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# THE LARGEST CLASS OF HEREDITARY SYSTEMS DEFINING A $C_0$ SEMIGROUP ON THE PRODUCT SPACE

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**1. Introduction.** The object of this paper is to characterize the largest class of autonomous linear hereditary differential systems which generates a strongly continuous semigroup of class $C_0$ on the product space $M^p = \mathbf{R}^n \times L^p(-h, 0)$, $1 \leqq p < \infty$, $0 < h \leqq +\infty$ ($\mathbf{R}$ is the field of real numbers and $L^p(-h, 0)$ is the space of equivalence classes of Lebesgue measurable maps $x:[-h, 0] \cap \mathbf{R} \to \mathbf{R}^n$ which are $p$-integrable in $[-h, 0] \cap \mathbf{R}$.) Our results extend and complete those of [**4**] and [**15**], [**16**] for linear hereditary differential equations possessing "finite memory" ($h < +\infty$) and those of [**14**], [**5**] and [**6**] in the "infinite memory case ($h = +\infty$)".

Consider the autonomous linear hereditary differential equation

$$(1.1)\qquad \begin{cases} \dot{x}(t) = L(x_t),\ t \geqq 0 \\ x(\theta) = \phi(\theta),\ \phi \text{ in } C(-h, 0), \end{cases}$$

where $x(t) \in \mathbf{R}^n$, $x_t:[-h, 0] \cap \mathbf{R} \to \mathbf{R}^n$ is defined as $x_t(\theta) = x(t + \theta)$, $C(-h, 0)$ is the space of bounded continuous functions $[-h, 0] \cap \mathbf{R} \to \mathbf{R}^n$ and $L:C(-h, 0) \to \mathbf{R}^n$ is a continuous linear map.

For $h$ finite it is well-known (cf. [**10**], [**11**], [**12**]) that the family of continuous linear transformations $S(t):C(-h, 0) \to C(-h, 0)$, $t \geqq 0$

$$(1.2)\qquad S(t)\phi = x_t$$

forms a strongly continuous semigroup of class $C_0$. Its infinitesimal generator is of the form

$$(1.3)\qquad \mathscr{D}(A) = \{\phi \in C^1(-h, 0): L(\phi) = \dot{\phi}(0)\}$$

$$(1.4)\qquad (A\phi)(\theta) = \begin{cases} L\phi, & \theta = 0 \\ \dot{\phi}(\theta), & -h \leqq \theta < 0 \end{cases},$$

where $\dot{\phi}$ denotes the derivative of $\phi$ and $C^1(-h, 0)$ is the space of functions $\phi$ in $C(-h, 0)$ with a derivative $\dot{\phi}$ in $C(-h, 0)$. For $h$ infinite the result is not true since bounded continuous functions on $[-\infty, 0]$ are not uniformly continuous. In 1972, Barbu and Grossman [**2**] have shown

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that the result can be recovered by replacing the space $C(-\infty, 0)$ by its closed subspace $C_l(-\infty, 0)$ of all bounded continuous functions on $]-\infty, 0]$ for which the limit exists at $-\infty$; such functions are uniformly continuous and the semigroup of left translations on $C_l(-\infty, 0)$ is continuous (cf. [**13**]).

For systems with finite memory, Borisovic and Turbabin [**4**] have shown that under three additional hypotheses on the map $L$, (1.1) still makes sense for initial conditions in the product space $M^p = \mathbf{R}^n \times L^p(-h, 0)$, $1 \leqq p < \infty$, that is;

$$(1.5) \quad \begin{cases} \dot{x}(t) = L(x_t),\ t \geqq 0 \\ x(0) = \phi^0,\ x(\theta) = \phi^1(\theta),\ \phi = (\phi^0, \phi^1) \in M^p. \end{cases}$$

More precisely, they have shown that the family of continuous linear transformations $S(t): M^p \to M^p$, $t \geqq 0$

$$(1.6) \quad S(t)(\phi^0, \phi^1) = (x(t), x_t)$$

forms a strongly continuous semigroup of class $C_0$. Its infinitesimal generator is now of the form

$$(1.7) \quad \mathscr{D}(A) = \{(\phi(0), \phi): \phi \in W^{1,p}(-h, 0)\}$$

$$(1.8) \quad A\phi = (L\phi, \dot{\phi}),$$

where $W^{1,p}(-h, 0)$ is the Sobolev space of all $\phi$ in $L^p(-h, 0)$ with a distributional derivative $\dot{\phi}$ in $L^p(-h, 0)$.

Analogous results in $M^2$ were given in 1972 in [**9**, pp. 301–304] for time-varying systems (that is, an evolution operator $S(t, s)$ rather than the semigroup $S(t)$) with both finite and infinite memory but with $L$ of the special form

$$(1.9) \quad L\phi = \sum_{i=0}^{N} A_i \phi(\theta_i) + \int_{-h}^{0} A_{01}(\theta)\phi(\theta)d\theta,$$

where $N \geqq 0$ is an integer, $a > 0$ is a finite real,

$$-\infty \leqq -h \leqq -a = \theta_N < \ldots < \theta_{i+1} < \theta_i < \ldots < \theta_0 = 0$$

are reals, $A_i$, $i = 0, \ldots, N$, is a family of $n \times n$ matrices and $A_{01}(\theta)$ is an $n \times n$ matrix of bounded measurable functions on $[-h, 0] \cap \mathbf{R}$.

In 1974 detailed proofs were given by R. K. Miller [**14**] for $M^p$ and $L$ of the form

$$(1.10) \quad L\phi = M\phi(0) + \int_{-\infty}^{0} K(\theta)\phi(\theta)d\theta$$

for an $n \times n$ matrix $K$ of functions in $L^1(-\infty, 0)$. Properties of the adjoint semigroup and its relation to the semigroup of Barbu and Grossman [**2**] were announced in 1975 in [**5**]. Detailed proofs were pro-

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vided in 1976 in [**6**]. At the beginning of that paper they construct the semigroup on $M^p$, $1 \leqq p < \infty$ for systems with infinite memory characterized by a continuous linear map $L: C(-\infty, 0) \to \mathbf{R}^n$ subject to the three conditions of [**4**] plus an additional one.

In 1977, R. B. Vinter [**15**], [**16**] showed that for systems with finite memory the three conditions imposed in [**4**] were redundant. Shortly after that an alternate proof of that fact was provided in [**7**] by using the structural operator associated with the map $L$.

As a result of this discussion, the problem of characterizing the largest class of $L$'s was raised by R. B. Vinter and J. Zabczyk. Their obvious candidate was the family $\mathscr{L}$ of all continuous linear map

$$(1.11) \quad L: W^{1,p}(-h, 0) \to \mathbf{R}^n,$$

since the very special form of the infinitesimal generator indicates that $\mathscr{L}$ is the largest possible family.

*Notation*. Given $-\infty \leqq a \leqq b \leqq +\infty$, $L^p(a, b)$ is the Banach space of all equivalence classes of Lebesgue measurable maps $[a, b] \cap \mathbf{R} \to \mathbf{R}^n$ which are $p$-integrable ($1 \leqq p < \infty$) or essentially bounded ($p = \infty$); the corresponding norms will be written $\| \ \|_p$. $C(a, b)$ is the Banach space of all bounded continuous maps $[a, b] \cap \mathbf{R} \to \mathbf{R}^n$ endowed with the sup norm $\| \cdot \|_{C(a,b)}$. The norm on the product space $M^p$ is defined as

$$(1.12) \quad \|\phi\|_{M^p}^p = |\phi^0|_{\mathbf{R}^n}^p + \|\phi^1\|_p^p.$$

$L^p_{\text{loc}}(a, b)$ will be the space of all equivalence classes of Lebesgue measurable maps $[a, b] \cap \mathbf{R} \to \mathbf{R}^n$ which are $p$-integrable ($1 \leqq p < \infty$) or essentially bounded ($p = \infty$) on each compact subset of $[a, b] \cap \mathbf{R}$. For an integer $m > 0$, $C^m(a, b)$ will denote the Banach space of all bounded continuous maps $[a, b] \cap \mathbf{R} \to \mathbf{R}^n$ for which the first $m$ derivatives are bounded and continuous in $[a, b] \cap \mathbf{R}$; for $1 \leqq p \leqq \infty$, $W^{1,p}(a, b)$ will denote the Sobolev space of all functions in $L^p(a, b)$ with a distributional first derivative in $L^p(a, b)$.

**2. Main results.** Consider the differential equation (1.1) where $L$ belongs to the family of continuous linear maps

$$(2.1) \quad L: W^{1,p}(-h, 0) \to \mathbf{R}^n.$$

For $1 \leqq p < \infty$, it is always possible to associate with $L$ two $n \times n$ matrices, $A_1(\theta)$ and $A_2(\theta)$, of functions in $L^q(-h, 0)$, $p^{-1} + q^{-1} = 1$,

$$(2.2) \quad L\phi = \int_{-h}^{0} [A_1(\theta)\phi(\theta) + A_2(\theta)\dot{\phi}(\theta)]d\theta.$$

Fix $p$, $1 \leqq p < \infty$. Given a continuous function $f: [0, \infty[ \to \mathbf{R}^n$ and an

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initial function $\phi$ in $W^{1,p}(-h, 0)$, consider the following equation

$$(2.3)\quad \begin{cases} \dot{x}(t) = \int_{-h}^{0} [A_1(\theta)x(t+\theta) + A_2(\theta)\dot{x}(t+\theta)]d\theta + f(t), & t \geqq 0, \\ x(\theta) = \phi(\theta), \quad \theta \in [-h, 0] \cap \mathbf{R}. \end{cases}$$

By integrating both sides of (2.3) and changing the order of integration in the term $\dot{x}(t+\theta)$ we obtain the integral equation

$$(2.4)\quad x(t) = \phi(0) + \int_0^t ds \int_{-h}^0 d\theta A_1(\theta)x(s+\theta) + \int_{-h}^0 d\theta A_2(\theta)\int_0^t ds\, \dot{x}(s+\theta) + \int_0^t f(s)ds.$$

It can be transformed into

$$(2.5)\quad \begin{cases} x(t) = \phi(0) + \int_0^t ds \int_{-h}^0 d\theta A_1(\theta)x(s+\theta) \\ \qquad + \int_{-h}^0 d\theta A_2(\theta)[x(t+\theta) - x(\theta)] + \int_0^t f(s)ds \\ x(\theta) = \phi(\theta) \text{ in } [-h, 0] \cap \mathbf{R}, \end{cases}$$

and further generalized to: for all $t \geqq 0$

$$(2.6)\quad x(t) = \phi^0 + \int_{-h}^0 \left[A_1(\theta)\int_0^t ds \begin{cases} x(s+\theta), & s+\theta \geqq 0 \\ \phi^1(s+\theta), & \text{otherwise} \end{cases} + A_2(\theta)\begin{cases} x(t+\theta) - \phi^1(\theta), & t+\theta \geqq 0 \\ \phi^1(t+\theta) - \phi^1(\theta), & \text{otherwise} \end{cases}\right]d\theta + \int_0^t f(s)ds,$$

where now $\phi = (\phi^0, \phi^1)$ and $f$ can be picked in $M^p$ and $L^1_{\text{loc}}(0, \infty)$, respectively.

THEOREM. *Fix $1 \leqq p < \infty$, two $n \times n$ matrices $A_1$ and $A_2$ of functions in $L^q(-h, 0)$, $p^{-1} + q^{-1} = 1$, and the map $L$ defined by (2.2).*

(i) *Given $\phi = (\phi^0, \phi^1)$ in $M^p$ and $f$ in $L^1_{\text{loc}}[0, \infty[$, equation (2.6) has a unique continuous solution $x: [0, \infty[ \to \mathbf{R}^n$. Moreover for all $T > 0$, there exists a constant $c(T) > 0$ such that*

$$(2.7)\quad \|x\|_{C(0,T)} \leqq c(T)(\|\phi\|_{M^p} + \|f\|_{L^1(0,T)}).$$

(ii) *Given $\phi$ in $W^{1,p}(-h, 0)$ and a continuous function $f: [0, \infty[ \to \mathbf{R}^n$, equation (2.3) has a unique continuously differentiable solution $x: [0, \infty[ \to \mathbf{R}^n$ which coincides with the solution of (2.6) corresponding to $(\phi^0, \phi^1) = (\phi(0), \phi)$.*

(iii) *When $f = 0$, define for each $t \geqq 0$ the continuous linear map $S(t): M^p \to M^p$*

$$(2.8)\quad S(t)\phi = (x(t), x_t).$$

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*The family* $\{S(t): t \geqq 0\}$ *forms a strongly continuous semigroup of class* $C_0$ *and its infinitesimal generator is characterized as follows:*

$$(2.9)\quad \mathscr{D}(A) = \{(\phi(0), \phi): \phi \in W^{1,p}(-h, 0)\},\ A\phi = (L\phi, \dot{\phi}).$$

(iv) *For all* $\phi$ *in* $M^p$ *and* $f$ *in* $L^1_{\mathrm{loc}}(0, \infty)$,

$$(2.10)\quad (x(t), x_t) = S(t)\phi + \int_0^t S(t-s)\tilde{f}(s)ds,$$

*where*

$$(2.11)\quad \tilde{f}(t) = (f(t), 0).$$

*Proof.* (i) Fix $T > 0$ and $x$ in $C(0, T)$. Define the linear map $Mx$: $[0, T] \to \mathbf{R}^n$ as the right hand side of equation (2.6); by hypothesis on $A_2$ and $\phi^1$, $Mx$ is continuous. For all $x$ and $y$ in $C(0, T)$:

$$(2.12)\quad (My)(t) - (Mx)(t)$$
$$= \int_0^t ds \int_{-h}^0 d\theta A_1(\theta) \begin{cases} y(s+\theta) - x(s+\theta), & s+\theta \geqq 0 \\ 0, & \text{otherwise} \end{cases}$$
$$+ \int_{-h}^0 d\theta A_2(\theta) \begin{cases} y(t+\theta) - x(t+\theta), & t+\theta \geqq 0 \\ 0, & \text{otherwise} \end{cases}.$$

But

$$|(My)(t) - (Mx)(t)|$$
$$\leqq \int_0^t ds \|A_1\|_q \left\{ \int_{-h}^0 d\theta \left| \begin{cases} y(s+\theta) - x(s+\theta), & s+\theta \geqq 0 \\ 0, & \text{otherwise} \end{cases} \right|^p \right\}^{1/p}$$
$$+ \|A_2\|_q \left\{ \int_{-h}^0 d\theta \left| \begin{cases} y(t+\theta) - x(t+\theta), & t+\theta \geqq 0 \\ 0, & \text{otherwise} \end{cases} \right|^p \right\}^{1/p}$$

and after a change of variable

$$(2.13)\quad |(My)(t) - (Mx)(t)| \leqq \int_0^t ds \|A_1\|_q \left\{ \int_0^s dr |y(r) - x(r)|^p \right\}^{1/p} + \|A_2\|_q \left\{ \int_0^t dr |y(r) - x(r)|^p \right\}^{1/p}.$$

Choose a constant $c > 0$ such that

$$(2.14)\quad pc = (T\|A_1\|_q + \|A_2\|_q)^p \quad \text{and} \quad g_\alpha(t) = \exp\left(\frac{c}{\alpha^p} t\right),$$

for some arbitrary parameter $\alpha$, $0 < \alpha < 1$. Then

$$\int_0^s \left| \frac{y(r) - x(r)}{g_\alpha(r)} \right|^p g_\alpha(r)^p dr \leqq \max_{[0,t]} \left| \frac{y(r) - x(r)}{g_\alpha(r)} \right|^p \int_0^s g_\alpha(r)^p dr.$$

But for all $s$ in $[0, t]$

$$\int_0^s g_\alpha(r)^p dr = \int_0^s \exp\left(\frac{pcr}{\alpha^p}\right) dr \leqq \frac{\alpha^p}{pc} g_\alpha(s)^p \leqq \frac{\alpha^p}{pc} g_\alpha(t)^p.$$

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Therefore the right hand side of inequality (2.13) can be majorized by

$$\left\{T\|A_1\|_q \frac{\alpha}{(pc)^{1/p}} g_\alpha(t) + \|A_2\|_q \frac{\alpha}{(pc)^{1/p}} g_\alpha(t) \max_{[0,t]} \left|\frac{y(r)-x(r)}{g_\alpha(r)}\right| = \alpha g_\alpha(t)\|y-x\|_{C_\alpha[0,t]},\ \|z\|_{C_\alpha[0,t]} = \max_{[0,t]}\left|\frac{z(r)}{g_\alpha(r)}\right|\right\}.$$

Finally for all $t$ in $[0, T]$

$$\left|\frac{(My)(t)-(Mx)(t)}{g_\alpha(t)}\right| \leqq \alpha\|y-x\|_{C_\alpha[0,t]}$$

and

(2.15) $\|My - Mx\|_{C_\alpha[0,T]} \leqq \alpha\|y-x\|_{C_\alpha[0,T]}$.

Thus $M$ is a contraction and necessarily equation (2.6) has a unique solution in $C[0, T]$. (Notice that the $\alpha$-norm $\|\cdot\|_{C_\alpha[0,t]}$ is equivalent to the usual norm $\|\cdot\|_{C[0,t]}$. This technique is borrowed from [**3**].) Inequality (2.7) is established by the same technique.

(ii) Substitute for $x(t)$ in equation (2.3) the expression

(2.16) $$x(t) = \phi(0) + \int_0^t \dot{x}(s)ds,$$

where we assume that $\dot{x}$ is continuous. We obtain

(2.17) $$\dot{x}(t) = \int_{-h}^0 \begin{cases} A_1(\theta)\left[\phi(0) + \int_0^{t+\theta}\dot{x}(s)ds\right] + A_2(\theta)\dot{x}(t+\theta), & t+\theta \geqq 0 \\ A_1(\theta)\phi(t+\theta) + A_2(\theta)\dot{\phi}(t+\theta), & t+\theta < 0 \end{cases} d\theta + f(t).$$

By changing the order of integration and changing variables, (2.17) can be rewritten as

(2.18) $\dot{x}(t) = (N\dot{x})(t) + \Phi(t)$,

where

(2.19) $$(N\dot{x})(t) = \int_{\max\{t-h,0\}}^t A(s-t)\dot{x}(s)ds + \Phi(t)$$

(2.20) $$A(\alpha) = A_2(\alpha) + \int_\alpha^0 A_1(\theta)d\theta, \quad \alpha \in [-h, 0] \cap \mathbf{R},$$

(2.21) $$\Phi(t) = \int_{-h}^0 \begin{cases} A_1(\theta)\phi(0), & t+\theta \geqq 0 \\ A_1(\theta)\phi(t+\theta) + A_2(\theta)\dot{\phi}(t+\theta), & t+\theta < 0 \end{cases} d\theta + f(t).$$

Note that $\Phi$ is continuous and that $N\dot{x}$ is continuous whenever $\dot{x}$ is continuous. Now proceed as in the proof of part (i) and show that for each $T > 0$ (2.18) has a unique fixed point $\dot{x}$ in $C(0, T)$. Most steps are

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analogous. Here choose the constant $c > 0$ such that

(2.22) $\quad pc = \|A\|_q{}^p.$

This establishes that (2.3) has a unique continuously differentiable solution $x$. But we have already seen that, by integrating (2.3) from 0 to $t$ and regrouping terms, $x$ verifies equation (2.5). From part (i) and by uniqueness of solution to (2.6), $x$ coincides with the (unique) solution of (2.6) corresponding to the initial condition $(\dot{\phi}^0, \phi^1) = (\phi(0), \phi)$, $\phi$ in $W^{1,p}(-h, 0)$, and the continuous function $f$. It is also easy to show that for all $T > 0$

(2.23) $\quad \|\dot{x}\|_{C[0,T]} \leqq c(T)[\|\phi\|_{W^{1,p}} + \|f\|_{C(0,T)}]$

for some constant $c(T) > 0$.

(iii) By definition of $x_t$ and $S(t)$ and inequality (2.7), it is readily seen that $\{S(t): t \geqq 0\}$ is a strongly continuous semigroup of class $C_0$. For $\phi$ in $\mathscr{D}(A)$ the map

$$t \rightarrow \frac{d}{dt} S(t)\phi = S(t)A\phi : [0, T] \rightarrow M^p$$

is continuous. In particular the $\mathbf{R}^n$-component,

$$t \rightarrow \frac{d}{dt} x(t) = [S(t)A\phi]^0 : [0, T] \rightarrow \mathbf{R}^n$$

is continuous. So $x$ belongs to $C^1(0, T)$. By definition

$$x_t(\theta) - \phi^1(\theta) = \begin{cases} x(t+\theta) - \phi^1(\theta) \ , & t + \theta \geqq 0 \\ \phi^1(t+\theta) - \phi^1(\theta), & \text{otherwise} \end{cases}$$

$$= \begin{cases} x(t+\theta) - \phi^0, t + \theta \geqq 0 \\ \phi^0 - \phi^0 \qquad , \text{otherwise} \end{cases} + \begin{cases} \phi^0 - \phi^1(\theta) \qquad , t + \theta \geqq 0 \\ \phi^1(t+\theta) - \phi^1(\theta), \text{otherwise} \end{cases}.$$

For a fixed small $\epsilon > 0$ and all $t \leqq \epsilon$

(2.24) $\quad x_t - \phi^1 = \hat{x}_t - \hat{x}_0 + \hat{\phi}_t - \hat{\phi}_0,$

where the functions $\hat{x}$ and $\hat{\phi}$:$[-h, +\infty[ \cap \mathbf{R} \rightarrow \mathbf{R}^n$ are defined as

$$(2.25) \quad \hat{x}(s) = \begin{cases} 0 & , \ s \in [-h, 0] \cap \mathbf{R} \\ x(s) - \phi^0 & , \ 0 < s \leqq \epsilon \\ (x(s) - \phi^0)\left(\dfrac{2\epsilon - s}{\epsilon}\right), & \epsilon < \theta \leqq 2\epsilon \\ 0 & , \ \theta > 2\epsilon \end{cases},$$

$$\hat{\phi}(\theta) = \begin{cases} \phi^1(\theta) & , \ \theta \in [-h, 0] \cap \mathbf{R} \\ \phi^0 & , \ 0 < \theta \leqq \epsilon \\ \phi^0\left(\dfrac{2\epsilon - \theta}{\epsilon}\right), & \epsilon < \theta \leqq 2\epsilon \\ 0 & , \ \theta > 2\epsilon \end{cases}.$$

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Since $x$ belongs to $C^1(0, 2\epsilon]$, then $\hat{x}$ belongs to $W^{1,\infty}(-h, +\infty)$ and necessarily

$$(2.26) \quad \lim_{t\to 0^+} \frac{\hat{x}_t - \hat{x}_0}{t} \text{ exists in } L^p(-h, +\infty), \quad 1 \leqq p \leqq \infty$$

(that is, $\hat{x}$ belongs to the domain of the semigroup of left translations on $L^p(-h, +\infty)$). A fortiori

$$(2.27) \quad \lim_{t\to 0^+} \frac{\hat{x}_t - \hat{x}_0}{t} \text{ exists in } L^p(-h, 0), \quad 1 \leqq p \leqq \infty.$$

But $\phi = (\phi^0, \phi^1)$ belongs to $\mathscr{D}(A)$ if and only if

$$(2.28) \quad \lim_{t\to 0^+} \frac{(x(t), x_t) - (\phi^0, \phi^1)}{t} \text{ exists.}$$

In view of (2.24) and (2.27), if $\phi$ belongs to $\mathscr{D}(A)$, then we conclude that

$$(2.29) \quad \lim_{t\to 0^+} \frac{\hat{\phi}_t - \hat{\phi}_0}{t} \text{ exists in } L^p(-h, 0).$$

But if limit (2.29) exists, then

$$(2.30) \quad \lim_{t\to 0^+} \frac{\hat{\phi}_t - \hat{\phi}_0}{t} \text{ exists in } L^p(-h, +\infty).$$

Again, condition (2.30) is equivalent to saying that $\hat{\phi}$ in $L^p(-h, +\infty)$ belongs to the domain $W^{1,p}(-h, +\infty)$ of the semigroup of left translations on $L^p(-h, +\infty)$. It is well known that an element of $W^{1,p}(-h, +\infty)$ is necessarily continuous on $[-h, +\infty[ \cap \mathbf{R}$ (cf. [**1**, Theorem 5.4]). Finally for $\phi$ in $\mathscr{D}(A)$

$$(2.31) \quad \begin{cases} \hat{\phi} \in W^{1,p}(-h, 0) \Rightarrow \phi^1 \in W^{1,p}(-h, 0) \\ \phi^1(0) - \phi^0 = 0. \end{cases}$$

Conversely if $\phi = (\phi^0, \phi^1)$ in $M^p$ satisfies (2.31), then limits (2.30) and (2.29) exists. Moreover from part (ii) the solution $x$ of (2.6) will be continuously differentiable and necessarily limits (2.26) and (2.27) exist. In view of (2.24), limit (2.28) exists and $\phi$ belongs to $\mathscr{D}(A)$. This completes the characterization of $\mathscr{D}(A)$. The last item is the second identity (2.9). For $\phi$ in $\mathscr{D}(A)$ equation (2.6) is equivalent to (2.4) and by direct computation

$$(A\phi)^0 = L\phi;$$

similarly

$$(A\phi)^1 = \lim_{t\to 0^+} \frac{x_t - \phi}{t} = \lim_{t\to 0^+} \frac{\hat{x}_t - \hat{x}_0}{t} + \lim_{t\to 0^+} \frac{\hat{\phi}_t - \hat{\phi}_0}{t} = 0 + \dot{\phi} = \dot{\phi}.$$

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(iv) Now, by standard techniques, it is easy to show that identity (2.10) holds for all $\phi$ in $\mathscr{D}(A)$ and $f$ in $C(0, T)$. To establish it for all $\phi$ in $M^p$ and $f$ in $L^1(0, T)$, we pick approximating sequences $\{\phi_n\}$ in $\mathscr{D}(A)$ and $\{f_n\}$ in $C(0, T)$. By continuity, for all $t \geqq 0$

$$(x_n(t), (x_n)_t) = S(t)\phi_n + \int_0^t S(t-s)\tilde{f}_n(s)ds \to S(t)\phi + \int_0^t S(t-s)\tilde{f}(s)ds$$

and by continuity of the solution $x_n$ of (2.6) with respect to the data $\phi_n$ and $f_n$ (cf. (2.7))

$$x_n \to x \text{ in } C(0, T) \Rightarrow \forall t \geqq 0 (x_n(t), (x_n)_t) \to (x(t), x_t) \text{ in } M^p.$$

This establishes (2.10) and completes the proof of the theorem.

**3. Conclusions.** For all $p$, $1 \leqq p < \infty$, and all $h$, $0 < h \leqq +\infty$, the linear injection

(3.1) $\quad W^{1,p}(-h, 0) \to C(-h, 0)$

is continuous (cf. [**1**, Theorem 5.4]). Thus the restriction to $W^{1,p}(-h, 0)$ of any continuous linear map

(3.2) $\quad L: C(-h, 0) \to \mathbf{R}^n$

is continuous for the $W^{1,p}(-h, 0)$-topology and the conclusions of the theorem apply for all $p$, $1 \leqq p < \infty$.

This shows that the additional hypotheses given in [**4**] and [**6**] are redundant. The system associated with a continuous linear map of the type (3.2) always forms a strongly continuous semigroup of class $C_0$ on $M^p$ for all $p$, $1 \leqq p < \infty$. So, in most situations, it is sufficient to work with the Hilbert space $M^2$ and avoid the non-reflexive Banach space $M^1$ (e.g. adjoint semigroup, stability, optimal control, etc.).

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