## Solution of the Cubic Equation.

By M. Edouard Collignon.

(Abstract.)

§ 1. The roots of the cubic equation

$$x^3 + px + q = 0$$

are given by the formula

$$x = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}.$$

This solution is of little practical use when the roots of the cubic are all real and unequal, that is, when $\frac{q^2}{4} + \frac{p^3}{27}$ is negative (*the Irreducible Case of Cardan's Solution*).

The object of this paper is to show how the real roots of the cubic may be found by making use of tables of values of the functions $x^3$, $x^3 + x$, $x^3 - x$.

§ 2. By substituting $\theta x'$ for $x$ in the equation

$$x^3 + px + q = 0,$$

and making $\left| \frac{p}{\theta^2} \right| = 1,$

that is, choosing $\theta = \pm \sqrt{p},$

the given equation can be reduced to one of the three forms

$$\text{I.} \begin{cases} x^3 + x = \text{A}, \\ x^3 \phantom{+ x} = \text{A}, \\ x^3 - x = \text{A}; \end{cases}$$ where A is positive.

The abscissæ of the points in which the straight line $y = \text{A}$ cuts the three curves

$$y = x^3 + x,$$
$$y = x^3 \quad ,$$
$$y = x^3 - x,$$

are the real roots of the equations I.

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FIGURE 1.

If $A < MP$, Pbeing a turning-point of the curve $y = x^3 - x$, the equation

$$x^3 - x = A$$

has three real roots;

if $A = MP$, the equation has three real roots, two of which are equal;

if $A > MP$, the equation has only one real root.

§ 3. Discussion of the curves

$$y = x^3 - x, \qquad (1)$$
$$y = x^3 \quad , \qquad (2)$$
$$y = x^3 + x. \qquad (3)$$

(1) The ordinate for the curve $y = x^3$ is the mean of the ordinates for the other two curves; the same is true of $\frac{dy}{dx}$ and of $\int y dx$.

(2) Hence the following theorem:—

The tangents drawn to the three curves at points which have the same abscissa are concurrent.

FIGURE 2.

If $x = OM$, the tangents intersect at a point F on the $y$-axis,

$$\text{and} \qquad OF = -2OM^3$$
$$= -2MP_2.$$
$$\text{Hence} \qquad FH = 2HP_2,$$
$$\text{and} \qquad OH = 2HM.$$

We have thus a method of drawing the tangents at $P_1$, $P_2$ and $P_3$.

Take $MH = \frac{1}{3}MO$; join $P_2H$ and produce it to cut $Oy$ in F; join $FP_1$ and $FP_3$.

(3) It may be easily shown that the area of the triangle $P_1FP_3 = OM^2$; and that the area bounded by OM, $MP_2$ and the curve $y = x^3$, namely, $\int_0^x x^3 dx = \frac{3}{4}$ of the rectangle $MP_2 . MH$.

(4) If $\phi_1$, $\phi_2$, $\phi_3$ be the angles made with $Ox$ by $P_1F$, $P_2F$, and $P_3F$, and $\psi$ the $\angle P_1FP_3$, we have

$$\psi = \phi_3 - \phi_1, \quad \tan\phi_1 = 3x^2 - 1,$$
$$\tan\phi_2 = 3x^2 \quad ,$$
$$\tan\phi_3 = 3x^2 + 1,$$

from which we find

$$\tan\psi = \frac{2}{9x^4} = \frac{2}{\tan^2\phi_2}.$$

FIGURE 3.

§ 4. $\mu_1$, $\mu$ are turning-points on the curve $y = x^3 - x$; their coordinates are

$$\pm\frac{1}{\sqrt{3}}, \ \mp\frac{2}{3\sqrt{3}}, \ \text{or} \ \pm 0{\cdot}57735, \ \mp 0{\cdot}38490.$$

The tangent at $\mu_1$ is parallel to $Ox$, hence for this point the angle $P_1FP_3$ (see last figure) is $\tan^{-1}2$; and therefore $\tan\phi_2 = 1$, that is, the tangent to $y = x^3$ makes an angle of 45° with $Ox$.

If in the equation $x^3 - x = A$,

$$A = P\mu = 0{\cdot}38490 \ldots,$$

the roots are OP, OP and OP′;

now $$OP = -\frac{1}{\sqrt{3}},$$

and since the sum of the roots is zero,

$$\therefore \quad OP' = \frac{2}{\sqrt{3}}.$$

The *Irreducible Case* corresponds to the region of the curve between the two points $\mu'$ and $\mu_1'$ whose coordinates are

$$\pm\frac{2}{\sqrt{3}}, \ \pm\frac{2}{3\sqrt{3}}, \ \text{that is,} \ \pm 1{\cdot}1547, \ \pm 0{\cdot}385.$$

§ 5. Tables may easily be drawn up giving the values of $x^3 - x$, $x^3$ and $x^3 + x$ for a certain number of values of $x$; and from these tables we can find values of $x$ between which the roots of the cubic must lie.

Taking, for example, the cubic

$$x^3 - x = \frac{1}{\sqrt{7}} = 0{\cdot}378\ldots,$$

since $\frac{1}{\sqrt{7}}$ is less than $\frac{2}{3\sqrt{3}}$, that is, 0·385, (see § 4)

the given equation must have 3 real roots, one positive and two negative; from a table giving the values of $x^3 - x$ for values of $x$ between 0 and 1·5 differing by ·1, (or from the curve $y = x^3 - x$ itself) we find that one root lies between

$$x = -0{\cdot}6 \text{ and } x = -0{\cdot}7,$$

since for $x = 0{\cdot}6,\ x^3 - x = -0{\cdot}384$

and for $x = 0{\cdot}7,\ x^3 - x = -0{\cdot}357$;

a second root lies between

$$x = -0{\cdot}5 \text{ and } x = -0{\cdot}6,$$

since for $x = 0{\cdot}5,\ x^3 - x = -0{\cdot}375$

and for $x = 0{\cdot}6,\ x^3 - x = -0{\cdot}384$;

the third root lies between

$$x = 1{\cdot}1 \text{ and } x = 1{\cdot}2.$$

Closer approximations to the roots of the cubic can now be found in several ways.

(1) By completing the table of values of $x^3 \pm x$ for values of $x$ lying between the limiting values obtained for the roots,

(2) or by a special method, such as the following (Newton's method).

Take, for example, the equation

$$x^3 + x = \text{A};$$

suppose A so great that $x$ is small compared with $x^3$; a first approximation to $x$ will thus be got by taking $x^3 = \text{A}$, which gives

$$x = \sqrt[3]{\text{A}} = a, \text{ say.}$$

FIGURE 4.

Let $\text{OP} = a$, then $\text{PM} = a^3 = \text{A}$,

and $\text{PN} = a^3 + a$;

the root sought is OR, where $\text{RL} = \text{PM} = \text{A}$.

Draw NB a tangent at N to $y = x^3 + x$;
let $x_1 = OP_1 =$ abscissa of B.

$$\tan NBM = 3a^2 + 1,$$

$$\therefore \quad BM = \frac{a}{3a^2+1};$$

$$\therefore \quad x_1 = OP_1 = a - \frac{a}{3a^2+1} = \frac{3a^3}{3a^2+1}.$$

This is a closer approximation to the root.

Repeat the process by drawing the tangent at B′, the point in which $P_1B$ meets the curve $y = x^3 + x$; let it cut ML at B″ and let $x_2 = OP_2$ be abscissa of B″.

It can easily be shown that

$$x_2 = \frac{2x_1^3 + a^3}{3x_1^2 + 1};$$

$x_2$ is a third approximation to the root.

Similarly we find

$$x_3 = \frac{2x_2^3 + a^3}{3x_2^2 + 1}$$

$$\cdots\cdots \quad \cdots\cdots\cdots$$

$$x_{n+1} = (2x_n^3 + a^3)/(3x_n^2 + 1).$$

A similar method can be applied to the equation

$$x^3 - x = A$$

and gives the following approximations to $x$:—

$$x = \sqrt[3]{A} = a, \text{ say},$$

$$x_1 = \frac{3a^3}{3a^2 - 1},$$

$$x_2 = \frac{2x_1^3 + a^3}{3x_1^2 - 1},$$

$$\cdots\cdots$$

$$x_{n+1} = \frac{2x_n^3 + a^3}{3x_n^2 - 1}.$$

*Example*:—To find approximations to the roots of the equation

$$x^3 - 2x - 5 = 0.$$

Put $\theta x' = x$;

we get $$\theta^3 x'^3 - 2\theta x' - 5 = 0,$$

that is, $$x'^3 - \frac{2}{\theta^2}x' = \frac{5}{\theta^3}.$$

Choosing $$\theta = \sqrt{2},$$

we get $$x'^3 - x' = \frac{5}{2\sqrt{2}} = 1{\cdot}7678.$$

$$1{\cdot}7678 \text{ is } > \frac{2}{3\sqrt{3}};$$

the equation has therefore only one real root and it is positive.

From the table of values of $x^3 - x$ we find that the root lies between 1·4 and 1·5; completing the table for values of $x$ between 1·4 and 1·5 we find that $x' = 1{\cdot}48$ to the nearest hundredth.

By applying the method explained above we find the following closer approximations to the root

$$1{\cdot}481, \quad 1{\cdot}4811, \quad 1{\cdot}48107.$$

Taking $$x' = 1{\cdot}48107$$

we get
$$\begin{aligned} x &= \sqrt{2} \times 1{\cdot}48107 \\ &= 2{\cdot}09455. \end{aligned}$$

Closer approximations can easily be found.

Lagrange and Newton, both of whom solved this equation, give

$$x = 2{\cdot}09455147.$$

When the only real root $a$ of a cubic $x^3 - x - A = 0$ has been found, the two imaginary roots can be deduced.

Let them be $\alpha \pm i\beta$.

The sum of the three roots being zero, we have

$$a + 2\alpha = 0;$$

$$\therefore \quad \alpha = -\frac{a}{2};$$

the product of the roots being A, we have

$$a(\alpha^2 + \beta^2) = A,$$

and ∴ $$\beta = \pm\sqrt{\frac{A}{a} - \alpha^2} = \sqrt{\frac{A}{a} - \frac{a^2}{4}}.$$

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§ 6. *Equations of higher degree than the 3rd.*

An equation of the 4th degree may be reduced to one of the forms

$$x^4 + x^2 + px + q = 0,$$
$$x^4 \qquad + px + q = 0,$$
$$x^4 - x^2 + px + q = 0,$$

so that the real roots may be found by drawing the curves $y = x^4 \pm x^2$ and $y = x^4$ and the straight line $y = -(px+q)$.

The equation of the 5th degree can be brought to one of the three forms

$$x^5 - x^3 + px^2 + qx + r = 0,$$
$$x^5 + x^3 + px^2 + qx + r = 0,$$
$$x^5 \qquad + px^2 + qx + r = 0.$$

The real roots will be the abscissæ of the points of intersection of one of the curves

$$y = x^5 - x^3, \quad y = x^5 + x^3, \quad y = x^5$$

and the parabola

$$y = -(px^2 + qx + r).$$

In general the problem of finding the real roots of an equation of the $m$th degree is reduced to that of finding the points of intersection of a curve of degree $m-3$ and one of the curves

$$y = x^m - x^{m-2}, \quad y = x^m, \quad y = x^m + x^{m-2}.$$

§ 7. *Discussion of the curves*

$$y = x^m - x^{m-2}, \quad y = x^m, \quad y = x^m + x^{m-2}.$$

(1) If $y_1$, $y_2$, $y_3$ be the 3 ordinates for the same abscissa,

$$2y_2 = y_1 + y_3,$$

$$2\frac{dy_2}{dx} = \frac{dy_1}{dx} + \frac{dy_3}{dx},$$

and $$2\int_0^x y_2 dx = \int_0^x y_1 dx + \int_0^x y_3 dx.$$

(2) If $x = 1$, $y_1 = 0$ and $\frac{dy_1}{dx} = 2$; this value of $\frac{dy_1}{dx}$ is independent of $m$, so that the curves $y = x^m - x^{m-2}$, for all values of $m$, have a common tangent at the point (1, 0).

(3) At the origin, if $m > 3$, the curves touch the axis of $x$; if $m$ is even the three curves are symmetrical with respect to $Oy$; if $m$ is odd the curves are symmetrical with respect to the origin which must therefore be a point of inflexion on each of the curves; the radius of curvature is infinite at the origin if $m > 2$.

(4) For the curve $y = x^m - x^{m-2}$

$$\frac{dy}{dx} = 0 \text{ where } mx^{m-1} - (m-2)x^{m-3} = 0.$$

This equation has $m-3$ roots equal to 0, which define the point of contact of the curve and $Ox$ (if $m > 3$); and two other roots $x = \pm\sqrt{\dfrac{m-2}{m}}$; at the points whose abscissæ are $\pm\sqrt{\dfrac{m-2}{m}}$ the tangent to the curve is parallel to $Ox$; the value of the ordinate at these points is given by

$$y = \left(\frac{m-2}{m}\right)^{\frac{m}{2}} - \left(\frac{m-2}{m}\right)^{\frac{m}{2}-1}.$$

For the locus of the points at which the tangents to the curves $y = x^m - x^{m-2}$ (for different values of $m$) are parallel to $Ox$ we have the equation

$$y = x^{\frac{2}{1-x^2}} - x^{\frac{2x^2}{1-x^2}}.$$

(5) It may be shown as in the case of the curves

$$y = x^3 - x, \quad y = x^3, \quad y = x^3 + x$$

that the tangents to the curves

$$y = x^m - x^{m-2}, \quad y = x^m, \quad y = x^m + x^{m-2}$$

at points which have the same abscissa, are concurrent, intersecting at

$$\left(\frac{m-3}{m-2}x, \; -\frac{2x^m}{m-2}\right);$$

the locus of this point for different values of $x$ is the curve

$$y = -\frac{2(m-2)^{m-1}}{(m-3)^m}x^m.$$

If $m = 3$, this equation becomes $x = 0$ (the particular case already noticed).

For the curve $y = x^m$, the subtangent is $\dfrac{x}{m}$, a given fraction of the abscissa. This leads to a method of drawing the tangents to the three curves at points which have a common abscissa.

Let OP be the abscissa and $M_1$, $M_2$, $M_3$ the three points in which the ordinate through P cuts the curves. Take PG (along $xO$) $=\dfrac{OP}{m}$ and join $GM_2$; this line will touch $y=x^m$.

Let F be the point of intersection of the tangents and OH its abscissa; then $OH=\dfrac{m-3}{m-2}x=\dfrac{m-3}{m-2}OP$ and $PH=\dfrac{OP}{m-2}$; besides $\dfrac{M_2F}{M_2G}=\dfrac{PH}{PG}=\dfrac{m}{m-2}$; so that to get F, produce $M_2G$ a distance $=\dfrac{2}{m-2}M_2G$; then join $FM_1$, $FM_3$; these lines touch the curves $y=x^m-x^{m-2}$ and $y=x^m+x^{m-2}$.

The area of the triangle $M_1FM_3=\frac{1}{2}M_1M_3 \,.\, HP$

$$=\frac{x^{m-1}}{m-2},$$

$$=\frac{m-1}{m-2}\int_0^x x^{m-2}dx.$$

(6) *Radius of curvature.*

For the curve $y=x^m$ we find

$$\rho=\frac{\left(1+m^2\dfrac{y^2}{x^2}\right)^{3/2}}{m(m-1)\dfrac{y}{x^2}}.$$

FIGURE 5.

Let M be a point on the curve, MG the tangent at M, MP the ordinate, MN the normal.

$$GP=\frac{x}{m};$$

$$\therefore\quad 1+m^2\frac{y^2}{x^2}=1+\frac{MP^2}{GP^2}=\frac{GM^2}{GP^2};$$

$$\therefore\quad \rho=\frac{m}{m-1}\,\frac{GM^3}{GP.PM}.$$

Draw GR, $\perp$ to MG to meet MP produced at R.

Then $$GM^2 = MR \,.\, MP\,;$$

$$\therefore\quad \rho = \frac{m}{m-1}\,\frac{MR.GM}{GP}$$

$$= \frac{MR}{\cos\beta} \times \frac{m}{m-1}\,;\ (\text{where } \beta = \angle MGx).$$

Draw RS parallel to $Ox$ to meet MN at S.

Then $$\rho = \frac{MR}{\cos\beta} \times \frac{m}{m-1} = MS \times \frac{m}{m-1}.$$

Hence to find C the centre of curvature, produce NM to C so that $MC = \frac{m}{m-1} \times MS$.

In the case of the parabola $y = x^2$, $\rho = 2MS$, so that S lies on the directrix ($y = -\frac{1}{4}$) of the parabola.

§ 8. To construct the curve $y = xf(x)$ from the curve $y = f(x)$.

Figure 6.

Let OM be the curve $y = f(x)$, M a point on it whose abscissa is OP.

Take, in the direction XO, a length $PQ = 1$.

Join MQ. Draw ON parallel to QM to meet PM at N.

Then $$\frac{PN}{PM} = \frac{PO}{PQ}\,;$$

$$\therefore\quad PN = \frac{PO\,.\,PM}{PQ} = xf(x).$$

Hence N is a point on the curve $y = xf(x)$.

If we can draw the tangent at $(x, f(x))$ it will be possible to draw the tangent at $(x, xf(x))$.

Let $z = xf(x)$ be the equation of the curve constructed from $y = f(x)$.

Putting $z = xy$, we get

$$\frac{dz}{dx} = y + x\frac{dy}{dx},$$

that is, $$\tan\beta = y + x\tan\alpha$$

where $$\tan\beta = \frac{dz}{dx} \text{ and } \tan\alpha = \frac{dy}{dx}.$$

Hence if OS be drawn parallel to MR, the tangent at M to $y=f(x)$, to cut PM produced at S and SS′ be taken $=$ PM, PS′ will $=y+x\tan\alpha$; if S′Q be joined, since PQ $=1$, S′Q will be parallel to the tangent at N to the curve $z=xf(x)$. The tangent required is ∴ NT, drawn parallel to S′Q.

§ 9. *Equations of higher degree than the 5th.*

To find the real roots of an equation of the $m$th degree we have to find the points of intersection of one of the curves

$$y=x^m-x^{m-2},\quad y=x^m,\quad y=x^m+x^{m-2}$$

and of a parabolic curve of degree $m-3$ at most.

A combination of this method and of the following will often simplify the solution. It consists in substituting for the curve whose equation is

$$y=\mathrm{A}x^m+\mathrm{B}x^{m-1}+\ldots+\mathrm{G}x^{m-n}+\mathrm{H}x^{m-n-1}+\ldots+\mathrm{P}$$

the curve whose equation is

$$z=\frac{x^{m-n}(\mathrm{A}x^n+\mathrm{B}x^{n-1}+\ldots+\mathrm{G})}{\mathrm{H}x^{m-n-1}+\ldots+\mathrm{P}}$$

and drawing the straight line $z=-1$, to cut it.

The solution of an equation of degree $m$ may thus be reduced to the solution of equations of much lower degrees.

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## On the Teaching of Geometry.

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