ON A RELATION BETWEEN INJECTORS AND CERTAIN COMPLEMENTED CHIEF FACTORS OF FINITE SOLUBLE GROUPS

Dedicated to the memory of Hanna Neumann

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1. Introduction

The Fitting class $\mathfrak{S}_\pi$ of finite soluble π-groups, where π is an arbitrary set of primes, has the property that each complement of an $\mathfrak{S}_\pi$-avoided, complemented chief factor of any finite soluble group G contains an $\mathfrak{S}_\pi$-injector of G. In other words, each $\mathfrak{S}_\pi$-avoided, complemented chief factor of G is $\mathfrak{S}_\pi$-complemented in the sense of Hartley (see [2]).

In general, for a Fitting class $\mathfrak{X}$ of finite soluble groups, none of the complements of an $\mathfrak{X}$-avoided, complemented chief factor of a finite soluble group G may contain an $\mathfrak{X}$-injector of G, as an example in Section 2 of [3] shows. As in [3], we will call an $\mathfrak{X}$-avoided, complemented chief factor of G a partially $\mathfrak{X}$-complemented chief factor of G if at least one of its complements contains an $\mathfrak{X}$-injector of G. Moreover,

DEFINITION. A Fitting class $\mathfrak{X}$ of finite soluble groups will be said to have the property (Λ) ((Λ^*)) if in each finite soluble group G every $\mathfrak{X}$-avoided, complemented chief factor of G is necessarily a partially $\mathfrak{X}$-complemented (an $\mathfrak{X}$-complemented) chief factor of G.

For the rest of the terminology used here we refer the readers to Hartley [2]. All groups considered here are finite and soluble.

Our main purpose of this note is to show that

THEOREM 1.1. *A Fischer class has the property* (Λ) *if and only if it is the Fischer class of* π*-groups for some suitable set* π *of primes.*

In general, one can have a Fitting class which has the property (Λ) but which is not $\mathfrak{S}_\pi$ for any set π of primes. The normal Fitting class $\mathfrak{H}$ defined in Satz 3.2 of Blessenohl and Gaschütz [1] provides an example of such a Fitting class.

The $\mathfrak{H}$-injector V of any group G has index at most 2 in G. Hence, if R/S is an $\mathfrak{H}$-avoided chief factor of G, then $V = VS$ complements R/S in G, and so R/S is also a partially $\mathfrak{H}$-complemented chief factor of G, Thus, $\mathfrak{H}$ has the property (Λ), but it is easy to check that $\mathfrak{H}$ is not $\mathfrak{S}_\pi$ for any set π of primes.

Theorem 1.1 is proved in Section 3 and in Section 2 we discuss Fitting classes with the property (Λ^*).

2. Fitting classes with the property (Λ^*).

In this section, we show that a Fitting class with the property (Λ^*) is necessarily $\mathfrak{S}_\pi$ for some suitable set π of primes.

THEOREM 2.1. *Let $\mathfrak{F}$ be a Fitting class with the property* (Λ^*). *Then* $\mathfrak{F} = \mathfrak{S}_\pi$ *for some suitable set π of primes.*

PROOF. Let π be the uniquely determined set of primes such that $\mathfrak{N}_\pi \subseteq \mathfrak{F} \subseteq \mathfrak{S}_\pi$ where $\mathfrak{N}_\pi$ is the class of all finite nilpotent π-groups (see Remark 1 of Section 3.3 in Hartley [2]). We show that $\mathfrak{F} = \mathfrak{S}_\pi$. Assume to the contrary that $\mathfrak{F} \subset \mathfrak{S}_\pi$ and let $G \in S_\pi \backslash \mathfrak{F}$ be of minimal order. Since both $\mathfrak{S}_\pi$ and $\mathfrak{F}$ are Fitting classes, it is clear that G has a unique maximal normal subgroup M of index p, say, which belongs to $\mathfrak{F}$. Consider the group $H = G \times G/M$. Clearly $M \times G/M$ is the $\mathfrak{F}$-injector of H. Let G^* be the subset of H which consists of all elements (x, xM), where $x \in G$. Then $G^* \lhd H$ and $H = GG^*$. In particular, $G^* \cap G = M$. Thus, G^* complements G/M in H. Since G/M is an $\mathfrak{F}$-avoided, complemented, and hence $\mathfrak{F}$-complemented chief factor of H, it follows then that G^* contains the $\mathfrak{F}$-injector $M \times G/M$ of H. But this is impossible. Hence, we must have $G \in \mathfrak{F}$, and so $\mathfrak{F} = \mathfrak{S}_\pi$, as required.

In view of Theorem 2.1 and the remark at the beginning of Section 1, we immediately have

COROLLARY 2.2. *A Fitting class has the property* (Λ^*) *if and only if it is the Fitting class of π-groups for some suitable set π of primes.*

3. Proof of the main theorem

In order to prove Theorem 1.1 we will need the following lemma.

LEMMA 3.1. *Let $\mathfrak{F}$ be a Fischer class with the property* (Λ), *let* $\mathfrak{S}_p \subseteq \mathfrak{F}$ *and let G be a semidirect product of an $\mathfrak{F}$-group A by a cyclic group* $B = \langle b \rangle$ *of order p^n, $n \geqq 1$. Then G is an $\mathfrak{F}$-group.*

PROOF. Let $C = \langle c \rangle$ be a cyclic group of order p^{n+1}, let $H = B \times C$ and let K be the subgroup of H generated by bc^p. Consider the twisted wreath product (see Neumann [4]) W of A by H over $B \times K$ with the action of $B \times K$ on A being defined as follows: Let B act on A as in the semidirect product G

of A by B, and let K act trivially on A. Since H is abelian, it is easy to check that K acts trivially on the base group $D = A_1 \times A_c \times \cdots \times A_{c^{p-1}}$ which is the direct product of p copies of A indexed by the coset representatives $\{1, c, \cdots, c^{p-1}\}$ of $B \times K$ in H, and also A_{c^i} is B-invariant and $[A_{c^i}]B \cong G$ for $i = 0, 1, \cdots, p-1$. In particular, $D \times K$ is contained in the $\mathfrak{F}$-injector V of W. But then, we must have that $DB/D\langle b^p \rangle$ is an $\mathfrak{F}$-covered chief factor of W; for, otherwise, it would be an $\mathfrak{F}$-avoided, complemented chief factor of W which is not partially $\mathfrak{F}$-complemented in W since $DK\langle b^p \rangle$ is not contained in any complement of $DB/D\langle b^p \rangle$ in W. Thus, V covers $DB/D\langle b^p \rangle$. However, since $D\langle b^p \rangle/D$ is the Frattini subgroup of DB/D, it follows now that V, in fact, covers DB/D, and so $V \geqq DB$. In particular, $DB \in \mathfrak{F}$ since $DB \lhd\lhd V$. Finally, since A_1 is B-invariant, and hence also DB-invariant, since A_1B/A_1 is a p-group and since $\mathfrak{F}$ is a Fischer class, it follows that $G \cong [A_1]B \in \mathfrak{F}$, and the lemma is proved.

We can now complete the proof of Theorem 1.1 as follows:

Proof of theorem 1.1. In view of the remark at the beginning of Section 1, it remains to show that if $\mathfrak{F}$ is a Fischer class with the property (Λ), then $\mathfrak{F}$ is the Fischer class of π-groups for some set π of primes. Let π be the uniquely determined set of primes such that $\mathfrak{N}_\pi \subseteq \mathfrak{F} \subseteq \mathfrak{S}_\pi$ (see the proof of Theorem 2.1) We will show that $\mathfrak{F} = \mathfrak{S}_\pi$. Assume to the contrary that $\mathfrak{F} \subset \mathfrak{S}_\pi$, and let $G \in \mathfrak{S}_\pi \backslash \mathfrak{F}$ be of minimal order. Then G has a unique maximal normal subgroup M which lies in $\mathfrak{F}$. Let $|G: M| = p$ and let $x \in G$ be of p-power order such that $\langle M, x \rangle = G$. Consider the semidirect product W of G by a cyclic group $\langle \alpha \rangle$ of order $p^n = |x|$, the order of x in G, with the action of $\langle \alpha \rangle$ on G being given by $g^\alpha = g^x$ for each $g \in G$. Clearly M is $\langle \alpha \rangle$-invariant, and so, by Lemma 3.1, $[M]\langle \alpha \rangle \in \mathfrak{F}$. Similarly, $[M]\langle \alpha x \rangle \in \mathfrak{F}$. But then

$$W = [M]\langle \alpha, \alpha x \rangle \in N_0\mathfrak{F} = \mathfrak{F},$$

whence, in particular, $G \in S_N\mathfrak{F} = \mathfrak{F}$, a contradiction. With this contradiction, the proof is complete.

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