GAP SERIES ON GROUPS AND SPHERES

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Introduction. Let G be a compact abelian group and E a subset of its dual group Γ. A function $f \in L^1(G)$ is called an *E-function* if $\hat{f}(\gamma) = 0$ for all $\gamma \notin E$ where

$$\hat{f}(\gamma) = \int_G f(x)\gamma(-x)\, dx, \qquad \gamma \in \Gamma;$$

dx is the Haar measure on G. A trigonometric polynomial that is also an E-function is called an *E-polynomial.*

DEFINITION. *E is a Sidon set if there is a finite constant B depending on E such that*

$$(1) \qquad \sum_{\gamma \in \Gamma} |\hat{f}(\gamma)| \leqslant B||f||_\infty \quad \textit{for every E-polynomial } f.$$

In §1 we discuss the sufficient arithmetic condition considered by Stečkin **(7)**, Hewitt and Zuckerman **(3)**, and Rudin **(6)**, which assures that E is a Sidon set. The hypotheses and conclusion are slightly improved. In particular it is shown that the characteristic function of such a Sidon set may be uniformly approximated by Fourier–Stieltjes transforms. This enables us to prove that the union of such a Sidon set and any other Sidon set is again a Sidon set.

Section 2 deals with the analogous question on spheres. S_2 will denote the surface of the unit sphere in Euclidean 3-space. If a function f on S_2 is integrable with respect to ordinary Lebesgue measure, then f is associated with a series of surface spherical harmonic polynomials:

$$(2) \qquad S[f](x) = \sum_{n=0}^{\infty} \hat{f}_n(x) \qquad \textbf{(1, } \text{Chapter 11}\textbf{)}.$$

If E is a subset of the natural numbers, then f is an E-function provided $\hat{f}_n = 0$ for all $n \notin E$. f is a polynomial if $\hat{f}_n = 0$ except for finitely many n. If f satisfies both, it is an E-polynomial. It is shown that there is no infinite set E and finite constant B such that

$$\sum_{n=0}^{\infty} ||\hat{f}_n||_\infty \leqslant B||f||_\infty \quad \text{for every } E\text{-polynomial } f.$$

We also show that there is no infinite-dimensional closed rotation-invariant subspace of $L^1(S_2)$ contained in $L^2(S_2)$.

If X is a locally compact space, $M(X)$ will be the space of all complex-valued regular Borel measures on X with finite total variation. For $\mu \in M(X)$, $||\mu||$ denotes the total variation of μ.

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1. Sidon sets for compact abelian groups.

1.1. The following two theorems concerning analytic properties of Sidon sets are well known (**5**, pp. 121, 123).

THEOREM 1.1. *Let E be a subset of the discrete group Γ. The following are equivalent:*

(a) *E is a Sidon set.*

(b) *Every bounded E-function has an absolutely convergent Fourier series.*

(c) *Every continuous E-function has an absolutely convergent Fourier series.*

(d) *For every bounded function ϕ on E there is a measure $\mu \in M(G)$ such that $\tilde{\mu}(\gamma) = \phi(\gamma)$ for all $\gamma \in E$.*

(e) *For every function ϕ on E that vanishes at infinity there is a function $f \in L^1(G)$ such that $\tilde{f}(\gamma) = \phi(\gamma)$ for all $\gamma \in E$.*

THEOREM 1.2. *A set E in the discrete group Γ is a Sidon set if to every function ϕ on E with $\phi(\gamma) = \pm 1$ there is a measure $\mu \in M(G)$ with*

$$\sup_{\gamma \in E} |\tilde{\mu}(\gamma) - \phi(\gamma)| < 1. \tag{3}$$

A set E is a Sidon set if and only if every countable subset of E is a Sidon set. Thus we can restrict ourselves to countable groups Γ.

DEFINITION 1.3. *Let $E \subset \Gamma$ and $\gamma_1, \gamma_2, \ldots$ be an enumeration of the elements of E. $R_s(E, \gamma)$ is the number of representations of γ in the form*

$$\gamma = \pm \gamma_{n_1} \pm \gamma_{n_2} \pm \ldots \pm \gamma_{n_s}, \qquad n_1 < n_2 < \ldots < n_s. \tag{4}$$

0 will denote the trivial character.

Rudin (**5**, p. 124) proves the following

THEOREM 1.4. *Let $E \subset \Gamma$ satisfy the following:*

(a) *If $\gamma \in E$ and $2\gamma \neq 0$, then $-\gamma \notin E$.*

(b) *There is a finite constant B and a decomposition of E into a finite union of disjoint sets $E_1, E_2, \ldots, E_t$, such that*

$$R_s(E_j, \gamma) \leqslant B^s \qquad (1 \leqslant j \leqslant t;\ s = 1, 2, 3, \ldots) \tag{5}$$

for all $\gamma \in E$ and for $\gamma = 0$. Then E is a Sidon set.

Stečkin, (**7**, p. 394) proves this for the circle T, provided (5) holds for all $\gamma \in Z$, the integers. Hewitt and Zuckerman (**3**) have shown it when $B = 1$. It is possible to omit (a) from the hypotheses, to weaken (b), and to strengthen the conclusion.

THEOREM 1.5. *Let $E \subset \Gamma$ and $0 < B < \infty$ be such that*

$$R_s(E, 0) \leqslant B^s \qquad (s = 1, 2, \ldots). \tag{6}$$

If $\phi(\gamma) = \pm 1$ *on* $E \cup (-E)$, *then for every* $\epsilon > 0$ *there exists* $\nu \in M(G)$ *such that*

$$\text{(7)} \qquad \begin{aligned} &|\tilde{\nu}(\gamma)| < \epsilon && (\gamma \notin E \cup (-E)),\\ &|\tilde{\nu}(\gamma) - \phi(\gamma)| < \epsilon && (\gamma \in E \cup (-E)). \end{aligned}$$

We shall show that (6) implies that there is a finite constant B_1 such that

$$\text{(8)} \qquad R_s(E, \gamma) \leqslant B_1^{\,s} \qquad (s = 1, 2, \ldots) \qquad \text{for all } \gamma \in \Gamma.$$

It follows from Theorem 1.2 and the conclusion of Theorem 1.5 that if E satisfies (6), then $E \cup (-E)$ is a Sidon set. It also is an immediate consequence that if E is the finite union of sets each of which satisfies (6), then $E \cup (-E)$ is a Sidon set. It is not known if every Sidon set is of this type. It is not even known if the union of two Sidon sets is always a Sidon set. However, it follows from (7) that if E is a set as in Theorem 1.5, then there are measures in $M(G)$ whose Fourier–Stieltjes transforms uniformly approximate the characteristic function of E in Γ.

This will allow us to prove

THEOREM 1.6. *If F is a Sidon set and E is a Sidon set of the type of* 1.5, *then* $E \cup F$ *is a Sidon set.*

1.2. *Proofs.*

LEMMA 1.7. *Let* $E \subset \Gamma$ *and* $1 \leqslant B < \infty$ *be such that*

$$R^s(E, 0) \leqslant B^s \ (s = 1, 2, \ldots).$$

Assume $\gamma \in E$ *and* $2\gamma \neq 0$ *implies* $-\gamma \notin E$. *Then*

$$\text{(9)} \qquad \sum_{s=1}^{\infty} (2B)^{-s} R_s(E, \gamma) \leqslant 2 \qquad \textit{for all } \gamma \in \Gamma.$$

It follows from (9) that

$$R_s(E, \gamma) \leqslant 2(2B)^s \qquad (s = 1, 2, \ldots; \gamma \in \Gamma).$$

Proof. Let $\beta = (2B)^{-1}$ and $\gamma_1, \gamma_2, \ldots$ be the elements of E. Let

$$f_k(x) = \begin{cases} 1 + \beta\gamma_k(x) + \beta\overline{\gamma_k(x)} & \text{if } 2\gamma_k \neq 0,\\ 1 + \beta\gamma_k(x) & \text{if } 2\gamma_k = 0, \end{cases}$$

and form the Riesz products

$$\text{(10)} \qquad P_N(x) = \prod_{k=1}^{N} f_k(x).$$

Since $\beta \leqslant \frac{1}{2}$ and $|\gamma_k(x)| = 1$, $P_N(x) \geqslant 0$. Expanding (10) we obtain

$$P_N(x) = 1 + \sum_{\gamma \in \Gamma} C_N(\gamma)\gamma(x)$$

where

$$|C_N(\gamma)| \leqslant \sum_{s=1}^{N} \beta^s \sum 1;$$

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the inner summation runs over all $\gamma_{n_1}, \gamma_{n_2}, \ldots, \gamma_{n_s}$ satisfying (4). In particular

$$|C_N(0)| \leqslant \sum_{s=1}^{N} \beta^s R_s(E, 0) \leqslant \sum_{s=1}^{N} (\beta B)^s \leqslant 1.$$

Since $P_N \geqslant 0$, $||P_N||_1 = 1 + C_N(0) \leqslant 2$. Thus

$$|\tilde{P}_N(\gamma)| \leqslant 2 \qquad \text{for all } \gamma \in \Gamma. \tag{11}$$

For $\gamma \neq 0$, $\tilde{P}_N(\gamma) = C_N(\gamma)$. Fix γ and let $N \to \infty$. It is easily seen that

$$\lim_{N\to\infty} C_N(\gamma) = \sum_{s=1}^{\infty} \beta^s R_s(E, \gamma).$$

Hence by (11),

$$\sum_{s=1}^{\infty} \beta^s R_s(E, \gamma) \leqslant 2 \qquad \text{for all } \gamma \in \Gamma.$$

Proof of Theorem 1.5. The proof follows closely that of Rudin (**5**, p. 125). Without loss of generality we may assume that $B \geqslant 1$, $0 \notin E$, and that $\gamma \in E$, $2\gamma \neq 0$ implies $-\gamma \notin E$.

By assumption, $R_s(E, 0) \leqslant B^s$ $(s = 1, 2, \ldots)$ so that by Lemma 1.7 we may assume (for a different B)

$$R_s(E, \gamma) \leqslant B^s \qquad (\gamma \in \Gamma; s = 1, 2, \ldots). \tag{12}$$

Let ϕ be a function on $E \cup (-E)$ such that $\phi(\gamma) = \pm 1$. Write $E = E^1 \cup E^2$ where

$$E^1 = \{\gamma : \gamma \in E \text{ and } \phi(\gamma) = \phi(-\gamma)\}$$

and

$$E^2 = \{\gamma : \gamma \in E \text{ and } \phi(\gamma) = -\phi(-\gamma)\}.$$

Let $\beta = (KB^2)^{-1}$ for some $K \geqslant 2$ and define

$$g(\gamma) = \begin{cases} \beta\phi(\gamma) & \text{if } \gamma \in E^1, \\ i\beta\phi(\gamma) & \text{if } \gamma \in E^2. \end{cases} \tag{13}$$

Let $\gamma_1, \gamma_2, \ldots$ be the elements of E_j $(j = 1, 2)$ and put

$$f_k(x) = \begin{cases} 1 + g(\gamma_k)\gamma_k(x) + \overline{g(\gamma_k)}(-\gamma_k)(x) & \text{if } 2\gamma_k \neq 0, \\ 1 + g(\gamma_k)\gamma_k(x) & \text{if } 2\gamma_k = 0. \end{cases} \tag{14}$$

Form the Riesz products

$$P_N(x) = \prod_{k=1}^{N} f_k(x).$$

Then as in (**5**, p. 125) a subsequence of $\{P_N\}$ converges weakly to a positive measure $\mu_j \in M(G)$ with the following properties:

(a) $\|\mu_j\| \leqslant \sup|\tilde{P}_N(0)| \leqslant 1 + \sum_2^\infty \beta^s R_s(E, 0)$.

(b) $$|\tilde{\mu}_j(\gamma_k) - g(\gamma_k)| \leqslant \sup_N |\tilde{P}_N(\gamma_k) - g(\gamma_k)| \leqslant \sum_2^\infty \beta^s R_s(E, \gamma_k) \qquad \text{if } \gamma_k \in E^j.$$

(c) $|\tilde{\mu}_j(-\gamma_k) - g(\gamma_k)| \leqslant \sum_2^\infty \beta^s R_s(E, \gamma_k)$ if $\gamma_k \in E^j$.

(d) $|\tilde{\mu}_j(\gamma)| \leqslant \sum_2^\infty \beta^s R_s(E, \gamma)$ if $\gamma \notin E^j \cup (-E^j) \cup \{0\}$.

But by (12)

$$\sum_2^\infty \beta^s R_s(E, \gamma) \leqslant \sum_2^\infty (\beta B)^s = \frac{(\beta B)^2}{1 - \beta B} < (K(K-1)B^2)^{-1}$$

so that if $\mu = \mu_1 - i\mu_2$, then by (13)

$$|\tilde{\mu}(\gamma) - \beta\phi(\gamma)| \leqslant 2(B^2K(K-1))^{-1} \qquad \text{if } \gamma \in E \cup (-E)$$

and

$$|\tilde{\mu}(\gamma)| \leqslant 2(B^2K(K-1))^{-1} \qquad \text{if } \gamma \notin E \cup (-E) \cup \{0\}.$$

Let $\nu = \mu/\beta$. Then

$$(15) \qquad \begin{array}{ll} |\tilde{\nu}(\gamma) - \phi(\gamma)| \leqslant 2(K-1)^{-1} & \text{if } \gamma \in E \cup (-E), \\ |\tilde{\nu}(\gamma)| \leqslant 2(K-1)^{-1} & \text{if } \gamma \notin E \cup (-E) \cup \{0\}. \end{array}$$

Given $\epsilon > 0$, choose K so large that $2(K-1)^{-1} < \epsilon$; then by adding a constant multiple of Haar measure to ν, we obtain the desired measure.

Proof of Theorem 1.6. Let F be any Sidon set and E a Sidon set as in Theorem 1.5. We may assume that $E = E \cup (-E)$, $E \cap F = \emptyset$, and $0 \notin E \cup F$. Given $\epsilon > 0$, the theorem above shows that there is a measure $\mu_\epsilon \in M(G)$ such that

$$(16) \qquad \sup_{\gamma \in \Gamma} |\tilde{\mu}_\epsilon(\gamma) - \phi(\gamma)| < \epsilon$$

where ϕ is the characteristic function of E.

Let b be a function on $E \cup F$ such that $b(\gamma) = \pm 1$. By Theorem 1.2 (d), there is $\mu_1 \in M(G)$ such that $\tilde{\mu}_1(\gamma) = b(\gamma)$ for all $\gamma \in F$. Similarly, there is $\mu_2 \in M(G)$ such that

$$\tilde{\mu}_2(\gamma) = -\mu_1(\gamma) + b(\gamma)$$

for all $\gamma \in E$. Let $\mu = \mu_1 + \mu_2 * \mu_\epsilon$ where

$$(17) \qquad \epsilon < \tfrac{1}{2} \min [\|\mu_2\|^{-1}, (\|\mu_1\| + 1)^{-1}].$$

Then

(18) $|\tilde{\mu}(\gamma) - b(\gamma)| = |\mu_2 * \mu_\epsilon(\gamma)| \leqslant ||\mu_2||\epsilon < \frac{1}{2}$ for $\gamma \in F$

and

(19) $$|\tilde{\mu}(\gamma) - b(\gamma)| = |\tilde{\mu}_1(\gamma) - b(\gamma) + (-\tilde{\mu}_1(\gamma) + b(\gamma))\tilde{\mu}_\epsilon(\gamma)| \leqslant |1 - \tilde{\mu}_\epsilon(\gamma)||\tilde{\mu}_1(\gamma) - b(\gamma)| < \tfrac{1}{2} \quad \text{for } \gamma \in E.$$

By Theorem 1.2, $E \cup F$ is a Sidon set.

1.3. Remarks. The following gives an equivalent statement for the hypotheses of Theorem 1.5. If there is $\gamma^* \in E$ such that $R_s(E, \gamma^*) \leqslant B^s$, $s = 1, 2, 3, \ldots$, then $R_s(E, 0) \leqslant 3B^{s+1}$, $s = 1, 2, \ldots$. For suppose

(20) $$0 = \sum_1^s \pm\gamma_{nk}, \qquad \gamma_{nk} \in E; n_1 < n_2 < \ldots < n_s.$$

Then there are two possibilities. If $\pm\gamma^*$ appears in the sum in (20), then we have a way of writing

$$\pm\gamma^* = \sum_1^{s-1} \pm\gamma_{nk}, \qquad n_1 < n_2 < \ldots < n_{s-1}.$$

There are at most $2R_{s-1}(E, \gamma^*)$ of these. If $\pm\gamma^*$ does not appear in (20), then by adding γ^* to each side we have a way of writing

$$\gamma^* = \sum_1^{s+1} \pm\gamma_{nk}, \qquad n_1 < n_2 < \ldots < n_k.$$

There are at most $R_{s+1}(E, \gamma^*)$ of these. Thus

$$R_s(E, 0) \leqslant 2R_{s-1}(E, \gamma^*) + R_{s+1}(E, \gamma^*) \leqslant 2B^{s-1} + B^{s+1} \leqslant 3B^{s+1}.$$

In the same way it can be shown that the condition for Theorem 1.5 is invariant when E is translated by an element of Γ (**3**, p. **7**).

2. Sidon sets for S_2.

2.1. If $f \in L^1(S_2)$, then f is associated with a series of harmonic polynomials

(21) $$S[f]x = \sum_{n=0}^{\infty} \tilde{f}_n(x)$$

where

(22) $$\tilde{f}_n(x) = (2n + 1) \int_{S_2} P_n(\langle x, y\rangle)f(y)\, dy.$$

P_n are the Legendre polynomials given by

(23) $$(1 - 2v\cos\theta + v^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} v^n P_n(\cos\theta).$$

$\langle x, y\rangle$ is the scalar product of x and y as vectors in E_3.

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Define $\mathfrak{P}_n$ to be the set of all such $\tilde{f}_n$. It is well known that $\mathfrak{P}_n$ contains the function $f(x) = P_n(\langle x, y_0\rangle)$ for each y_0 in S_2 and that $\mathfrak{P}_n$ is the smallest rotation-invariant subspace of $L^2(S_2)$ containing $P_n(\langle x, y_0\rangle)$. Also if $f \in \mathfrak{P}_n$, then

$$f(x) = (2n+1) \int_{S_2} P_n(\langle x, y\rangle) f(y)\, dy. \tag{24}$$

In particular,

$$P_n(\langle x, z\rangle) = (2n+1) \int_{S_2} P_n(\langle x, y\rangle) P_n(\langle z, y\rangle)\, dy. \tag{25}$$

If $x \in S_2$, x' will denote the point antipodal to x, i.e. $\langle x, x'\rangle = -1$.

The question may be asked: Does there exist an infinite set of integers E and a finite constant B such that if f is an E-polynomial on S_2, then

$$\sum_{n=1}^{\infty} \|\tilde{f}_n\|_\infty \leqslant B\|f\|_\infty? \tag{26}$$

The answer is negative. For assume that (26) holds for every E-polynomial and let f be a bounded E-function. Let $\sigma_N{}^2(f; x)$ be the second Cesàro means of

$$S(f) = \sum_{n=0}^{\infty} \tilde{f}_n.$$

Then

$$\sigma_N{}^2(f; x) = \sum_{n=0}^{N} \tilde{f}_n(x) a(N; n) = \int_{S_2} f(y) K_N(\langle x, y\rangle)\, dy \tag{27}$$

where $a(N; n) \to 1$ as $N \to \infty$, $K_N \geqslant 0$, and

$$\int_{S_2} K_N(\langle x, y\rangle)\, dy = 1$$

(cf. **2**, p. 81). Thus $\|\sigma_N{}^2(f)\|_\infty \leqslant \|f\|_\infty$. But $\sigma_N{}^2(f)$ is an E-polynomial so that by (26)

$$\sum_{n=0}^{\infty} \|\sigma_N{}^2(f)_n\|_\infty \leqslant B\|\sigma_N{}^2(f)\|_\infty \leqslant B\|f\|_\infty. \tag{28}$$

Letting $N \to \infty$, we see from (27) and (28) that (26) must hold for all bounded E-functions. This is impossible by

THEOREM 2.1. *Suppose E is an infinite set of integers. Then there is a bounded E-function f on S_2 such that $\|\tilde{f}_{n_k}\|_\infty = 1$ for an infinite number of $n_k \in E$. Furthermore, f can be chosen so that it is continuous except at two points.*

Proof. Choose a sequence of distinct points of S_2 converging to some point $x_0 \in S_2$; say $x_1, x_2, \ldots$. Choose a neighbourhood U_k about x_k so small that if U'_k is the set of points antipodal to U_k, then none of the U_k and U'_j overlap. By (**4**, p. 311) we can choose $n_k \in E$ so large that

$$|P_{n_k}(\langle x, x_k\rangle)| \leqslant 2^{-k} \qquad \text{for } x \notin U_k \cup U'_k. \tag{29}$$

Then

$$\sum_{k=1}^{\infty} P_{n_k}(\langle x, x_k \rangle)$$

converges uniformly on compact sets of S_2 that miss x_0 and x'_0. Furthermore, since each $x \in S_2$ is in at most one $U_k \cup U'_k$, (29) implies that

$$\left| \sum_{k=1}^{\infty} P_{n_k}(\langle x, x_k \rangle) \right| \leqslant 1 + \sum_{k=1}^{\infty} 2^{-k} = 2. \tag{30}$$

Since $||P_n||_\infty = 1$,

$$f(x) = \sum_{k=1}^{\infty} P_{n_k}(\langle x, x_k \rangle)$$

is the desired function.

A set of integers $\{n_k\}$ for which there is λ with

$$\frac{n_{k+1}}{n_k} > \lambda > 1 \qquad (k = 1, 2, 3, \ldots)$$

is called a *Hadamard* set. If E is a Hadamard set, it is not possible to find a continuous function satisfying the conclusion of Theorem 2.1.

THEOREM 2.2. *If E is a Hadamard set, then every continuous E-function has a uniformly convergent Laplace series. That is, if f is an E-function, then*

$$\sum_{n=0}^{N} \tilde{f}_n(x) \to f(x)$$

uniformly as $N \to \infty$.

In particular $||\tilde{f}_n||_\infty \to 0$ as $n \to \infty$, for such a function.

Proof. Gronwall (**4**, p. 351) proves that the first Cesàro means of the Laplace series of a continuous function f on S_2 converges to f uniformly. By a theorem of Kolmogoroff (**8**, p. 79), a uniformly Cesàro summable series with its support on a Hadamard set has uniformly convergent partial sums.

It is always possible to find a continuous E-function such that $\sum ||f_n||_\infty = \infty$. We need only consider

$$f(x) = \sum \frac{1}{k} \cdot P_{n_k}(\langle x, x_k \rangle)$$

where $\{n_k\}$ and $\{x_k\}$ are as in the proof of Theorem 2.1.

2.2. If E is a subset of the discrete abelian group Γ and E is a Sidon set, then every E-function $f \in L^1(G)$ is also in $L^p(G)$ $(1 \leqslant p < \infty)$ (cf. **5**, p. 128). Since every infinite compact abelian group G has Sidon sets (**5**, p. 126), this shows that there are infinite-dimensional closed translation-invariant subspaces of $L^1(G)$ contained in $L^2(G)$. Hewitt and Zuckerman (**3**, p. 15) consider this problem (without the condition of being translation-invariant) when G is not necessarily abelian.

We may consider the same problem on S_2: Does there exist an infinite-dimensional closed rotation-invariant subspace of $L^1(S_2)$ that is contained in $L^2(S_2)$? The answer is negative.

We shall show that there exists a sequence $\{Y_n\}$ $(Y_n \in \mathfrak{P}_n)$ such that

$$\frac{||Y_n||_2}{||Y_n||_1} > C.\, n^{1/4} \tag{31}$$

for some positive constant C. If a closed rotation-invariant subspace X of $L^1(S_2)$ contains a function f with $\tilde{f}_n \neq 0$, then X contains all of $\mathfrak{P}_n$ and hence Y_n. If $X \subset L^2(S_2)$, then $|| \; ||_1$ and $|| \; ||_2$ are equivalent norms on X so that there is a finite constant B with

$$||f||_2 \leqslant B||f||_1 \qquad \text{for all } f \in X. \tag{32}$$

If X is infinite-dimensional, it must contain infinitely many of the Y_n. Equations (31) and (32) then give a contradiction.

The Y_n are defined by

$$Y_n(\theta, \phi) = \cos n\phi \, (\sin \theta)^n. \tag{33}$$

$Y_n \in \mathfrak{P}_n$ (**4**, pp. 95, 122). It is easy to calculate

$$\begin{aligned} ||Y_n||_2^2 &= \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{\pi} (\cos n\phi)^2 (\sin \theta)^{2n} \sin \theta \, d\theta d\phi \\ &= \frac{1}{4\sqrt{\pi}} \frac{\Gamma(n+1)}{\Gamma(n+3/2)} \end{aligned} \tag{34}$$

and

$$\begin{aligned} ||Y_n||_1 &= \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{\pi} |\cos n\phi| (\sin \theta)^n \sin \theta \, d\theta d\phi \\ &= (\pi)^{-(3/2)} \frac{\Gamma(\frac{1}{2}n+1)}{\Gamma(\frac{1}{2}n+3/2)}. \end{aligned} \tag{35}$$

It is known that

$$\frac{\Gamma(t) t^{\frac{1}{2}}}{\Gamma(t+\frac{1}{2})} \to c \neq 0 \qquad \text{as } t \to \infty.$$

Thus (34) and (35) imply (31).

These results, appropriately modified, hold also for the surface of the unit sphere in Euclidean K space, $K > 3$.

References

1. A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher transcendental functions*, Vol. I (New York, 1953).
2. L. Féjer, *Über die Laplacesche Reihe*, Math. Ann., *67* (1909), 76–109.
3. E. Hewitt and H. S. Zuckerman, *Some theorems on lacunary Fourier series, with extensions to compact groups*, Trans. Amer. Math. Soc., *93* (1959), 1–19.

4. E. W. Hobson, *The theory of spherical and ellipsoidal harmonics* (New York, 1955).
5. W. Rudin, *Fourier analysis on groups* (New York, 1962).
6. ——— *Trigonometric series with gaps*, J. Math. Mech., *9* (1960), 203–228.
7. S. B. Stečkin, *On absclute convergence of Fourier series*, Izv. Akad. Nauk SSSR, Ser. Mat., *20* (1956), 385–412.
8. A. Zygmund, *Trigonometric series*, 2nd ed., Vol. I (Cambridge, 1959).

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