Can. J. Math. Vol. 46 (3), 1994 pp. 619–633

DEGREE OF APPROXIMATION BY RATIONAL FUNCTIONS WITH PRESCRIBED NUMERATOR DEGREE

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ABSTRACT. We prove a Jackson type theorem for rational functions with prescribed numerator degree: For continuous functions $f: [-1,1] \longrightarrow \mathbb{R}$ with ℓ sign changes in $(-1,1)$, there exists a real rational function $R_{\ell,n}(x)$ with numerator degree ℓ and denominator degree at most n, that changes sign exactly where f does, and such that

$$\|f - R_{\ell,n}\|_{L_\infty[-1,1]} \le C(\ell+1)^2 \omega_\phi\Big(f; \frac{1}{n}\Big).$$

Here C is independent of f, n and ℓ, and ω_ϕ is the Ditzian-Totik modulus of continuity. For special functions such as $f(x) = \operatorname{sign}(x)|x|^\alpha$, we consider improvements of the Jackson rate.

1. **Introduction.** In [6], Levin and Saff have investigated the degree of uniform approximation on a finite interval I, of real functions f continuous on I, by reciprocals of real and complex polynomials. They have immediately made the observation that in order to approximate f by reciprocals of real polynomials, f must keep a fixed sign in I. Thus they were compelled to use complex polynomials whenever f had a sign change in I. Their work has been extended in part in [4] and in [1], but always under the assumption that f has no sign changes in I. We shall allow the function f to have finitely many sign changes in I, but rather than approximating by reciprocals of polynomials (an impossible task), we shall have in the numerator a fixed polynomial of degree ℓ that changes sign with f. For this type of rational approximation, we prove a Jackson type estimate. We also investigate some special functions that admit a much better degree of approximation than is guaranteed by the Jackson rate.

For simplicity of notation, we take $I = [-1,1]$. We state our main results in Section 2 and their proofs are given in Sections 4 and 5. In Section 3, we state and prove some auxiliary results, notably Theorem 3.1, which is crucial to the proofs, and which we believe is of independent interest. We conclude with some examples in Section 6.

The first author's research was completed while visiting Witwatersrand University.

Received by the editors September 28, 1992.

AMS subject classification: 41A20, 41A25.

Key words and phrases: rational approximation, real rational approximation, Jackson theorems, moduli of continuity, prescribed numerator degree.

2. **Main results.** Let π_n denote the set of polynomials of degree at most n with real coefficients. Our main result is the following Jackson-type theorem involving the Ditzian-Totik modulus of continuity of f in $[-1,1]$. Namely (see [2]), with

$$\phi(x) := \sqrt{1-x^2}, \quad x \in [-1,1],$$

we set for $t > 0$,

$$\omega_\phi(f;t) = \sup_{0<h\le t} \sup_{x\pm\frac{h}{2}\phi(x)\in[-1,1]} \left| f\left(x+\frac{h}{2}\phi(x)\right) - f\left(x-\frac{h}{2}\phi(x)\right)\right|.$$

THEOREM 2.1. *There exists an absolute constant C with the following property: If $f \in C[-1,1]$ changes sign exactly ℓ times in $(-1,1)$, say at $b_1, b_2, \ldots, b_\ell$, then for each $n \ge 1$, there exists a polynomial $p_n \in \pi_n$, having the same sign as f in $(b_\ell, 1)$, and such that for $x \in [-1,1]$,*

$$\left| f(x) - \frac{(x-b_1)(x-b_2)\cdots(x-b_\ell)}{p_n(x)} \right| \le C(\ell+1)^2 \omega_\phi\left(f;\frac{1}{n}\right). \tag{2.1}$$

Note that in particular C is independent of f, n, ℓ, $b_1, b_2, \ldots, b_\ell$ and x. For $\ell = 0$, the theorem reduces to the result of Leviatan, Levin and Saff [4]. In fact, our proof depends very heavily on the result of [4]. Obviously, the minimal dependence on the number of sign changes ℓ in the right-hand side of (2.1) would be of interest. Can $(\ell+1)^2$ be dropped altogether?

If $f(x) = \text{sign}(x)|x|^\alpha$, $0 < \alpha < 1$, then by Theorem 2.1 f is uniformly approximable by $x/p_n(x)$ at the rate of $n^{-\alpha}$. However, if $\alpha > 1$, then (2.1) only guarantees the rate n^{-1}. Our next result shows that we can do much better if $\alpha > 1$, and that in fact, for $\alpha > 1$, the rate $n^{-\alpha}$ is attainable and is best possible. (The case $\alpha = 1$ needs no discussion). Let us set

$$E_{1n}(f) := \inf \|f - R_{1n}\|_{L_\infty[-1,1]},$$

where the infimum is taken over all rational functions of type $(1,n)$. Then we have:

THEOREM 2.2. *Let $\alpha > 0$. Then there exists an absolute constant C such that if*

$$f(x) := \text{sign}(x)|x|^\alpha, \quad x \in [-1,1],$$

then for $n \ge 1$,

$$E_{1n}(f) \le C^{\max\{1,\alpha\}} n^{-\alpha}. \tag{2.2}$$

Moreover for $\alpha > 1$, (2.2) is best possible in the sense that there exists a constant A_α such that for $n \ge 1$,

$$E_{1n}(f) \ge A_\alpha n^{-\alpha}. \tag{2.3}$$

It is interesting to compare Theorem 2.2 with Theorem 2.3 in [6]. Our technique in proving (2.2) for $\alpha \in (0,1]$ is based on the technique of Section 3, and that for $\alpha > 1$ is based on an idea of Levin and Saff [6]. For $\alpha \in (0,2]$, it is possible to give a different proof based on the Dombrowski-Nevai formula (applied in the special case of the Legendre weight) [3], but we omit this.

3. **Auxiliary lemmas.** We need the following auxiliary result, which is of independent interest:

THEOREM 3.1. *There is an absolute constant $C > 0$ with the following property: Let $-1 < b_1 < b_2 < \cdots < b_\ell < 1$, and set*

$$\rho(x) := \prod_{j=1}^{\ell}(x - b_j).$$

Then there exists for $n \geq 3\ell$, a polynomial $S(x)$ of degree $\leq n$ such that for $x \in [-1,1]$,

$$0 \leq 1 - \frac{|\rho(x)|}{S(x)} \leq \min\left\{1, \frac{C\ell}{n}\sum_{j=1}^{\ell}\frac{\sqrt{1-b_j^2}}{|x-b_j|}\right\}. \tag{3.1}$$

We emphasise that C is independent of n and ℓ and the location of $b_1, b_2 \cdots b_\ell$. The proof of Theorem 3.1 requires a few lemmas. In the sequel, we use $\sim$ in the following sense: If $\{c_n\}_{n=1}^{\infty}$ and $\{d_n\}_{n=1}^{\infty}$ are sequences of real numbers, we write $c_n \sim d_n$ if there exist positive constants C_1 and C_2 such that

$$C_1 \leq c_n/d_n \leq C_2, \quad n \geq 1.$$

Similar notation is used for functions and sequences of functions. We also use C, C_1, C_2, to denote constants independent of n and $P \in \pi_n$. The same symbol does not necessarily denote the same constant in different occurrences.

LEMMA 3.2. *There exists n_1 and C such that for $n \geq n_1$ and $|b| \leq \cos(\frac{\pi}{2n})$, there exists a polynomial $V_{n,b}$ of degree $\leq n-1$, such that*

$$V_{n,b}(b) = 1 = \|V_{n,b}\|_{L_\infty[-1,1]}, \tag{3.2}$$

and

$$|V_{n,b}(t)| \leq C\frac{\sqrt{1-b^2}}{n|t-b|}, \quad t \in [-1,1] \setminus \{b\}. \tag{3.3}$$

PROOF. $V_{n,b}$ was constructed in Proposition 13.1 in [5]. ∎

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LEMMA 3.3. *For $n \geq n_1$ and $|b| \leq \cos(\frac{\pi}{2n})$, define*

$$R_{n,b}(t) := \int_b^t V_{n,b}^2(u)\,du \Big/ \int_b^1 V_{n,b}^2(u)\,du, \quad t \in \mathbb{R}. \tag{3.4}$$

Set

$$c_{n,b} := \int_{-1}^b V_{n,b}^2(u)\,du \Big/ \int_b^1 V_{n,b}^2(u)\,du. \tag{3.5}$$

(i) $R_{n,b}(u)$ has the same sign as $u-b$.
(ii) Uniformly for $n \geq n_1$ and $|b| \leq \cos(\frac{\pi}{2n})$,

$$c_{n,b} \sim 1. \tag{3.6}$$

(iii) There exists C such that for $n \geq n_1$ and $|b| \leq \cos(\frac{\pi}{2n})$,

$$\|R_{n,b}\|_{L_\infty[-1,1]} \leq C; \tag{3.7}$$

$$|R_{n,b}(t) - 1| \leq C\frac{\sqrt{1-b^2}}{n|t-b|}, \quad t \in (b,1); \tag{3.8}$$

and

$$|R_{n,b}(t) + c_{n,b}| \leq C\frac{\sqrt{1-b^2}}{n|t-b|}, \quad t \in (-1,b). \tag{3.9}$$

PROOF. (i) This follows directly from the definition of $R_{n,b}$.
(ii) By the Bernstein and Markov inequalities, for $s \in [-1,1]$,

$$|V'_{n,b}(s)| \leq \frac{n}{\sqrt{1-s^2}}\|V_{n,b}\|_{L_\infty[-1,1]} = \frac{n}{\sqrt{1-s^2}} =: 1/\chi_n(s),$$

say. We first show that

$$|s-b| \leq \frac{1}{4}\chi_n(b) \text{ implies } |V'_{n,b}(s)| \leq 2/\chi_n(b). \tag{3.10}$$

For this it suffices to show that

$$|s-b| \leq \frac{1}{4}\chi_n(b) \text{ implies } 1/\chi_n(s) \leq 2/\chi_n(b). \tag{3.11}$$

Now for such s, note that

$$\begin{aligned} 1-s^2 &= 1-b^2+b^2-s^2 \\ &\geq 1-b^2-\frac{1}{2}\chi_n(b) \\ &= \chi_n(b)\Big\{n\sqrt{1-b^2}-\frac{1}{2}\Big\} \geq \chi_n(b)\Big\{n\sin\frac{\pi}{2n}-\frac{1}{2}\Big\} \geq \frac{1}{2}\chi_n(b), \end{aligned}$$

by the inequality $\sin v \geq \frac{2}{\pi}v$, $v \in [0, \frac{\pi}{2}]$. Then

$$\chi_n^2(b)/\chi_n^2(s) = 1 + \frac{s^2 - b^2}{1 - s^2} \leq 1 + \frac{\chi_n(b)/2}{\chi_n(b)/2} = 2.$$

So we have (3.11) and then (3.10). Then for $|t - b| \leq \frac{1}{4}\chi_n(b)$, there exists s between t and b such that

$$\begin{aligned} 1 \geq V_{n,b}(t) &= V_{n,b}(b) - V'_{n,b}(s)(t - b) \\ &\geq 1 - \big(2/\chi_n(b)\big)\Big(\frac{1}{4}\chi_n(b)\Big) = 1/2, \end{aligned}$$

by (3.10). Then

$$\int_b^{b+\frac{1}{4}\chi_n(b)} V_{n,b}^2(u)\,du \sim \int_{b-\frac{1}{4}\chi_n(b)}^b V_{n,b}^2(u)\,du \sim \chi_n(b). \tag{3.12}$$

Note that

$$\begin{aligned} 1 - \Big(b + \frac{1}{4}\chi_n(b)\Big) &\geq (1 - b)\Big(\frac{1+b}{2}\Big) - \frac{1}{4}\chi_n(b) \\ &= \frac{1}{2}\chi_n(b)\Big\{n\sqrt{1 - b^2} - \frac{1}{2}\Big\} > 0, \end{aligned}$$

so $b + \frac{1}{4}\chi_n(b) < 1$. Similarly, $b - \frac{1}{4}\chi_n(b) > -1$. Moreover, by (3.2) and (3.3),

$$\int_{-1}^1 V_{n,b}^2(u)\,du \leq \int_{-1}^1 \min\Big\{1, C_1\frac{\chi_n(b)}{|u - b|}\Big\}^2 du \leq \chi_n(b)\int_{-\infty}^{\infty} \min\{1, C_1/s\}^2\,ds, \tag{3.13}$$

by the substitution $s = (u - b)/\chi_n(b)$. So we have (3.6).

(iii) Now for $u \in [-1, 1]$,

$$\begin{aligned} |R_{n,b}(u)| &\leq \int_{-1}^1 V_{n,b}^2(u)\,du \Big/ \int_b^1 V_{n,b}^2(u)\,du \\ &\leq \int_{-1}^1 V_{n,b}^2(u)\,du \Big/ \int_b^{b+\frac{1}{4}\chi_n(b)} V_{n,b}^2(u)\,du \leq C_1, \end{aligned}$$

by (3.12) and (3.13). So we have (3.7). Next, for $t > b$, (3.2), (3.3) and (3.12) yield

$$\begin{aligned} |R_{n,b}(t) - 1| &= \Big|\int_t^1 V_{n,b}^2(u)\,du \Big/ \int_b^1 V_{n,b}^2(u)\,du\Big| \\ &\leq \int_t^1 \min\Big\{1, C_1\frac{\chi_n(b)}{(u - b)}\Big\}^2 du \Big/ \big(C_2\chi_n(b)\big) \\ &\leq \chi_n(b)\int_{(t-b)/\chi_n(b)}^{\infty} \min\{1, C_1/s\}^2\,ds \Big/ \big(C_2\chi_n(b)\big) \\ &\leq C_3\chi_n(b)/(t - b) = C_3\frac{\sqrt{1 - b^2}}{n|t - b|}, \end{aligned}$$

so we have (3.8). The proof of (3.9) is similar, one estimates for $t < b$

$$|R_{n,b}(t) + c_{n,b}| = \Big|\int_{-1}^t V_{n,b}^2(u)\,du \Big/ \int_b^1 V_{n,b}^2(u)\,du\Big|. \qquad \blacksquare$$

The main part of Theorem 3.1 is proved in the following lemma, which is the case $\ell = 1$ of Theorem 3.1:

LEMMA 3.4. *There exists C_1 with the following property: For $n \geq 1$ and $b \in (-1,1)$, there exist polynomials $S_{n,b}$ of degree $\leq 2n+1$ that are positive in $[-1,1]$ and such that*

$$\|S_{n,b}\|_{L_\infty[-1,1]} \leq C_1, \tag{3.14}$$

and for $u \in [-1,1]$,

$$0 \leq 1 - \frac{|u-b|}{S_{n,b}(u)} \leq \min\left\{1, C_1 \frac{\sqrt{1-b^2}}{n|u-b|}\right\}. \tag{3.15}$$

C_1 is independent of n, b and u.

PROOF. We consider two ranges of b:

CASE I: $|b| \leq \cos(\frac{\pi}{2n})$. Let n_1 be as in the previous lemma. For $n < n_1$, we can take $S_{n,b}$ to be the constant polynomial $S_{n,b} = 2$. It is easy to see that (3.14) and (3.15) are satisfied for $n < n_1$: For

$$0 \leq 1 - \frac{|u-b|}{2} \leq 1$$

and

$$\frac{\sqrt{1-b^2}}{n|u-b|} \geq \frac{\sin\big(\pi/(2n_1)\big)}{2n_1}$$

and so we need only make C_1 in (3.15) large enough (depending only on n_1). So in the sequel, let us assume that $n \geq n_1$. We define a linear function

$$L_n(t) = 2\left(\frac{t + c_{n,b}}{1 + c_{n,b}}\right) - 1,$$

which has the property

$$L_n(-c_{n,b}) = -1; \quad L_n(1) = 1.$$

This linear map will compensate for the fact that $R_{n,b}$ approximates $-c_{n,b}$ and not -1 in $[-1,b)$. Note that

$$|L_n'(t)| = \frac{2}{1 + c_{n,b}} \leq 2,$$

a bound independent of n, b and t. Let us define

$$S_{n,b}(u) := (u-b)L_n\big(R_{n,b}(u)\big) + K\sqrt{1-b^2}/n,$$

where K will be chosen sufficiently large later. Let C have the meaning in (3.7), so that $C \geq 1$. Now for $u \in [b,1]$,

$$\begin{aligned} \Big|(u-b)L_n\big(R_{n,b}(u)\big) - |u-b|\Big| &= |u-b|\,\Big|L_n\big(R_{n,b}(u)\big) - L_n(1)\Big| \\ &\leq |u-b|\,\|L_n'\|_{L_\infty[-C,C]}|R_{n,b}(u) - 1| \\ &\leq 2C\sqrt{1-b^2}/n, \end{aligned}$$

by (3.8). Let us choose $K > 2C$. Then

$$(K-2C)\sqrt{1-b^2}/n \leq S_{n,b}(u) - |u-b| \leq (K+2C)\sqrt{1-b^2}/n,$$

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and hence

$$0 \le 1 - \frac{|u-b|}{S_{n,b}(u)} \le \frac{(K+2C)\sqrt{1-b^2}/n}{S_{n,b}(u)}.$$

Since $S_{n,b}(u) > |u-b|$, we obtain (3.15). Next, suppose that $u \in [-1, b)$. Recall that $L_n(-c_{n,b}) = -1$. Then

$$\begin{aligned}\big|(u-b)\,L_n\big(R_{n,b}(u)\big) - |u-b|\,\big| &= |u-b|\,\big|L_n\big(R_{n,b}(u)\big) - L_n(-c_{n,b})\big| \\ &\le |u-b|\,\|L_n'\|_{L_\infty[-C,C]}\,|R_{n,b}(u)+c_{n,b}| \\ &\le 2C\sqrt{1-b^2}/n,\end{aligned}$$

by (3.9). The rest of the proof is the same as for $u > b$.

CASE II: $1 > |b| > \cos(\frac{\pi}{2n})$. Let us suppose that $1 > b > \cos(\frac{\pi}{2n})$, and set

$$S_{n,b}(s) := b - s + \epsilon_n,$$

where

$$\epsilon_n := 2\pi\sqrt{1-b^2}/n.$$

Note that

$$\sqrt{1-b^2} < \sin\Big(\frac{\pi}{2n}\Big) < \frac{\pi}{2n},$$

so

$$(1-b)/\epsilon_n < (1-b^2)/\epsilon_n = n\sqrt{1-b^2}/(2\pi) < 1/4.$$

Then for $s < b$,

$$S_{n,b}(s) = |s-b| + \epsilon_n,$$

so

$$1 - \frac{|s-b|}{S_{n,b}(s)} = \frac{\epsilon_n}{|s-b|+\epsilon_n}.$$

The right-hand side is non-negative and admits the upper bound

$$\min\Big\{1, \frac{\epsilon_n}{|s-b|}\Big\} = \min\Big\{1, \frac{2\pi\sqrt{1-b^2}}{n|s-b|}\Big\},$$

so we have (3.15) in this case. Next, for $s \in (b, 1)$,

$$|s-b| \le (1-b) < \epsilon_n/4,$$

so

$$S_{n,b}(s) = -|s-b| + \epsilon_n > \frac{3}{4}\epsilon_n \ge 3|s-b|,$$

and then

$$0 \le 1 - \frac{|s-b|}{S_{n,b}(s)} \le 1.$$

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Finally for such s,

$$\frac{\sqrt{1-b^2}}{n|s-b|} \geq \frac{\sqrt{1-b^2}}{n|1-b|} \geq \frac{1}{n\sqrt{1-b^2}} \geq \frac{2}{\pi},$$

so in this case (3.15) holds trivially. Finally,

$$\|S_{n,b}\|_{L_\infty[-1,1]} \leq 2 + \epsilon_n \leq 2 + 2\pi,$$

so (3.14) also holds. ∎

We can proceed to the

PROOF OF THEOREM 3.1. Recall that we assume $n \geq 3\ell$. We let $[x]$ denote the greatest integer $\leq x$, and with $S_{n,b}$ as in the previous lemma, set

$$S(x) := \prod_{j=1}^{\ell} S_{[n/(3\ell)],b_j}(x).$$

Then by the previous lemma, S is positive in $[-1,1]$ and has degree at most

$$\ell\big(2[n/(3\ell)] + 1\big) \leq \frac{2}{3}n + \ell \leq \frac{2}{3}n + \frac{1}{3}n = n.$$

Moreover, from (3.15), for $1 \leq j \leq \ell$ and $x \in [-1,1]$,

$$S_{[n/(3\ell)],b_j}(x) \geq |x - b_j|,$$

so

$$S(x) \geq \prod_{j=1}^{\ell} |x - b_j| = |\rho(x)|,$$

and hence

$$0 \leq 1 - \frac{|\rho(x)|}{S(x)} \leq 1.$$

Next, (3.15) also gives

$$\begin{aligned} 1 - \frac{|\rho(x)|}{S(x)} &= 1 - \prod_{j=1}^{\ell}\left(1 - \left(1 - \frac{|x-b_j|}{S_{[n/(3\ell)],b_j}(x)}\right)\right) \\ &\leq 1 - \prod_{j=1}^{\ell}\left(1 - \min\left\{1, C_2\ell\frac{\sqrt{1-b_j^2}}{n|x-b_j|}\right\}\right) \\ &\leq \sum_{j=1}^{\ell} \min\left\{1, C_2\ell\frac{\sqrt{1-b_j^2}}{n|x-b_j|}\right\} \end{aligned}$$

where we have used the inequality

$$1 - \prod_{j=1}^{\ell}(1 - y_j) \leq \sum_{j=1}^{\ell} y_j,$$

which is valid for $y_j \in [0,1]$, $1 \leq j \leq \ell$. (This is easily proved by induction on ℓ). Together with our earlier bound, this completes the proof of Theorem 3.1. ∎

We need two lemmas concerning the ϕ-modulus of continuity:

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LEMMA 3.5. *There exists an absolute constant C such that for $s,t \in [-1,1]$ and $f \in C[-1,1]$,*

$$|f(s)-f(t)| \min\left\{1, \frac{\sqrt{1-s^2}}{n|s-t|}\right\} \leq C\omega_\phi\left(f;\frac{1}{n}\right). \tag{3.16}$$

PROOF. Recall that $\phi(t) = \sqrt{1-t^2}$, $t \in [-1,1]$. Let Δ denote the left-hand side of (3.16), and let

$$a := (s+t)/2; \quad |s-t| =: h\phi(a).$$

If firstly $h \leq 1/n$, we have

$$\begin{aligned}\Delta \leq |f(s)-f(t)| &= \left|f\left(a+\frac{h}{2}\phi(a)\right) - f\left(a-\frac{h}{2}\phi(a)\right)\right| \\ &\leq \omega_\phi(f;h) \leq \omega_\phi\left(f;\frac{1}{n}\right).\end{aligned}$$

If $h > 1/n$, we have

$$\Delta \leq \frac{|f(s)-f(t)|}{n|s-t|}\sqrt{1-s^2} \leq \frac{\omega_\phi(f;h)}{nh\phi(a)}\phi(s). \tag{3.17}$$

Here by homogeneity properties of $\omega_\phi(f;h)$ [2, Theorem 4.1.2, p. 38] and as $nh > 1$,

$$\omega_\phi(f;h)/(nh) = \omega_\phi\left(f;nh\frac{1}{n}\right)/(nh) \leq C\omega_\phi\left(f;\frac{1}{n}\right).$$

Also assuming that, for example, $a \geq 0$, we have

$$\begin{aligned}\phi(a)^2 = 1-a^2 \geq 1-a &= \frac{1}{2}(1-s) + \frac{1}{2}(1-t) \\ &\geq \frac{1}{2}(1-s) \geq \frac{1}{4}(1-s^2) = \frac{1}{4}\phi(s)^2.\end{aligned}$$

Substituting these last two estimates into (3.17) yields (3.16) for $h > 1/n$. ∎

LEMMA 3.6. *There exists an absolute constant C with the following property: If $f \in C[-1,1]$ has a zero in $[-1,1]$, then*

$$|f(x)| \leq \omega_\phi(f;2\sqrt{2}) \quad \forall x \in [-1,1].$$

PROOF. Let $f(b) = 0$. Fix $x \in [-1,1]$ and write $a := \frac{1}{2}(x+b)$; $h\phi(a) := |x-b|$. Then

$$|f(x)| = |f(x)-f(b)| = \left|f\left(a+\frac{h}{2}\phi(a)\right) - f\left(a-\frac{h}{2}\phi(a)\right)\right| \leq \omega_\phi(f;h).$$

We now estimate h. Note that the last part of the previous proof shows that

$$\phi(a) \geq \frac{1}{2}\phi(x); \quad \phi(a) \geq \frac{1}{2}\phi(b).$$

If firstly x and b have the same sign, say both are non-negative, then

$$h = \left|\frac{(1-x)-(1-b)}{\phi(a)}\right| \leq \frac{\max\{1-x, 1-b\}}{\phi(a)}$$
$$\leq \max\left\{2\frac{1-x}{\phi(x)}, 2\frac{1-b}{\phi(b)}\right\} \leq 2\max\left\{\frac{1-x^2}{\phi(x)}, \frac{1-b^2}{\phi(b)}\right\} \leq 2.$$

Similarly if both x and b are negative. If x and b have opposite sign, and say a is non-negative, then

$$\phi(a)^2 = 1-a^2 \geq 1-a = \frac{1}{2}(1-x) + \frac{1}{2}(1-b) \geq \frac{1}{2},$$

as $1-x$ and $1-b$ are non-negative, and at least one is ≥ 1, so

$$h = |x-b|/\phi(a) \leq 2\sqrt{2}.$$ ∎

4. **Jackson type estimates (Proof of Theorem 2.1).**

PROOF OF THEOREM 2.1. We consider two ranges of n:

CASE I: $n \geq 6\ell$. Let $-1 < b_1 < b_2 < \cdots < b_\ell < 1$ and $f \in C[-1,1]$ change sign exactly at b_j, $1 \leq j \leq \ell$. We may assume that f is positive in $(b_\ell, 1)$. Let

$$\rho(x) := \prod_{j=1}^{\ell}(x-b_j).$$

Note that if $p_n(x)$ is a polynomial positive in $(-1,1)$, then

$$\left|f(x) - \frac{\rho(x)}{p_n(x)}\right| = \left||f(x)| - \frac{|\rho(x)|}{p_n(x)}\right|,$$

so this suggests that we set

$$p_n(x) := S_{[n/2]}(x)\, U_{[n/2]}(x),$$

where $S_{[n/2]}(x)$ is the polynomial of degree $\leq n/2$ of Theorem 3.1 approximating $|\rho(x)|$. (Recall $n/2 \geq 3\ell$, so Theorem 3.1 is applicable). We choose $U_{[n/2]}(x)$ to be a polynomial of degree $\leq n/2$ such that

$$\left||f(x)| - \frac{1}{U_{[n/2]}(x)}\right| \leq C\omega_\phi\left(|f|; \frac{1}{n}\right). \tag{4.1}$$

Here C is an absolute constant, and $U_{[n/2]}$ exists by Theorem 1 in [4]. Since

$$\omega_\phi(|f|;t) \leq \omega_\phi(f;t) \leq 2\omega_\phi(|f|;t), \quad t \in (0,1],$$

we can replace $|f|$ by f in the right-hand side of (4.1). According to Theorem 3.1,

$$0 \leq 1 - \frac{|\rho(x)|}{S_{[n/2]}(x)} \leq \min\left\{1, \frac{C\ell}{n}\sum_{j=1}^{\ell}\frac{\sqrt{1-b_j^2}}{|x-b_j|}\right\}, \quad x \in [-1,1].$$

Then

$$\left|f(x) - \frac{\rho(x)}{p_n(x)}\right| = \left||f(x)| - \frac{|\rho(x)|}{p_n(x)}\right|$$
$$= \left||f(x)|\left(1 - \frac{|\rho(x)|}{S_{[n/2]}(x)}\right) + \frac{|\rho(x)|}{S_{[n/2]}(x)}\left(|f(x)| - \frac{1}{U_{[n/2]}(x)}\right)\right|$$
$$\leq |f(x)| \min\left\{1, \frac{C\ell}{n}\sum_{j=1}^{\ell}\frac{\sqrt{1-b_j^2}}{x-b_j}\right\} + C\omega_\phi\left(f; \frac{1}{n}\right),$$

as $|\rho|/S_{[n/2]} \leq 1$ and by (4.1). Here

$$\text{(4.2)} \quad |f(x)| \min\left\{1, \frac{C\ell}{n}\sum_{j=1}^{\ell}\frac{\sqrt{1-b_j^2}}{|x-b_j|}\right\} \leq C_1\ell\sum_{j=1}^{\ell}|f(x)-f(b_j)| \min\left\{1, \frac{\sqrt{1-b_j^2}}{n|x-b_j|}\right\}$$
$$\leq C_2\ell^2\omega_\phi\left(f; \frac{1}{n}\right),$$

by Lemma 3.5 and as $f(b_j) = 0$, $1 \leq j \leq \ell$. We have the required estimate (2.1).

CASE II: $n < 6\ell$. Note that as f changes sign, $\omega_\phi(f;3) > 0$ and so we can choose $p_n(x)$ to be a positive constant so large that for $x \in [-1,1]$,

$$\left|f(x) - \frac{\rho(x)}{p_n(x)}\right| \leq |f(x)| + \frac{2^\ell}{p_n(x)} \leq 2\omega_\phi(f;3),$$

by Lemma 3.6. By the homogeneity property of ω_ϕ, we obtain

$$2\omega_\phi(f;3) = 2\omega_\phi\left(f; 18\ell\cdot\frac{1}{6\ell}\right) \leq C\ell\omega_\phi\left(f; \frac{1}{6\ell}\right) \leq C\ell\omega_\phi\left(f; \frac{1}{n}\right),$$

as $n < 6\ell$. Here C is an absolute constant. Again, we have (2.1). ∎

We remark that the last few lines of Case I of the proof easily admits the following improved estimate for the right-hand side of (4.2):

$$\leq C_2\ell\omega_\phi\left(f; \frac{\ell}{n}\right),$$

which leads to a marginal improvement of (2.1).

5. **Approximation of** $\operatorname{sign}(x)|x|^\alpha$ **(Proof of Theorem 2.2).** Since for $0 < \alpha \leq 1$, the assertion (2.2) follows easily from Theorem 2.1, we shall restrict our attention to the case $\alpha > 1$. We first prove (2.2) and then (2.3).

PROOF OF (2.2) OF THEOREM 2.2. We follow an idea of Levin and Saff [6]. Let

$$\tau_n(t) := t^{\alpha-2}\big(T_n(t)/t\big)^{2k},$$

where n is odd, T_n is the Chebyshev polynomial of degree n, and k is the smallest integer satisfying $k \geq \alpha - \frac{1}{2}$. Since $\alpha > 1$, τ_n is integrable on [0, 1], so we set

$$p_n(x) := \frac{1}{C_n x^{\alpha-1}}\int_0^x \tau_n(t)\,dt, \quad x > 0,$$

where

$$C_n := \int_0^1 \tau_n(t)\,dt,$$

and $p_n(-x) = p_n(x)$. As p_n is an even polynomial on $[-1,1]$, we may consider $x \geq 0$. Firstly,

$$\frac{1}{p_n(x)} - x^{\alpha-1} = x^{\alpha-1}\int_x^1 \tau_n(t)\,dt \Big/ \int_0^x \tau_n(t)\,dt \geq 0,$$

and so

$$(5.1)\qquad \begin{aligned}\Delta := \left|f(x) - \frac{x}{p_n(x)}\right| &= x\left(\frac{1}{p_n(x)} - x^{\alpha-1}\right)\\ &= x^{\alpha}\int_x^1 \tau_n(t)\,dt \Big/ \int_0^x \tau_n(t)\,dt.\end{aligned}$$

We now proceed exactly as in [6], but provide the details, to avoid confusion because of the different relation between α and k here to that of [6]. Note the following estimates [6]:

$$\left|\frac{T_n(t)}{t}\right| \leq \min\left\{\frac{1}{t}, n\right\}, \qquad t \in (0,1].$$

$$\left|\frac{T_n(t)}{t}\right| \geq \frac{2n}{\pi}, \qquad t \in \left(0, \sin\left(\frac{\pi}{2n}\right)\right].$$

These readily yield

$$(5.2)\qquad \tau_n(t) \leq \begin{cases} t^{\alpha-2}n^{2k}, \\ t^{\alpha-2-2k}, \end{cases} \qquad t \in (0,1]$$

and

$$(5.3)\qquad \tau_n(t) \geq \left(\frac{2}{\pi}\right)^{2k} n^{2k} t^{\alpha-2}, \qquad t \in \left(0, \sin\left(\frac{\pi}{2n}\right)\right].$$

CASE I: $x \in [0, \sin(\frac{\pi}{2n})]$. Here

$$\begin{aligned}\int_x^1 \tau_n(t)\,dt &\leq \left[\int_0^{1/n} + \int_{1/n}^1\right]\tau_n(t)\,dt\\ &\leq \int_0^{1/n} n^{2k}t^{\alpha-2}\,dt + \int_{1/n}^1 t^{\alpha-2-2k}\,dt\\ &= \frac{n^{2k+1-\alpha}}{\alpha-1} + \frac{n^{2k+1-\alpha}}{2k+1-\alpha} \leq 2\frac{n^{2k+1-\alpha}}{\alpha-1}\end{aligned}$$

as

$$(5.4)\qquad 2k+1-\alpha \geq \alpha.$$

Next

$$\int_0^x \tau_n(t)\,dt \geq \left(\frac{2}{\pi}\right)^{2k} n^{2k}\int_0^x t^{\alpha-2}\,dt = \left(\frac{2}{\pi}\right)^{2k} n^{2k}\frac{x^{\alpha-1}}{\alpha-1}.$$

Then from (5.1),

$$0 \leq \Delta \leq \left(\frac{\pi}{2}\right)^{2k} 2n^{1-\alpha}x \leq \left(\frac{\pi}{2}\right)^{2k} 2n^{1-\alpha}\frac{\pi}{2n} = \left(\frac{\pi}{2}\right)^{2k}\pi n^{-\alpha}.$$

CASE II: $x \in (\sin(\frac{\pi}{2n}), 1]$. Here

$$\int_x^1 \tau_n(t)\,dt \le \int_x^1 t^{\alpha-2-2k}\,dt \le \frac{x^{\alpha-1-2k}}{2k+1-\alpha} \le \frac{x^{\alpha-1-2k}}{\alpha},$$

by (5.4). Also

$$\begin{aligned}\int_0^x \tau_n(t)\,dt &\ge \int_0^{\sin(\pi/(2n))} \tau_n(t)\,dt \\ &\ge \left(\frac{2}{\pi}\right)^{2k} n^{2k} \int_0^{1/n} t^{\alpha-2}\,dt \ge \left(\frac{2}{\pi}\right)^{2k} \frac{n^{2k-\alpha+1}}{\alpha-1}.\end{aligned}$$

Then

$$\begin{aligned}0 < \Delta &\le \left(\frac{\pi}{2}\right)^{2k} \frac{\alpha-1}{\alpha} n^{-\alpha} (nx)^{-[2k+1-2\alpha]} \\ &\le \left(\frac{\pi}{2}\right)^{2k} n^{-\alpha},\end{aligned}$$

as $nx \ge 1$ and $2k+1-2\alpha \ge 0$. ∎

PROOF OF (2.3) OF THEOREM 2.2. First observe that if $R(x) = (x+b)/Q(x)$ is a best approximation to $f(x) = \operatorname{sign}(x)|x|^\alpha$ from the real rational functions of type $(1, n)$, then

$$\begin{aligned}E_{1n}(f) &= \|f - R\|_{L_\infty[-1,1]} \\ &= \|-[f(-x) - R(-x)]\|_{L_\infty[-1,1]} \\ &= \big\|f(x) - \big(-R(-x)\big)\big\|_{L_\infty[-1,1]}.\end{aligned}$$

By uniqueness of the best approximation, we have $R(x) = -R(-x)$. Then

$$\frac{Q(-x)}{Q(x)} = \frac{x-b}{x+b}.$$

If $b \neq 0$, then this implies that $Q(x) = (x+b)P(x)$, which in turn yields

$$\frac{-P(-x)}{P(x)} = 1,$$

so P is odd. Then $R(x) = 1/P(x)$ has a pole at 0, a contradiction. So $b = 0$, and

$$R(x) = x/Q(x).$$

Since R is odd, it follows that Q is even, that is

$$R(x) = x/S_n(x^2),$$

where $S_n(\cdot)$ is a polynomial of degree $\le n/2$. Then

$$\begin{aligned}E_{1n}(f) &= \|x^\alpha - x/S_n(x^2)\|_{L_\infty[0,1]} \\ &= \|t^{\alpha/2} - \sqrt{t}/S_n(t)\|_{L_\infty[0,1]}.\end{aligned}$$

Assume that as $n \longrightarrow \infty$ through a subsequence,

$$\delta_n := E_{1n}(f) n^{\alpha} \longrightarrow 0.$$

Then for such n, and for $t \in [0,1]$,

$$t^{\alpha/2} - \delta_n n^{-\alpha} \leq \sqrt{t}/S_n(t) \leq t^{\alpha/2} + \delta_n n^{-\alpha}. \tag{5.5}$$

It then easily follows for such n that

$$\|S_n\|_{L_\infty[n^{-2},1]} \leq C_1 n^{\alpha-1},$$

and by the Bernstein-Walsh inequality,

$$\|S_n\|_{L_\infty[0,1]} \leq C_2 \|S_n\|_{L_\infty[n^{-2},1]} \leq C_3 n^{\alpha-1}. \tag{5.6}$$

But fixing $\eta > 0$ and setting $t = (\eta/n)^2$ in (5.5) yields

$$1/S_n\big((\eta/n)^2\big) = n^{-\alpha+1}\eta^{\alpha-1}\big(1 + O(\delta_n)\big)$$

and hence, for large enough n,

$$S_n\big((\eta/n)^2\big) \geq n^{\alpha-1}\eta^{1-\alpha}/2.$$

For large enough n, and fixed but small enough η, this contradicts (5.6) since $1 - \alpha < 0$. So we have completed the proof of Theorem 2.2. ■

6. **An example.**

Let

$$f(x) := \prod_{j=1}^{\ell} \operatorname{sign}(x - b_j)|x - b_j|^{\alpha_j},$$

where $-1 \leq b_1 < b_2 < \cdots < b_\ell \leq 1$ and $\alpha_j > 0$, $1 \leq j \leq \ell$. If $b_1 > -1$ and $b_\ell < 1$ and we let

$$\alpha := \min_{1 \leq j \leq \ell} \alpha_j, \quad \beta := \max_{1 \leq j \leq \ell} \alpha_j,$$

then for each $n \geq 1$, there exists a polynomial $p_n \in \pi_n$ such that

$$\left| f(x) - \frac{(x - b_1)(x - b_2)\cdots(x - b_\ell)}{p_n(x)} \right| = Cn^{-\alpha}, \tag{6.1}$$

where $C = C(\beta, \ell)$ depends only on β and ℓ. However, if $b_1 = -1$ or $b_\ell = 1$ or both, then in the left-hand side of (6.1) we need take only the product of the factors $x - b_j$ corresponding to $-1 < b_j < 1$. At the same time in the definition of α, we may replace α_1 by $2\alpha_1$ or α_ℓ by $2\alpha_\ell$, respectively, or both. This follows from the following inequality, which is easily proved by induction: For any positive polynomials p_{jn}, $1 \leq j \leq \ell$, we have

$$\left| f(x) - \frac{(x - b_1)(x - b_2)\cdots(x - b_\ell)}{p_{1n}(x)p_{2n}(x)\cdots p_{\ell n}(x)} \right|$$
$$\leq 2^{(\ell-1)\beta+1} \sum_{j=1}^{\ell} \left\| \operatorname{sign}(x - b_j)|x - b_j|^{\alpha_j} - \frac{x - b_j}{p_{jn}(x)} \right\|_{L_\infty[-1,1]}.$$

To each of the terms involving b_j, $-1 < b_j < 1$, apply the estimate of Theorem 2.4, while if $b_1 = -1$ or $b_\ell = 1$, we use the Levin-Saff estimate ((2.7) in [6]). Note that in applying any of these estimates, we first apply a linear transformation taking b_j into 0. Obviously if $\alpha > 1$, (6.1) is better than the rate guaranteed by Theorem 2.1.

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