisms, their derivatives at any point are vector space isomorphisms, so we conclude that the rank of $(f \cdot g^{-1})'(u)$ equals the rank of $(f \cdot g^{-1})'(e) = f'(e) - g'(e)$ (see Fact 6.6), as desired.

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By the theorem on constant rank functions, Proposition 4.11, we conclude that there are nonempty open sets $U \subset G$ and $V \subset H$, balls B_1, B_2 and B_3 around the origin and definable strict diffeomorphisms $\varphi_1 : U \to B_1 \times B_2$ and $\varphi_2 : V \to B_1 \times B_3$, such that $f(U) \subset V$ and $\varphi_2(f \cdot g^{-1}) = \varphi_1 \alpha$ for $\alpha : B_1 \times B_2 \to B_1 \times B_3$ the function $(x,y) \mapsto (x,0)$. Note that the dimension of B_2 is equal to k. Translating in G, we may assume $e \in U$. More precisely, from the formula $(f \cdot g^{-1})L_u = (L_{f(u)} R_{g(u)^{-1}})(f \cdot g^{-1})$ discussed before, if $u \in U$ maps to $(0,0)$ under φ_1, then $e \in u^{-1}U$, so we may replace $(U, V, \varphi_1, \varphi_2)$ by $(u^{-1}U, f(u)^{-1}Vg(u), \varphi_1 L_u, \varphi_2 L_{f(u)} R^{-1}_{g(u)})$. Note also that $\varphi_1(e) = (0,0)$.

In this case, we obtain $\{0\} \times B_2 = \varphi_1(Z \cap U)$, so the local dimension of Z at e is the local dimension of $\{0\} \times B_2 \subset B_1 \times B_2$ at $(0,0)$, which is the dimension of B_2 and is as in the statement. □

In the particular case where, in the notation of the previous statement, $\dim(\ker(f'(e) - g'(e))) = \dim(G)$, we get the following:

Corollary 6.12. *Suppose U and V are definable strictly differentiable local Lie groups, and let $g, f : U \to V$ be definable strictly differentiable local Lie group morphisms. Then f and g are equal (as local Lie group morphisms) if and only if $f'(0) = g'(0)$.*

In particular, if G and H are definable strictly differentiable weak Lie groups and g, f are definable strictly differentiable Lie group morphisms, then f and g coincide in an open neighborhood of the identity e if and only if $f'(e) = g'(e)$.

The following two corollaries are not needed for the sequel, but may be interesting on their own right.

Corollary 6.13. *Suppose H_1 and H_2 are subgroups of the strictly differentiable definable weak Lie group G. Then $T_e(H_1 \cap H_2) = T_e(H_1) \cap T_e(H_2)$ as subspaces of $T_e(G)$.*

Proof. This is a consequence of Corollary 6.11. Indeed, we know H_1, H_2 and $H_1 \cap H_2$ are strictly differentiable definable weak Lie groups and the inclusion maps $H_1 \cap H_2 \to H_i$ and $H_i \to G$ are strictly differentiable immersions – for example, by Proposition 6.10 – so the statement makes sense. The diagonal map $\Delta : H_1 \cap H_2 \to H_1 \times H_2$ is the equaliser of the two projections $p_1 : H_1 \times H_2 \to G$ and $p_2 : H_1 \times H_2 \to G$. The kernel of $p_1'(e) - p_2'(e)$ is the image under the diagonal map of $T_e(H_1) \cap T_e(H_2)$. So by the equality of the dimensions in Corollary 6.11, we conclude $T_e(H_1 \cap H_2) = T_e(H_1) \cap T_e(H_2)$. □

Corollary 6.14. *If G is a definable strictly differentiable weak Lie group and H_1, H_2 are subgroups, then there is $U \subset G$ an open neighborhood of e such that $U \cap H_1 = U \cap H_2$ if and only if $T_e(H_1) = T_e(H_2)$.*

Proof. By Corollary 6.13, we get $T_e(H_3) = T_e(H_1) = T_e(H_2)$ for $H_3 = H_1 \cap H_2$. Then as the inclusion $H_3 \to H_1$ produces an isomorphism of tangent spaces at the identity, we conclude by the inverse function Theorem 4.4 that there is $U \subset G$ an open neighborhood of the identity, such that $U \cap H_1 = U \cap H_3$. Note that this also uses that the topology of

H_1 and H_3, which makes them strictly differentiable definable Lie groups, coincides with the subgroup topology coming from G; see Proposition 6.10.

Symmetrically, we have $U' \cap H_2 = U' \cap H_3$ for some open U'. □

Next, we give the familiar definition of the Lie bracket in $T_e(G)$ for the definable Lie group G, and we show it forms a Lie algebra. The exact same definition can also be applied to local Lie groups. For the sake of greater readability, we stick with the former setting.

Definition 6.15. Suppose G is a definable strictly differentiable weak Lie group. For $g \in G$, we consider the map $c_g : G \to G$ defined by $c_g(h) = ghg^{-1}$. Then c_g is a definable group morphism, and so it is strictly differentiable; see Proposition 5.7. So we have a map $\mathrm{Ad} : G \to \mathrm{Aut}_K(T_e(G))$, given by $g \mapsto c_g'(e)$. This is a definable map and, by the chain rule, a group morphism (for this recall that $c_g c_h = c_{gh}$). Therefore, Ad is strictly differentiable, and so its derivative at e gives a linear map $\mathrm{ad} : T_e(G) \to \mathrm{End}_K(T_e(G))$. In other words, this gives a bilinear map $(x,y) \mapsto \mathrm{ad}(x)(y)$, $T_e(G) \times T_e(G) \to T_e(G)$ denoted $(x,y) \mapsto [x,y]$.

This map is called the Lie bracket.

Proposition 6.16. *Let G be a definable weak T_2-Lie group. Let $0 \in U \subset K^n$ be an open set and $i : U \to G$ a T_2-diffeomorphism of U onto an open subset of G that sends 0 to e. Make U into a local definable group via i. Then under the identification $i'(0) : K^n \to T_e(G)$, the Lie bracket is characterised by the property $x \cdot y \cdot x^{-1} \cdot y^{-1} = [x,y] + O(x,y)^3$, for $x,y \in U$.*

Proof. We use the notation of Definitions 3.7 and 3.8. The function $f(x,y) = x \cdot y \cdot x^{-1}$ satisfies $f(0,y) = y$ and $f(x,0) = 0$, so its Taylor approximation of order 2 is of the form $f(x,y) = y + axy + O(x,y)^3$. Indeed, it is of the form $a_0 + a_1x + a_2y + a_3x^2 + a_4xy + a_5y^2 + O(x,y)^3$, and plugging $x = 0$ and using the uniqueness of the Taylor approximation, we get $a_0 = a_5 = 0$ and $a_2 = 1$, and a similar argument with $y = 0$ gives $a_1 = a_3 = 0$, so $f(x,y) = y + axy + O(x,y)^3$ as claimed.

From the definition of $\mathrm{Ad}(x)$, we get $f(x,y) = \mathrm{Ad}(x)y + O_x(y^2)$, where O_x means that the coefficient may depend on x.

Note that the definition of $\mathrm{ad}(x)$ gives $\mathrm{Ad}(x)(y) = y + [x,y] + O(x^2y)$. Indeed, we have $\mathrm{Ad}(x) = \mathrm{Ad}(0) + \mathrm{Ad}'(0)(x) + O(x^2) = I + \mathrm{ad}(x) + O(x^2)$, where I is the identity matrix, and evaluating at y, we conclude $\mathrm{Ad}(x)y = y + [x,y] + O(x^2y)$.

We conclude that $y + axy + O(x,y)^3 = y + [x,y] + O_x(y^2)$. This implies $[x,y] = axy$. See Lemma 3.16. Now from $x \cdot y \cdot x^{-1} = y + axy + O(y,x)^3$, and the formula $x \cdot y^{-1} = x - y + b_0x^2 + b_1xy + b_2y^2 + O(x,y)^3$ (see Fact 6.6), we get $x \cdot y \cdot x^{-1} \cdot y^{-1} = (x \cdot y \cdot x^{-1}) \cdot y^{-1} = axy + b_3y^2 + O(x,y)^3$. However, if $c(x,y) = x \cdot y \cdot x^{-1} \cdot y^{-1}$, then $c(0,y) = 0$ implies that $b_3 = 0$, as required. □

Lemma 6.17. *Let G be a definable weak T_3-Lie group. Let $0 \in U \subset K^n$ be an open set and $i : U \to G$ a T_3-diffeomorphism of U onto an open subset of G that sends 0 to e. Make U into a local definable group via i. Then under the identification $i'(0) : K^n \to T_e(G)$,*

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we have that $c(x,c(y,z)) = [x,[y,z]] + O(x,y,z)^4$ *for* $x,y,z \in U$*. Here, we denote* $c(x,y) = x \cdot y \cdot x^{-1} \cdot y^{-1}$*.*

Proof. First, we show that $c(x,y) = [x,y] + a_1xy^2 + a_2x^2y + O(x,y)^4$. Indeed, using Proposition 6.16 and the Taylor expansion of order 3 of c around the origin, we obtain $c(x,y) = [x,y] + a_0y^3 + a_1xy^2 + a_2x^2y + a_3x^3 + O(x,y)^4$. Taking $x = 0$ in this formula, we conclude using the uniqueness of Taylor coefficients that $a_0 = 0$. Similarly, taking $y = 0$, we conclude $a_3 = 0$, which finishes the proof of the claim.

Now we conclude that $c(x,c(y,z)) = [x,c(y,z)] + a_1xc(y,z)^2 + a_2x^2c(y,z) + O(x,c(y,z))^4$. Proposition 6.16 implies that $c(y,z) = O(y,z)^2$, so $a_1xc(y,z)^2 = O(x,y,z)^5$ and $a_2x^2c(y,z) = O(x,y,z)^4$. Note also that $c(y,z) = [y,z] + O(y,z)^3$ implies $[x,c(y,z)] = [x,[y,z]] + O(x,y,z)^4$. Putting all this together, we conclude $c(x,c(y,z)) = [x,[y,z]] + O(x,y,z)^4$, as required. □

Remark 6.18. Note that in a definable local strictly differentiable and P_1-Lie group, we have $x \cdot y = x + y + O(x,y)^2$. From this, it follows inductively that $x_1 \cdot \cdots \cdot x_n = x_1 + \cdots + x_n + O(x_1, \cdots, x_n)^2$.

Proposition 6.19. *Let* G *be a definable strictly differentiable weak Lie group. Then* $(T_e(G),[,])$ *is a Lie algebra.*

Proof. We have to prove $[x,x] = 0$ and the Jacobi identity. We will use the characterisation of Proposition 6.16 (we may assume G is T_3 by Proposition 6.4). $[x,x] = 0$ now follows immediately.

The idea of proof of the Jacobi identity is to express xyz as $f(x,y,z)zyx$ in two different ways using associativity: the first one permutes from left to right, and the second permutes yz and then permutes from left to right. The details follow.

Writing $c(x,y) = xyx^{-1}y^{-1}$, one has

$$xyz = c(x,y)yxz = c(x,y)yc(x,z)zx = c(x,y)c(y,c(x,z))c(x,z)yzx = c(x,y)([y,[x,z]] + O(x,y,z)^4)c(x,z)c(y,z)zyx = (c(x,y) + [y,[x,z]] + c(x,z) + c(y,z) + O(x,y,z)^4)zyx.$$

At the fourth equality, we use Lemma 6.17. At the last step, we use Remark 6.18 together with the estimate $c(x,y) = O(x,y)^2$ implied by Proposition 6.16 for the error.

However,

$$xyz = xc(y,z)zy = ([x,[y,z]] + O(x,y,z)^4)c(y,z)xzy = ([x,[y,z]] + O(x,y,z)^4)c(y,z)c(x,z)zxy = ([x,[y,z]] + O(x,y,z)^4)c(y,z)c(x,z)zc(x,y)yx = ([x,[y,z]] + O(x,y,z)^4)c(y,z)c(x,z)([z,[x,y]] + O(x,y,z)^4)c(x,y)zyx = ([x,[y,z]] + c(y,z) + c(x,z) + c(x,y) + [z,[x,y]] + O(x,y,z)^4)zyx.$$

From this, we get $[y,[x,z]] = [x,[y,z]] + [z,[x,y]] + O(x,y,z)^4$, and from the uniqueness of Taylor expansions, we obtain $[y,[x,z]] = [x,[y,z]] + [z,[x,y]]$, which is the Jacobi identity. □

Given a strictly differentiable definable weak Lie group G, we denote $Lie(G)$ the tangent space $T_e(G)$ considered as a Lie algebra with the Lie bracket $[x,y]$.

7. Definable fields

In this section, we prove that if L is a definable field in a 1-h-minimal valued field, then, as a definable field, L is isomorphic to a finite field extension of K. This result generalises [2, Theorem 4.2], where this is proved for real closed valued fields, and [12, Theorem 4.1] where this is proved for p-adically closed fields.

With the terminology and results we have developed in the previous section, the main ingredients of the proof are similar to those appearing in the classification of infinite fields definable in o-minimal fields, [10, Theorem 1.1].

Lemma 7.1. *Suppose K is a pregeometric field of characteristic 0, $L \subseteq K$ a definable subfield. Then $L = K$.*

Proof. L is a definable set which is infinite because the characteristic of K is 0. If $L \neq K$, we have a definable injection of L-vector spaces $L^2 \to K$, but L^2 has dimension 2 and K has dimension 1, so this is a contradiction. □

Lemma 7.2. *Suppose K is a pregeometric field of characteristic 0. Let F_1 and F_2 be finite extensions of K, and consider them as definable fields in K. If $\varphi : F_1 \to F_2$ is a definable field morphism, then φ is a morphism of K extensions. In other words, it is the identity when restricted to K.*

Proof. The set $\{x \in K : \varphi(x) = x\}$ is a definable subfield of K, so Lemma 7.1 gives the desired conclusion. □

Proposition 7.3. *Suppose K is 1-h-minimal and F is a definable field. Then F is isomorphic as a definable field to a finite extension of K. The forgetful functor from finite K-extensions to definable fields is an equivalence of categories.*

Proof. That the functor is full is Lemma 7.2.

Let F be a definable field. By Proposition 6.4, we have that $(F,+)$ is a definable strictly differentiable weak Lie group. If $a \in F$, the map $L_a : x \mapsto ax$ is a definable group morphism and so it is strictly differentiable, by the fullness in Proposition 6.4. We get a definable map $f : F \to M_n(K)$ defined as $a \mapsto L'_a(0)$. Here we are identifying the tangent space of F at 0 with K^n via taking a fixed chart around 0. By the chain rule we have $f(ab) = f(a)f(b)$ for all $a,b \in F$. Clearly $f(1) = 1$. Finally one has $f(a+b) = f(a)+f(b)$ (the derivative of multiplication $G \times G \to G$ in a Lie group is the sum map, see for instance Fact 6.6). We conclude that f is a ring map, and because F is a field it is injective. If we set $i : K \to M_n(K)$ given by $i(k) = kI$ where I is the identity matrix, then $i^{-1}f(F) \subset K$ is a definable subfield of K, and so by Lemma 7.1 one has $i(K) \subset f(F)$. So F/K is a finite field extension as required. □

8. One-dimensional groups are finite by abelian by finite

In this section, we prove that if K is a 1-h-minimal valued field and G is a one-dimensional group definable in K, then G is finite-by-abelian-by-finite. This generalises [13, Theorem 2.5], where it is proved that one-dimensional groups definable in p-adically closed fields

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are abelian-by-finite. This result is analogous to [11, Corollary 2.16], where it is shown that a one-dimensional group definable in an o-minimal structure is abelian-by-finite.

The proof here is not a straightforward adaptation of either since we do not assume NIP, making the argument more involved.

Definition 8.1. Let G be a group. We let C^w denote the set of elements $x \in G$ whose centraliser $c_G(x)$ has finite index in G.

Note that C^w is a characteristic subgroup of G.

Lemma 8.2. *Suppose G is an (abstract) group. Take C^w as in Definition 8.1, Z its center. Then C^w and Z are characteristic groups of G, and Z is commutative. Moreover, Z has finite index in G if and only if G is abelian-by-finite.*

When G is definable in a geometric theory, C^w and Z are definable. Also, $x \in C^w$ if and only if $\dim(c_G(x)) = \dim(G)$.

Proof. It is clear that C^w and Z are characteristic and that Z is abelian. So, in particular, if $[G:Z] < \infty$, then G is abelian-by-finite. However, if A is an abelian subgroup of G of finite index, then $A \subset C^w$, as $A \subset c_G(a)$ for every $a \in A$. If $a_1, \ldots, a_n$ are a set of representatives for left cosets of A in C^w, then $\bigcap_{k=1}^n c_G(a_k) \cap A \subset Z$, and as $a_k \in C^w$, the $c_G(a_k)$ have finite index in G, and so Z has finite index in G.

If G is definable in a geometric theory, note that $x \in C^w$ if and only if x^G, the orbit of G under conjugation, is finite. This is because the fibers of the map $x \mapsto x^g$ are cosets of $c_G(x)$. So C^w is definable because a geometric theory eliminates the exist infinity quantifier. We also get that if $\dim(c_G(x)) = \dim(G)$, then $c_G(x)$ is of finite index. □

We need the following standard observation:

Lemma 8.3. *Suppose $f: X \times Y \to Z$ is a function definable in a pregeometric theory. Denote $n = \dim(X)$ and $m = \dim(Y)$. Suppose for all $x \in X$, the nonempty fibers of the function $f_x(y) = f(x,y)$ have dimension m. Suppose that for all $y \in Y$, the nonempty fibers of the function $f_y(x) = f(x,y)$ have dimension n. Then f has finite image.*

Proof. We claim that the nonempty fibers of f have dimension $n+m$. Indeed, if $(x_0,y_0) \in X \times Y$, then $f^{-1}f(x_0,y_0)$ contains $\bigcup_{x \in f_{y_0}^{-1} f_{y_0}(x_0)} \{x\} \times f_x^{-1} f_x(y_0)$, so we conclude by the additivity of dimension. □

Lemma 8.4. *Suppose G is definable in a pregeometric theory and $G = C^w$. Then the image of the commutator map $c: G \times G \to G$ is finite.*

Proof. The commutator map $c(x,y)$ is constant when x is fixed and y varies over a right coset of $c_G(x)$, and it is constant when y is fixed and x varies over a right coset of $c_G(y)$. This implies that the image of c is finite; see Lemma 8.3. □

Lemma 8.5. *Suppose G is an n-dimensional group definable in a pregeometric theory such that $G = C^w$. Then there is a definable characteristic subgroup, G_1, of finite index*

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with a characteristic finite subgroup L, central in G_1, such that G_1/L is abelian. If Z is the center of G_1, then Z/L contains $(G_1/L)^m$, the m-th powers of G_1/L, for some m.

If the theory is NIP, then the center of G has finite index.

Proof. If the theory has NIP, then the center has finite index in G by Baldwin-Saxl (e.g., [14, Lemma 1.3]). Indeed, $Z = \bigcap_{g\in G} c_G(g)$ is an intersection of a definable family subgroups, each of which has finite index by the assumption $G = C^w$. So, as G is NIP, one gets that Z is the intersection of finitely many of the centralisers and $[G:Z] < \omega$.

In general, by Lemma 8.4, we know that $c(G,G)$ is finite. The centraliser of $c(G,G)$ is G_1 and has finite index in G by the hypothesis that $G = C^w$. Clearly, G_1 is characteristic in G, so we may replace G by G_1 and assume that $c(G,G)$ is contained in the center of G.

In this case, we prove that $c(G,G)$ generates a finite central characteristic group $L = D(G)$. Indeed, since $c(G,G)$ is central, simple computation shows $c(gh,x) = c(g,x)c(h,x)$ for all $g,h,x \in G$. It follows that $c(g,h)^m = c(g^m,h)$ is in $c(G,G)$. Thus, $c(g,h)$ has finite order. As $c(G,G)$ is central with elements of finite order, the group it generates is central and finite. It is obviously characteristic.

We also see that if m is the order of $D(G)$, then $g^m \in Z$ for all $g \in G$. This is because $c(g^m,h) = c(g,h)^m = 1$, so $(G/D(G))^m$ is contained in $Z/D(G)$ as required. □

Lemma 8.6. *Suppose G is an n-dimensional abelian group definable in a 1-h-minimal theory. Then the m-torsion of G is finite and $G^m \subset G$ is a subgroup of dimension n.*

Proof. The map $x \mapsto x^m$ is a definable group morphism with invertible derivative at the identity – see, for instance, Fact 6.6 – so by Corollary 6.11, we get that the m-torsion of G is finite, and so by additivity of dimension, $\dim(G^m) = \dim(G) = n$. □

Lemma 8.7. *Suppose G is an n-dimensional group definable in a 1-h-minimal field. Then C^w is the kernel of the map $\mathrm{Ad}: G \to GL_n(K)$.*

If the Lie algebra $\mathrm{Lie}(G)$ is abelian, then C^w has finite index.

Proof. The first statement follows from Corollary 6.12 and Lemma 8.2.

If the Lie bracket is abelian, then, by the definition of the Lie bracket, the derivative of Ad at e is 0. This means that $C^w = \ker(\mathrm{Ad})$ contains an open neighborhood of e by Corollary 6.12, so C^w is n-dimensional. By additivity of dimension, we conclude that C^w has finite index in G. □

Lemma 8.8. *Suppose G is an n-dimensional finite-by-abelian group definable in a 1-h-minimal theory. Then $G = C^w$, and the center of G has dimension n.*

Proof. Let H be a finite normal subgroup such that G/H is abelian. By elimination of finite imaginaries in fields, we have that G/H is definable. Also by Corollary 6.11, we see that the quotient map $p: G \to G/H$ induces an isomorphism of tangent spaces at the identity, and under this isomorphism $1 = \mathrm{Ad}(p(g)) = \mathrm{Ad}(g)$ for all $g \in G$. We conclude that $G = C^w$, by Lemma 8.7. By Lemma 8.5, we get characteristic groups $L \subset G_1 \subset G$ such that G/G_1 is finite, L is finite, G_1/L is abelian, and $Z(G_1)/L \supset (G_1/L)^m$ for some m. Note that as G_1 has finite index in G and $G = C^w$, we have that $Z(G)$ has finite index

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in $Z(G_1)$, and so $\dim(Z(G)) = \dim(Z(G_1))$. So we just have to see that $(G_1/L)^m$ has dimension n. This is, precisely, the conclusion of Lemma 8.6, concluding the proof of the lemma. □

Proposition 8.9. *Suppose K is 1-h-minimal. Suppose G is a strictly differentiable definable weak Lie group. Then* $\mathrm{Lie}(G)$ *is abelian if and only if G is finite-by-abelian-by-finite. In this case, G has characteristic definable subgroups $L \subset G_1 \subset G$ such that G/G_1 is finite, G_1/L is abelian, and L is finite and central in G_1. Also, if Z is the center of G_1, then Z is n-dimensional, and Z/L contains $(G_1/L)^m$ for some m.*

If K is NIP, then we may take $L = 1$.

Proof. First, if $\mathrm{Lie}(G)$ is abelian, then by Lemma 8.7, C^w has finite index in G. By Lemma 8.2, C^w is definable. It is readily checked that C^w satisfies the assumptions of Lemma 8.5. Putting everything together, we get that G is finite-by-abelian-by-finite. In the other direction, if G is finite-by-abelian-by finite, it has a finite index subgroup whose Lie algebra is commutative, by Lemma 8.8. But then $\mathrm{Lie}(G)$ is abelian, too.

So we now proceed under the assumption that $\mathrm{Lie}(G)$ is abelian. In this case, C^w has finite index in G (by Lemma 8.8), and as C^w is a characteristic subgroup (Lemma 8.2), we may apply lemma 8.5 to C^w to get L and G_1, as in the statement. That $Z(G_1)$ is n-dimensional follows from Lemma 8.8, and so does the fact that $(G_1/L)^m \subseteq Z/L$ for some m. In case G is NIP, the application of 8.5 is simpler, as we may take G_1 to be the center of C^w. □

Corollary 8.10. *Suppose G is a one-dimensional group definable in a 1-h-minimal valued field. Then G is finite-by-abelian-by-finite. If the theory is NIP, then G is abelian-by-finite.*

Proof. By Proposition 6.4, we get that G is a strictly differentiable definable weak Lie group. The result now follows from Proposition 8.9 because the only one-dimensional Lie algebra is abelian. □

In the NIP case, this corollary follows more directly from the fact that a definable group is definably weakly Lie. Indeed, this implies that there is an element $x \in G$ with $x^n \neq e$ (because the derivative of the map $x \mapsto x^n$ at e is $v \mapsto nv$, which is not equal to 0; see Fact 6.6). By $\aleph_0$-saturation, there is $x \in G$ such that the group generated by x is infinite. Then by [13, Remark 2.4], the double centraliser of x has finite index and is abelian. Indeed, note that if $a \in c_G(x)$, $c_G(a)$ contains the group generated by x, and so $\dim(c_G(a)) = 1$.

The last corollary shows that the classification of one-dimensional abelian groups definable in ACVF carried out in [1] extends to all definable 1-dimensional groups, for K of characteristic 0 (see also the main result of [7]). That is, since ACVF_0 is 1-h-minimal and NIP, 1-dimensional definable groups are abelian-by-finite, and the classification of definable 1-dimensional abelian groups of [1] applies. We do not know if this corollary is true in $\mathrm{ACVF}_{p,p}$. Similarly, the commutativity assumption is unnecessary in the classification of 1-dimensional groups definable in pseudo-local fields of residue characteristic 0. As those are pure henselian, they, too, are 1-h-minimal, so we may apply Proposition 8.9. To get the full result, we observe that though pseudo-local fields are not NIP, an inspection of the list of the definable 1-dimensional abelian groups obtained

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in [1] shows they are almost divisible (i.e., for all n, the map $x \mapsto nx$ has finite kernel). Therefore, in the notation of Proposition 8.9, the center of G_1 has finite index in G_1, and so every one-dimensional group is abelian-by-finite.

Question 8.11. If G is finite-by-abelian, does the center of G have finite index in G? This is true if the theory is NIP or if nA has finite index in A for every abelian definable group, by Lemmas 8.8 and 8.5.

Remark 8.12. $\mathrm{ACVF}_{p,p}$ does not fit into the framework of 1-h-minimality. However, many of the ingredients in previous sections translate to this setting. For example, K is geometric, a subset of K^n has dimension n if and only if it contains a nonempty open set, one-to-finite functions defined in an open set are generically continuous, functions definable in an open set are generically continuous, and K is definably spherically complete.

That one-to-finite functions are generically continuous follows from the fact that $\mathrm{acl}(a)$ coincides with the field-theoretic algebraic closure of a and by a suitable result about continuity of roots. That functions are generically continuous follows from the fact that $\mathrm{dcl}(a)$ is the Henselization of the perfect closure of a, so a definable function is definably piecewise a composition of rational functions, inverse of the Frobenius automorphism and roots of Hensel polynomials, all of these functions being continuous.

However, the inverse of the Frobenius is not differentiable anywhere, so Proposition 3.13 does not hold. Also, the Frobenius is an homeomorphism with 0 derivative, so, for example, Proposition 4.6 does not hold.

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