

DEGREE CONES AND MONOMIAL BASES OF LIE ALGEBRAS AND QUANTUM GROUPS

TEODOR BACKHAUS and XIN FANG

Mathematisches Institut, Universität zu Köln, Cologne, North Rhine-Westphalia, Germany
e-mails: tbackha@math.uni-koeln.de, xinfang.math@gmail.com

and GHISLAIN FOURIER

School of Mathematics and Statistics, University of Glasgow, United Kingdom
Institut für Algebra, Zahlentheorie und Diskrete Mathematik,
Leibniz Universität Hannover, Hannover, Lower Saxony, Germany
e-mail: fourier@math.uni-hannover.de

(Received 10 June 2016; revised 29 September 2016; accepted 30 October 2016;
first published online 20 March 2017)

Abstract. We provide \mathbb{N} -filtrations on the negative part $U_q(\mathfrak{n}^-)$ of the quantum group associated to a finite-dimensional simple Lie algebra \mathfrak{g} , such that the associated graded algebra is a skew-polynomial algebra on \mathfrak{n}^- . The filtration is obtained by assigning degrees to Lusztig’s quantum PBW root vectors. The possible degrees can be described as lattice points in certain polyhedral cones. In the classical limit, such a degree induces an \mathbb{N} -filtration on any finite-dimensional simple \mathfrak{g} -module. We prove for type A_n , C_n , B_3 , D_4 and G_2 that a degree can be chosen such that the associated graded modules are defined by monomial ideals, and conjecture that this is true for any \mathfrak{g} .

1. Introduction.

1.1. Motivation. Let \mathfrak{g} be a finite-dimensional simple Lie algebra with a triangular decomposition $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$. Let $U(\mathfrak{n}^-)$ be the corresponding universal enveloping algebra. Lusztig [18] introduced the canonical basis for the quantized enveloping algebra $U_q(\mathfrak{n}^-)$. Subsequently, Kashiwara [14] gave a different construction of this basis under the name global crystal basis. When the quantum parameter is specialized to 1, the canonical basis is specialized to a linear basis \mathcal{B} of $U(\mathfrak{n}^-)$. Let $V(\lambda)$ be the finite-dimensional irreducible representation of $U(\mathfrak{g})$ of highest weight λ and v_λ be a fixed highest weight vector. The canonical basis \mathcal{B} of $U(\mathfrak{n}^-)$ induces a canonical basis of $V(\lambda)$ by

$$\mathcal{B}_\lambda = \{b \cdot v_\lambda \mid b \in \mathcal{B}, b \cdot v_\lambda \neq 0\}.$$

This is one of the most important properties of the canonical basis.

In this paper, motivated by [7–9], we are interested in the existence of monomial bases \mathcal{E} of $U(\mathfrak{n}^-)$ satisfying the following properties:

- (P1) There exists an \mathbb{N} -filtration \mathcal{F} on $U(\mathfrak{n}^-)$ such that the associated graded algebra is the polynomial algebra $S(\mathfrak{n}^-)$; the set \mathcal{E} is a linear basis of the associated graded algebra.

(P2) Let $V^{\mathcal{F}}(\lambda)$ be the associated graded $S(\mathfrak{n}^-)$ -module with cyclic vector $v_\lambda^{\mathcal{F}}$,

$$\mathcal{E}_\lambda = \{b \cdot v_\lambda^{\mathcal{F}} \mid b \in \mathcal{E}, b \cdot v_\lambda^{\mathcal{F}} \neq 0\}$$

is a linear basis of $V^{\mathcal{F}}(\lambda)$, hence a linear basis of $V(\lambda)$.

An equivalent formulation of (P2) in terms of monomial ideals is:

(P2') find an \mathbb{N} -filtration \mathcal{F} on $U(\mathfrak{n}^-)$ such that for any dominant integral weight λ , the defining ideal of $\text{gr}^{\mathcal{F}} V(\lambda)$ is monomial.

By turning back to quantum groups, we may ask similar questions:

(Q1) Is there an \mathbb{N} -filtration of $U_q(\mathfrak{n}^-)$ such that the associated graded algebra is $S_q(\mathfrak{n}^-)$ (a skew-polynomial algebra on the vector space \mathfrak{n}^-).

1.2. Answering (Q1). The answer to (P1) is rather trivial. We fix a weighted basis of \mathfrak{n}^- , indexed by positive roots Δ^+ , and let $\mathbf{d} : \mathfrak{n}^- \rightarrow \mathbb{N}$ be a degree function on \mathfrak{n}^- such that for any basis elements $x, y \in \mathfrak{n}^-$:

$$\mathbf{d}(x) + \mathbf{d}(y) > \mathbf{d}([x, y]).$$

This induces an \mathbb{N} -filtration on $U(\mathfrak{n}^-)$ and the induced associated graded algebra is isomorphic to $S(\mathfrak{n}^-)$. We denote \mathcal{D} , called the *classical degree cone*, the real cone generated by all degree functions on \mathfrak{n}^- (respectively Δ^+) satisfying these inequalities.

To construct an \mathbb{N} -filtration on $U_q(\mathfrak{n}^-)$, it is not enough to consider its Chevalley generators F_1, \dots, F_n , since $U_q(\mathfrak{n}^-)$ is already a graded algebra for any grading on these generators, and the defining ideals of simple modules are seldom monomial.

There is another basis of $U_q(\mathfrak{n}^-)$ given by Lusztig [19], called quantum PBW basis. Let $\underline{w}_0 = s_{i_1} \dots s_{i_N}$ be a reduced decomposition of the longest Weyl group element. We associate a sequence of elements $F_{\beta_1}, \dots, F_{\beta_N} \in U_q(\mathfrak{n}^-)$, where $\{\beta_1, \dots, \beta_N\}$ is the set of positive roots and F_{β_i} is a quantum PBW root vector of weight $-\beta_i$. Then, Lusztig has shown that ordered monomials in $F_{\beta_1}, \dots, F_{\beta_N}$ form a linear basis of $U_q(\mathfrak{n}^-)$.

The naive approach of setting the degree of F_{β_i} to 1 for all β_i does not provide $\text{gr } U_q(\mathfrak{n}^-) \cong S_q(\mathfrak{n}^-)$ for the induced filtration:

EXAMPLE. Let $\mathfrak{g} = \mathfrak{sl}_4$ be of type A_3 . Fix the reduced decomposition $\underline{w}_0 = s_1 s_2 s_1 s_3 s_2 s_1$ of the longest element w_0 in the Weyl group of \mathfrak{g} . We denote by $F_{i \ i+1 \dots k}$ the quantum PBW root vector corresponding to the root $-(\alpha_i + \alpha_{i+1} + \dots + \alpha_k)$. The following relation holds in $U_q(\mathfrak{n}^-)$:

$$F_{23}F_{12} = F_{12}F_{23} - (q - q^{-1})F_2F_{123}.$$

In general, the commutation relations in $U_q(\mathfrak{n}^-)$ are given by the following Levendorskii–Soibelman (L–S for short) formula: for any $i < j$:

$$F_{\beta_i}F_{\beta_j} - q^{-(\beta_i, \beta_j)}F_{\beta_j}F_{\beta_i} = \sum_{n_{i+1}, \dots, n_{j-1} \geq 0} c(n_{i+1}, \dots, n_{j-1})F_{\beta_{i+1}}^{n_{i+1}} \dots F_{\beta_{j-1}}^{n_{j-1}}.$$

These commutation relations depend heavily on the choice of reduced decomposition \underline{w}_0 . For a given reduced decomposition \underline{w}_0 , we seek for degree functions on the set of

positive roots

$$\mathbf{d} : \Delta_+ \longrightarrow \mathbb{N},$$

such that letting $\text{deg}(F_\beta) = \mathbf{d}(\beta)$ for $\beta \in \Delta_+$ defines a filtered algebra structure on $U_q(\mathfrak{n}^-)$ and the associated graded algebra satisfies $\text{gr}^{\mathbf{d}} U_q(\mathfrak{n}^-) \cong S_q(\mathfrak{n}^-)$. Inspired by the L–S formula, we define for any reduced decomposition \underline{w}_0 the *quantum degree cone* $\mathcal{D}_{\underline{w}_0}^q$ by

$$\mathcal{D}_{\underline{w}_0}^q := \left\{ (d_\beta) \in \mathbb{R}_+^{|\Delta_+|} \mid \text{for any } i < j, d_{\beta_i} + d_{\beta_j} > \sum_{k=i+1}^{j-1} n_k d_{\beta_k} \text{ if } c(n_{i+1}, \dots, n_{j-1}) \neq 0 \right\}.$$

The first main theorem of this paper is:

THEOREM A. *Let \underline{w}_0 be a reduced decomposition. Then,*

- (1) *the set $\mathcal{D}_{\underline{w}_0}^q$ is a non-empty, open polyhedral cone;*
- (2) *a degree function $\mathbf{d} : \Delta_+ \longrightarrow \mathbb{N}$ defines a filtered algebra structure on $U_q(\mathfrak{n}^-)$ such that $\text{gr}^{\mathbf{d}} U_q(\mathfrak{n}^-) \cong S_q(\mathfrak{n}^-)$ if and only if $\mathbf{d} \in \mathcal{D}_{\underline{w}_0}^q \cap \mathbb{Z}^N$.*

It is natural to ask whether there is a uniform degree function \mathbf{d} which is compatible with every reduced decomposition. We will show that for a simple Lie algebra \mathfrak{g} , such a function exists if and only if the rank of \mathfrak{g} is less or equal than 2, i.e. for any \mathfrak{g} of rank larger than 2, we have

$$\bigcap_{\underline{w}_0 \in R(w_0)} \mathcal{D}_{\underline{w}_0}^q = \emptyset,$$

where $R(w_0)$ is the set of all reduced decompositions of w_0 .

Suppose \mathfrak{g} is of simply laced type and the reduced decomposition is adapted to an orientation of the associated Dynkin quiver. Using the Hall algebra realization of $U_q(\mathfrak{n}^-)$, the coordinates of the lattice points in the quantum degree cone have an interpretation as dimensions of certain homomorphism spaces for the particular Dynkin quiver [7].

1.3. Answering (P2). We turn from the quantum situation to the classical one and analyse the implication of the induced filtration for finite-dimensional simple modules.

Let $V(\lambda)$ be the simple module of highest weight λ . Since $V(\lambda) = U(\mathfrak{n}^-) \cdot v_\lambda$, any filtration on $U(\mathfrak{n}^-)$ induces a filtration on $V(\lambda)$.

Let $\mathbf{d} : \Delta_+ \rightarrow \mathbb{N}$ be a degree function for $U(\mathfrak{n}^-)$ such that $\text{gr}^{\mathbf{d}} U(\mathfrak{n}^-) \cong S(\mathfrak{n}^-)$. The associated graded module $\text{gr}^{\mathbf{d}} V(\lambda)$ of the induced filtration is a cyclic $S(\mathfrak{n}^-)$ -module. Hence, there exists an ideal $I^{\mathbf{d}}(\lambda) \subset S(\mathfrak{n}^-)$ such that $\text{gr}^{\mathbf{d}} V(\lambda) \cong S(\mathfrak{n}^-)/I^{\mathbf{d}}(\lambda)$.

Our second aim of the paper is to find monomial bases of $\text{gr}^{\mathbf{d}} V(\lambda)$. If the ideal $I^{\mathbf{d}}(\lambda)$ is monomial, there exists a unique monomial basis for $\text{gr}^{\mathbf{d}} V(\lambda)$. We will focus on this case in the paper.

The *global monomial set* \mathcal{S}_{gm} consists of all degree functions $\mathbf{d} : \Delta_+ \rightarrow \mathbb{N}$ such that for any dominant integral weight λ , $I^{\mathbf{d}}(\lambda)$ is a monomial ideal. As the other main result of this paper, all monomial bases appearing in the context of PBW filtration in the literature can be actually obtained through a degree in the global monomial set.

THEOREM B. *Let \mathfrak{g} be a simple Lie algebra of type A_n, C_n, B_3, D_4 or G_2 . Then, $\mathcal{S}_{\text{gm}} \neq \emptyset$.*

We provide a degree function in the global monomial set in each case (for the A_n -case this has been done already in [7]). Based on the evidence of several further examples, we conjecture:

CONJECTURE. (1) $\mathcal{S}_{\text{gm}} \neq \emptyset$ for any simple finite-dimensional Lie algebra \mathfrak{g} .
 (2) For any simply laced simple Lie algebra, there exists \underline{w}_0 such that $\mathcal{S}_{\text{gm}} \cap \mathcal{D}_{\underline{w}_0}^d$ is non-empty.

1.4. Remarks on the boundary of the classical degree cone. Let \mathfrak{g} be of type A_n . The boundary of \mathcal{D} , denoted by $\partial\mathcal{D}$, is defined as the difference of the closure of \mathcal{D} and its relative interior. Let $S(\partial\mathcal{D}) := \partial\mathcal{D} \cap \mathbb{Z}^{\Delta_+}$ be the lattice points in $\partial\mathcal{D}$.

Let $\mathbf{d} \in S(\partial\mathcal{D})$. Then, \mathbf{d} defines a filtration $\mathcal{F}^{\mathbf{d}}$ on $U(\mathfrak{n}^-)$. In general, the associated graded algebra is no longer the commutative algebra $S(\mathfrak{n}^-)$, but some algebra which is a degenerated version of $U(\mathfrak{n}^-)$ and admits a further degeneration to $S(\mathfrak{n}^-)$. This associated graded algebra is the universal enveloping algebra of the Lie algebra $\mathfrak{n}^{-,\mathbf{d}}$, which is a contraction the Lie bracket of \mathfrak{n}^- on the prescribed roots by \mathbf{d} (see [4] for details).

For $\lambda \in \mathcal{P}_+$, we can similarly define the associated graded module $V^{\mathbf{d}}(\lambda)$: It is a cyclic $U(\mathfrak{n}^{-,\mathbf{d}})$ -module with cyclic vector $v_{\lambda}^{\mathbf{d}}$. It is proved in [4] that the highest weight orbit

$$\mathcal{F}^{\mathbf{d}}(\lambda) := \overline{\exp(\mathfrak{n}^{-,\mathbf{d}}) \cdot [v_{\lambda}^{\mathbf{d}}]} \subset \mathbb{P}(V^{\mathbf{d}}(\lambda))$$

is a flat degeneration of the partial flag variety $\mathcal{F}(\lambda)$.

Moreover, for some $\mathbf{d} \in S(\partial\mathcal{D})$, it is conjectured in [5] that a monomial basis of the representation $V^{\mathbf{d}}(\lambda)$ can be parameterized by the lattice points in a chain-order polytope associated to a marked poset.

1.5. Organization of paper. In Section 2, we fix notations, introduce the classical degree cone, the local and global monomial sets. Quantum degree cones are defined in Section 3, where Theorem A is proved. We provide examples and properties of the quantum degree cone in Section 4. In Section 5, examples for local and global monomials sets are given and Theorem B is proved. We conclude with some examples on quantum degree cones in Section 6.

2. Lie algebras and the classical degree cones.

2.1. Notations and basic properties. Let \mathfrak{g} be a simple Lie algebra of rank n over the field of complex numbers \mathbb{C} . We fix a triangular decomposition $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ and a set of simple roots $\Pi = \{\alpha_1, \dots, \alpha_n\}$ of \mathfrak{g} . The set of positive roots of \mathfrak{g} will be denoted by Δ_+ with cardinality N . Let $\mathcal{Q}_+ = \sum_{i=1}^n \mathbb{N}\varpi_i$ be the root monoid. Let $\rho = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha$ be the half sum of positive roots. For $\alpha \in \Delta_+$, we pick a root vector f_{α} of weight $-\alpha$. Let $\varpi_i, i = 1, \dots, n$ be the fundamental weights, \mathcal{P} be the weight lattice and $\mathcal{P}_+ = \sum_{i=1}^n \mathbb{N}\varpi_i$ be the set of dominant integral weights. For a dominant integral weight $\lambda \in \mathcal{P}_+$, let $V(\lambda)$ be the finite-dimensional irreducible representation of \mathfrak{g} of highest weight λ and v_{λ} a highest weight vector. Let $U(\mathfrak{n}^-)$ be the enveloping algebra

of \mathfrak{n}^- and $S(\mathfrak{n}^-)$ be the symmetric algebra of \mathfrak{n}^- . For $\lambda = \sum_{i=1}^n \lambda_i \varpi_i \in \mathcal{P}_+$, denote the height of λ by $|\lambda| := \sum_{i=1}^n m_i$.

We define $\mathbb{R}^{\Delta_+} := \{f : \Delta_+ \rightarrow \mathbb{R} \text{ is a function}\}$. It is an \mathbb{R} -vector space of dimension N . Let $\mathbb{R}_{\geq 0}^{\Delta_+} \subset \mathbb{R}^{\Delta_+}$ be the set of functions taking positive values, we define similarly \mathbb{N}^{Δ_+} and \mathbb{Z}^{Δ_+} . A function $\mathbf{d} \in \mathbb{R}^{\Delta_+}$ is determined by its values $(d_\beta := \mathbf{d}(\beta))_{\beta \in \Delta_+}$. Once a sequence of positive roots $(\beta_1, \beta_2, \dots, \beta_N)$ is fixed, \mathbb{R}^{Δ_+} is identified with \mathbb{R}^N via: $\mathbf{d} \mapsto (d_{\beta_1}, d_{\beta_2}, \dots, d_{\beta_N})$.

Let W be the Weyl group of \mathfrak{g} with generators s_1, \dots, s_n and $w_0 \in W$ be the longest element. We denote $R(w_0)$ the set of all reduced decompositions of w_0 .

For any reduced decomposition $\underline{w}_0 = s_{i_1} \dots s_{i_N} \in R(w_0)$, we associate a convex total order on Δ_+ : for $1 \leq t \leq N$, we denote $\beta_t = s_{i_1} \dots s_{i_{t-1}}(\alpha_{i_t})$, then $\Delta_+ = \{\beta_1, \dots, \beta_N\}$ and $\beta_1 < \beta_2 < \dots < \beta_N$ is the desired convex total order. It is proved by Papi [20] that the above association induces a bijection between $R(w_0)$ and the set of all convex total orders on Δ_+ .

For the simple Lie algebra \mathfrak{sl}_{n+1} of type A_n and $1 \leq i \leq j \leq n$, we denote $\alpha_{i,j} := \alpha_i + \dots + \alpha_j$, then $\Delta_+ = \{\alpha_{i,j} \mid 1 \leq i \leq j \leq n\}$. For the simple Lie algebra of type B_n and $1 \leq i \leq j \leq n$, we denote $\alpha_{i,j} := \alpha_i + \dots + \alpha_j$ and for $\alpha_{i,\bar{j}} := \alpha_i + \dots + \alpha_n + \alpha_n + \dots + \alpha_j$, then $\Delta_+ = \{\alpha_{i,j}, \alpha_{k,\bar{l}} \mid 1 \leq i \leq j \leq n, 1 \leq k < l \leq n\}$. For the simple Lie algebra \mathfrak{sp}_{2n} of type C_n and $1 \leq i \leq j \leq n$, we denote $\alpha_{i,j} := \alpha_i + \dots + \alpha_j$ and $\alpha_{i,\bar{j}} := \alpha_i + \dots + \alpha_n + \dots + \alpha_j$, notice that $\alpha_{i,n} = \alpha_{i,\bar{n}}$, then $\Delta_+ = \{\alpha_{i,j}, \alpha_{i,\bar{j}} \mid 1 \leq i \leq j \leq n\}$.

For $\mathbf{s} = (s_\alpha)_{\alpha \in \Delta_+} \in \mathbb{N}^{\Delta_+}$, we denote $f^{\mathbf{s}} := \prod_{\alpha \in \Delta_+} f_\alpha^{s_\alpha} \in S(\mathfrak{n}^-)$. For any $\mathbf{d} \in \mathbb{N}^{\Delta_+}$, we denote $\text{deg}_{\mathbf{d}}(f^{\mathbf{s}}) := \sum_{\alpha \in \Delta_+} s_\alpha d_\alpha$.

2.2. The classical degree cone. We start with the classical degree cone.

DEFINITION 2.1. *The classical degree cone \mathcal{D} is defined by*

$$\mathcal{D} := \{\mathbf{d} \in \mathbb{R}_{\geq 0}^{\Delta_+} \mid \text{for any } \alpha, \beta, \gamma \in \Delta_+ \text{ such that } \alpha + \beta = \gamma, d_\alpha + d_\beta > d_\gamma\}.$$

EXAMPLE 2.2. The element \mathbf{e} defined by $e_\alpha = 1$ for all α is in \mathcal{D} for any simple Lie algebra.

By definition, \mathcal{D} is an open polyhedral cone. We let $S(\mathcal{D}) := \mathcal{D} \cap \mathbb{Z}^{\Delta_+}$ denote the set of lattice points in \mathcal{D} . For any $\mathbf{d} = (d_\beta)_{\beta \in \Delta_+} \in S(\mathcal{D})$, we define a filtration $\mathcal{F}^{\mathbf{d}}$ on $U(\mathfrak{n}^-)$ by

$$\mathcal{F}_s^{\mathbf{d}} U(\mathfrak{n}^-) := \text{span}\{f_{\gamma_1} f_{\gamma_2} \dots f_{\gamma_k} \mid \gamma_1, \dots, \gamma_k \in \Delta_+ \text{ such that } d_{\gamma_1} + d_{\gamma_2} + \dots + d_{\gamma_k} \leq s\}.$$

By the cyclicity, every irreducible representation $V(\lambda)$ admits a filtration arising from $\mathcal{F}^{\mathbf{d}}$:

$$\mathcal{F}_s^{\mathbf{d}} V(\lambda) := \mathcal{F}_s^{\mathbf{d}} U(\mathfrak{n}^-) \cdot v_\lambda.$$

Note that for $\mathbf{d} = \mathbf{e}$, this is the PBW filtration, which has been subject to a lot of researches in the past 10 years.

LEMMA 2.3. *For any $\mathbf{d} \in S(\mathcal{D})$, we have the following:*

- (1) $\mathcal{F}^{\mathbf{d}} := (\mathcal{F}_0^{\mathbf{d}} \subset \mathcal{F}_1^{\mathbf{d}} \subset \dots \subset \mathcal{F}_n^{\mathbf{d}} \subset \dots)$ defines a filtration on $U(\mathfrak{n}^-)$ whose associated graded algebra is isomorphic to $S(\mathfrak{n}^-)$.

(2) Let $V^{\mathbf{d}}(\lambda)$ be the graded module associated to the induced filtration. Then, $V^{\mathbf{d}}(\lambda)$ is a cyclic $S(\mathfrak{n}^-)$ -module.

Proof. The universal enveloping algebra $U(\mathfrak{n}^-)$ is a quotient of the tensor algebra $T(\mathfrak{n}^-)$ by the ideal generated by $x \otimes y - y \otimes x - [x, y]$ for all $x, y \in \mathfrak{n}^-$. In \mathfrak{n}^- , for $\alpha, \beta, \gamma \in \Delta_+$ with $\alpha + \beta = \gamma$, $[f_\alpha, f_\beta]$ is a multiple of f_γ ; if $\mathbf{d} \in \mathcal{D}$, we have $d_\alpha + d_\beta > d_\gamma$, which proves the first part of the lemma. The second part is clear. \square

Let $v_\lambda^{\mathbf{d}}$ be a cyclic vector in $V^{\mathbf{d}}(\lambda)$. By (2) of the lemma, the $S(\mathfrak{n}^-)$ -module map

$$\varphi : S(\mathfrak{n}^-) \rightarrow V^{\mathbf{d}}(\lambda), \quad x \mapsto x \cdot v_\lambda^{\mathbf{d}}$$

is surjective. We denote $I^{\mathbf{d}}(\lambda) := \ker \varphi$ and call it the defining ideal of $V^{\mathbf{d}}(\lambda)$.

2.3. The local and global monomial set. We are interested in some particular degrees such that the associated graded module admits “good” bases.

DEFINITION 2.4. *The local monomial set S_{lm} is defined by*

$$S_{\text{lm}} := \{ \mathbf{d} = (d_\beta)_{\beta \in \Delta_+} \in S(\mathcal{D}) \mid \text{for any } i = 1, 2, \dots, n, I^{\mathbf{d}}(\varpi_i) \text{ is a monomial ideal} \}.$$

REMARK 2.5. For any simple Lie algebra \mathfrak{g} , the local monomial set S_{lm} is non-empty. For example, one possibility is to linearly order a monomial basis of any fixed regular representation. The induced order will be in the local monomial set.

DEFINITION 2.6. *The global monomial set S_{gm} is defined by*

$$S_{\text{gm}} := \{ \mathbf{d} = (d_\beta)_{\beta \in \Delta_+} \in S(\mathcal{D}) \mid \text{for any } \lambda \in \mathcal{P}_+, I^{\mathbf{d}}(\lambda) \text{ is a monomial ideal} \}.$$

It is clear that $S_{\text{gm}} \subset S_{\text{lm}}$.

The main goal of this paper is to study the following questions:

- (1) Whether the global monomial set S_{gm} is non-empty? That is to say, does there exist a filtration on $U(\mathfrak{n}^-)$ arising from $\mathbf{d} \in \mathcal{D}$ such that for any finite-dimensional irreducible representation, the defining ideal of the associated graded module is monomial?
- (2) If the answer to the above question is affirmative, for any $\lambda \in \mathcal{P}_+$, we obtain a unique monomial basis for $V^{\mathbf{d}}(\lambda)$ parameterized by $S(\lambda) := \{ \mathbf{s} \in \mathbb{N}^{\Delta_+} \mid f^{\mathbf{s}} \cdot v_\lambda^{\mathbf{d}} \neq 0 \}$. Whether there exists a lattice polytope $P(\lambda)$ such that $S(\lambda)$ is exactly the lattice points in $P(\lambda)$?

2.4. Criteria for the local monomial set. We first give a criterion to decide whether $\mathbf{d} \in \mathcal{D}$ is contained in S_{lm} , which is useful in the rest of the paper.

We fix $\lambda \in \mathcal{P}_+$. For $\mu \in \mathcal{P}$ such that $r_\mu := \dim V(\lambda)_\mu \neq 0$, we denote

$$S_\mu := \left\{ \mathbf{s} \in \mathbb{N}^{\Delta_+} \mid f^{\mathbf{s}} \cdot v_\lambda \neq 0 \in V(\lambda) \text{ and } \sum_{\alpha \in \Delta_+} s_\alpha \alpha = \lambda - \mu \right\} :$$

this is a finite set. Suppose that $S_\mu = \{\mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_{m_\mu}\}$ with

$$\text{deg}_{\mathbf{d}}(f^{\mathbf{s}_1}) \leq \text{deg}_{\mathbf{d}}(f^{\mathbf{s}_2}) \leq \dots \leq \text{deg}_{\mathbf{d}}(f^{\mathbf{s}_{m_\mu}}).$$

Let $T_\mu = \{\mathbf{s}_k \mid f^{\mathbf{s}_k} \cdot v_\lambda \notin \text{span}\{f^{\mathbf{s}_1} \cdot v_\lambda, \dots, f^{\mathbf{s}_{k-1}} \cdot v_\lambda\}\}$. Then by construction, the set $\{f^{\mathbf{s}} \cdot v_\lambda \mid \mathbf{s} \in T_\mu\}$ is a basis.

LEMMA 2.7. *Let $\mu \in \mathcal{P}$. Suppose*

$$\mathbf{s}_k \notin T_\mu \Rightarrow \text{deg}(f^{\mathbf{s}_k}) > \text{deg}(f^{\mathbf{s}_l}) \text{ for all } \mathbf{s}_l \in T_\mu \text{ with } l < k,$$

then the defining ideal $I^{\mathbf{d}}(\lambda)$ is monomial.

Proof. It suffices to show that if $\mathbf{s}_k \notin T_\mu$, then $f^{\mathbf{s}_k} \in I^{\mathbf{d}}(\lambda)_\mu$. Indeed, by definition, $\mathbf{s}_k \notin T_\mu$ implies that $f^{\mathbf{s}_k} \cdot v_\lambda$ is a linear combination of $f^{\mathbf{s}_{i_1}} \cdot v_\lambda, \dots, f^{\mathbf{s}_{i_p}} \cdot v_\lambda$ with $\mathbf{s}_{i_q} \in T_\mu$. By assumption, $\text{deg}(f^{\mathbf{s}_k}) > \text{deg}(f^{\mathbf{s}_{i_q}})$, hence in the graded module, we have $f^{\mathbf{s}_k} \in I^{\mathbf{d}}(\lambda)_\mu$. □

The following corollary is a special case of the lemma; it will be used repeatedly when dealing with the examples.

COROLLARY 2.8. *The defining ideal $I^{\mathbf{d}}(\lambda)$ is monomial, if for any $\mu \in \mathcal{P}$ with $r_\mu \neq 0$:*

- (1) *if $r_\mu = 1$, $\text{deg}_{\mathbf{d}}(f^{\mathbf{s}_1}) < \text{deg}_{\mathbf{d}}(f^{\mathbf{s}_2})$;*
- (2) *if $r_\mu > 1$, $\#\{\text{deg}_{\mathbf{d}}(f^{\mathbf{s}_k}) \mid 1 \leq k \leq m_\mu\} = m_\mu$.*

2.5. How local and global monomial sets are related. We give a sufficient condition for an element in S_{lm} being contained in S_{gm} . Let $\mathbf{d} \in S_{\text{lm}}$ and for $1 \leq i \leq n$, $S^{\mathbf{d}}(\varpi_i) = \{\mathbf{a} \in \mathbb{N}^{\Delta_+} \mid f^{\mathbf{a}} \cdot v_{\varpi_i}^{\mathbf{d}} \neq 0\}$. For an integer $m \geq 1$, let $S^{\mathbf{d}}(\varpi_i)^{+m}$ denote the m -fold Minkowski sum of $S^{\mathbf{d}}(\varpi_i)$. We will write $S(\varpi_i)$ to instead $S^{\mathbf{d}}(\varpi_i)$ when the context is clear.

THEOREM 2.9. *For any $\lambda = m_1\varpi_1 + m_2\varpi_2 + \dots + m_n\varpi_n \in \mathcal{P}_+$, if $\#(S(\varpi_1)^{+m_1} + S(\varpi_2)^{+m_2} + \dots + S(\varpi_n)^{+m_n}) = \dim V(\lambda)$, then $\mathbf{d} \in S_{\text{gm}}$.*

The rest of this paragraph is devoted to the proof of this statement. This is based on the proof of [7, Theorem 3], which handles type A . For any $\tau = \sum_{i=1}^n r_i\varpi_i \in \mathcal{P}_+$, we define

$$S(\tau) := S(\varpi_1)^{+r_1} + S(\varpi_2)^{+r_2} + \dots + S(\varpi_n)^{+r_n}.$$

We want to show simultaneously that for $\lambda, \mu \in \mathcal{P}_+$,

- (1) $\{f^{\mathbf{s}} \cdot v_{\lambda+\mu}^{\mathbf{d}} \mid \mathbf{s} \in S(\lambda + \mu)\}$ is a basis of $V^{\mathbf{d}}(\lambda + \mu)$;
- (2) the defining ideal $I^{\mathbf{d}}(\lambda + \mu)$ is monomial.

The statements will be proved by induction on the height of $\lambda + \mu$. The height 1 case is the assumption $\mathbf{d} \in S_{\text{lm}}$. The induction step will be divided into several parts.

Let $<$ be a total order on $\{f_\beta \mid \beta \in \Delta_+\}$ refining the partial order defined by $\mathbf{d} = (d_\beta)_{\beta \in \Delta_+}$ and consider the induced lexicographical order on the monomials in $U(\mathfrak{n}^-)$. The following proposition is proved essentially in [10, Proposition 2.11]; in [7, Proposition 4], it is proved in detail for a particular degree function for type A_n , but the proof used there is valid for a general \mathbf{d} . Both of the following two propositions are independent of the assumptions in Theorem 2.9.

PROPOSITION 2.10. *For any $\lambda, \mu \in \mathcal{P}^+$ the set $\{f^{\mathbf{s}} \cdot (v_{\lambda}^{\mathbf{d}} \otimes v_{\mu}^{\mathbf{d}}) \mid \mathbf{s} \in S(\lambda + \mu)\}$ is linear independent in $V^{\mathbf{d}}(\lambda) \otimes V^{\mathbf{d}}(\mu)$.*

This set lies in the Cartan component of $V^{\mathbf{d}}(\lambda) \otimes V^{\mathbf{d}}(\mu)$ and since $|S(\lambda + \mu)| = \dim V(\lambda + \mu)$ this set is a basis of the Cartan component of $V(\lambda) \otimes V(\mu)$ and of $V(\lambda + \mu)$, respectively.

PROPOSITION 2.11. *If $\mathbf{s} \notin S(\lambda + \mu)$, then $f^{\mathbf{s}} \cdot v_{\lambda+\mu}^{\mathbf{d}} = 0$ in $V^{\mathbf{d}}(\lambda + \mu)$.*

Proof. We fix $\mathbf{s} \notin S(\lambda + \mu)$ and write

$$f^{\mathbf{s}} \cdot v_{\lambda+\mu} = \sum_{\mathbf{t} \in S(\lambda+\mu)} c_{\mathbf{t}} f^{\mathbf{t}} \cdot v_{\lambda+\mu} \text{ in } V(\lambda + \mu). \tag{1}$$

Since $V(\lambda + \mu) \subset V(\lambda) \otimes V(\mu)$, we have an expansion of the equation (1):

$$f^{\mathbf{s}} \cdot (v_{\lambda} \otimes v_{\mu}) = \sum_{\mathbf{t} \in S(\lambda+\mu), \mathbf{a}+\mathbf{b}=\mathbf{t}} c_{\mathbf{t}} c_{\mathbf{a},\mathbf{b}} f^{\mathbf{a}} \cdot v_{\lambda} \otimes f^{\mathbf{b}} \cdot v_{\mu} \text{ in } V(\lambda) \otimes V(\mu).$$

By replacing those $\mathbf{a} \notin S(\lambda)$ (resp. $\mathbf{b} \notin S(\mu)$) by a sum supported on $S(\lambda)$ (resp. $S(\mu)$), we obtain a unique expression. By induction, the corresponding monomials have strictly lower degrees than $\deg(f^{\mathbf{a}})$ (resp. $\deg(f^{\mathbf{b}})$). This implies that we have for all \mathbf{t} appearing in this unique expression $\deg(f^{\mathbf{t}}) < \deg(f^{\mathbf{s}})$. \square

PROPOSITION 2.12. *Suppose the assumptions of Theorem 2.9 are satisfied, then the set $\mathcal{B} = \{f^{\mathbf{s}} \cdot v_{\lambda+\mu}^{\mathbf{d}} \mid \mathbf{s} \in S(\lambda + \mu)\}$ is a basis of $V^{\mathbf{d}}(\lambda + \mu)$.*

Proof. By considering each filtration component, this is a direct consequence of Proposition 2.11. \square

We are left with proving (2), the monomiality of the annihilating ideal. This follows immediately from the following proposition.

PROPOSITION 2.13. *Suppose the assumptions of Theorem 2.9 are satisfied, then the defining ideal of the Cartan component of $V^{\mathbf{d}}(\lambda) \otimes V^{\mathbf{d}}(\mu)$ is monomial and there exists an $S(\mathfrak{n}^-)$ -module isomorphism from the Cartan component of $V^{\mathbf{d}}(\lambda) \otimes V^{\mathbf{d}}(\mu)$ to $V^{\mathbf{d}}(\lambda + \mu)$.*

Proof. We have for $\mathbf{s} \notin S(\lambda + \mu) = S(\lambda) + S(\mu)$:

$$f^{\mathbf{s}} \cdot (v_{\lambda} \otimes v_{\mu}) = \sum_{\mathbf{t}_1+\mathbf{t}_2=\mathbf{s}} f^{\mathbf{t}_1} \cdot v_{\lambda} \otimes f^{\mathbf{t}_2} \cdot v_{\mu} \text{ in } V(\lambda) \otimes V(\mu)$$

and $\mathbf{t}_1 \notin S(\lambda)$ or $\mathbf{t}_2 \notin S(\mu)$. Hence, by Proposition 2.11, we can conclude that either $f^{\mathbf{t}_1} \cdot v_{\lambda}^{\mathbf{d}} = 0$ in $V^{\mathbf{d}}(\lambda)$ or $f^{\mathbf{t}_2} \cdot v_{\mu}^{\mathbf{d}} = 0$ in $V^{\mathbf{d}}(\mu)$. We obtain

$$f^{\mathbf{s}} \cdot (v_{\lambda}^{\mathbf{d}} \otimes v_{\mu}^{\mathbf{d}}) = 0 \text{ in } V^{\mathbf{d}}(\lambda) \otimes V^{\mathbf{d}}(\mu). \tag{2}$$

Therefore, we have monomiality.

By Proposition 2.11, there is a surjective map of $S(\mathfrak{n}^-)$ -modules from the Cartan component of $V^{\mathbf{d}}(\lambda) \otimes V^{\mathbf{d}}(\mu)$ to $V^{\mathbf{d}}(\lambda + \mu)$, which is an isomorphism for dimension reasons. \square

This proves the monomiality statement (2) and hence Theorem 2.9, i.e. $\mathbf{d} \in \mathcal{S}_{\text{gm}}$.

3. Quantum groups and quantum degree cones.

3.1. Quantum groups. Let $C = (c_{ij})_{n \times n} \in \text{Mat}_n(\mathbb{Z})$ be the Cartan matrix of \mathfrak{g} and $D = \text{diag}(d_1, \dots, d_n) \in \text{Mat}_n(\mathbb{Z})$ be a diagonal matrix symmetrizing C . Thus, $A = DC = (a_{ij})_{n \times n} \in \text{Mat}_n(\mathbb{Z})$ is symmetric. Let $U_q(\mathfrak{g})$ be the corresponding quantum group over $\mathbb{C}(q)$: As an algebra, it is generated by E_i, F_i and $K_i^{\pm 1}$ for $i = 1, \dots, n$, subject to the following relations: For $i, j = 1, \dots, n$,

$$K_i K_i^{-1} = K_i^{-1} K_i = 1, \quad K_i E_j K_i^{-1} = q_i^{c_{ij}} E_j, \quad K_i F_j K_i^{-1} = q_i^{-c_{ij}} F_j,$$

$$E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},$$

and for $i \neq j$,

$$\sum_{r=0}^{1-c_{ij}} (-1)^r E_i^{(1-c_{ij}-r)} E_j E_i^{(r)} = 0, \quad \sum_{r=0}^{1-c_{ij}} (-1)^r F_i^{(1-c_{ij}-r)} F_j F_i^{(r)} = 0,$$

where

$$q_i = q^{d_i}, \quad [n]_q! = \prod_{i=1}^n \frac{q^n - q^{-n}}{q - q^{-1}}, \quad E_i^{(n)} = \frac{E_i^n}{[n]_{q_i}!} \quad \text{and} \quad F_i^{(n)} = \frac{F_i^n}{[n]_{q_i}!}.$$

Let $U_q(\mathfrak{n}^-)$ be the sub-algebra of $U_q(\mathfrak{g})$ generated by F_i for $i = 1, \dots, n$. For $\lambda \in \mathcal{P}_+$, we denote by $V_q(\lambda)$ the finite-dimensional irreducible representation of $U_q(\mathfrak{g})$ of highest weight λ and type 1 with highest weight vector v_λ .

When q is specialized to 1, the quantum group $U_q(\mathfrak{g})$ admits $U(\mathfrak{g})$ as its classical limit. In this limit, the representation $V_q(\lambda)$ is specialized to $V(\lambda)$.

3.2. PBW root vectors and commutation relations. Let $T_i = T''_{i,1}, i = 1, \dots, n$ be Lusztig’s automorphisms:

$$T_i(E_i) = -F_i K_i, \quad T_i(F_i) = -K_i^{-1} E_i, \quad T_i(K_j) = K_j K_i^{-c_{ij}},$$

for $i = 1, \dots, n$, and $j \neq i$,

$$T_i(E_j) = \sum_{r+s=-c_{ij}} (-1)^r q_i^{-r} E_i^{(s)} E_j E_i^{(r)}, \quad T_i(F_j) = \sum_{r+s=-c_{ij}} (-1)^r q_i^r F_i^{(r)} F_j F_i^{(s)}.$$

For details, see Chapter 37 in [19]. We fix a reduced decomposition $w_0 = s_{i_1} \dots s_{i_N} \in R(w_0)$ and let positive roots $\beta_1, \beta_2, \dots, \beta_N$ be defined as in Section 2.1. The quantum PBW root vector F_{β_i} associated to a positive root β_i is defined by

$$F_{\beta_i} = T_{i_1} T_{i_2} \dots T_{i_{i-1}}(F_{i_i}) \in U_q(\mathfrak{n}^-).$$

The PBW theorem of quantum groups affirms that the set

$$\{F^{\mathbf{s}} := F_{\beta_1}^{s_1} F_{\beta_2}^{s_2} \dots F_{\beta_N}^{s_N} \mid \mathbf{s} = (s_1, \dots, s_N) \in \mathbb{N}^N\}$$

forms a $\mathbb{C}(q)$ -basis of $U_q(\mathfrak{n}^-)$ ([19], Corollary 40.2.2).

The commutation relations between these quantum PBW root vectors are given by the following L–S formula: For any $i < j$,

$$F_{\beta_j} F_{\beta_i} - q^{-(\beta_i, \beta_j)} F_{\beta_i} F_{\beta_j} = \sum_{n_{i+1}, \dots, n_{j-1} \geq 0} c(n_{i+1}, \dots, n_{j-1}) F_{\beta_{i+1}}^{n_{i+1}} \dots F_{\beta_{j-1}}^{n_{j-1}}, \tag{3}$$

where $c(n_{i+1}, \dots, n_{j-1}) \in \mathbb{C}[q^{\pm 1}]$. We denote

$$M_{i,j} = \{F_{\beta_{i+1}}^{n_{i+1}} F_{\beta_{i+2}}^{n_{i+2}} \dots F_{\beta_{j-1}}^{n_{j-1}} \mid n_{i+1}\beta_{i+1} + n_{i+2}\beta_2 + \dots + n_{j-1}\beta_{j-1} = \beta_i + \beta_j\},$$

then for weight reasons, the sum in the right-hand side of (3) is supported inside $M_{i,j}$. Denote by $M_{i,j}^q \subset M_{i,j}$, the set of monomials which actually appear with a non-zero coefficient in the right-hand side of (3). It should be pointed out that the right-hand side of (3) largely depends on the chosen reduced decomposition. In general, it is hard to know which monomials appear in $M_{i,j}^q$.

Let us have a closer look on how these formulas depend on the reduced decomposition. Let $\underline{w}_0, \underline{w}'_0 \in R(w_0)$ be two reduced decompositions such that they are of form:

$$\underline{w}_0 = \underline{w}_L s_p s_q \underline{w}_R, \quad \underline{w}'_0 = \underline{w}_L s_q s_p \underline{w}_R$$

with $1 \leq p \neq q \leq n$ and $s_p s_q = s_q s_p$. We define $l = \ell(\underline{w}_L)$.

Let the convex total order on Δ_+ induced by \underline{w}_0 (resp. \underline{w}'_0) be

$$\beta_1 < \beta_2 < \dots < \beta_N \quad (\text{resp. } \beta'_1 < \beta'_2 < \dots < \beta'_N).$$

For $s \leq l$, the L–S formula (3) reads

$$F_{\beta_s} F_{\beta_{l+2}} - q^{(\beta_s, \beta_{l+2})} F_{\beta_{l+2}} F_{\beta_s} = \sum_{n_{s+1}, \dots, n_{l+1} \geq 0} c(n_{s+1}, \dots, n_{l+1}) F_{\beta_{s+1}}^{n_{s+1}} \dots F_{\beta_{l+1}}^{n_{l+1}}. \tag{4}$$

For $t \geq l + 3$, the L–S formula (3) reads

$$F_{\beta_t} F_{\beta_{l+1}} - q^{-(\beta_t, \beta_{l+1})} F_{\beta_{l+1}} F_{\beta_t} = \sum_{n_{l+2}, \dots, n_{t-1} \geq 0} c(n_{l+2}, \dots, n_{t-1}) F_{\beta_{l+2}}^{n_{l+2}} \dots F_{\beta_{t-1}}^{n_{t-1}}. \tag{5}$$

LEMMA 3.1. *In the formula (4), $n_{l+1} = 0$; in the formula (5), $n_{l+2} = 0$.*

Proof. We prove for example the first statement, the second one can be shown similarly.

First, notice that for any $i \neq l + 1, l + 2, \beta_i = \beta'_i, \beta_{l+1} = \beta'_{l+2}, \beta_{l+2} = \beta'_{l+1}$. The same argument can be applied to quantum PBW root vectors: Let $F_{\beta_1}, F_{\beta_2}, \dots, F_{\beta_N}$ (resp. $F'_{\beta_1}, F'_{\beta_2}, \dots, F'_{\beta_N}$) be the quantum PBW root vectors obtained from \underline{w}_0 (resp. \underline{w}'_0). Then, for any $i \neq l + 1, l + 2, F_{\beta_i} = F'_{\beta_i}, F_{\beta_{l+1}} = F'_{\beta_{l+2}}, F_{\beta_{l+2}} = F'_{\beta_{l+1}}$. For $s \leq l$, we apply the L–S formula to F'_{β_s} and $F'_{\beta_{l+1}}$, it gives

$$F'_{\beta_s} F'_{\beta_{l+1}} - q^{(\beta'_s, \beta'_{l+1})} F'_{\beta_{l+1}} F'_{\beta_s} = \sum_{m_{s+1}, \dots, m_l \geq 0} d(m_{s+1}, \dots, m_l) F'^{m_{s+1}}_{\beta_{s+1}} \dots F'^{m_l}_{\beta_l}.$$

Comparing it to (4) gives $n_{l+1} = 0$. □

3.3. Quantum degree cones. We fix in this paragraph a reduced decomposition $\underline{w}_0 \in R(w_0)$ and positive roots β_1, \dots, β_N obtained from \underline{w}_0 as explained in Section 2.1.

DEFINITION 3.2. *The quantum degree cone associated to \underline{w}_0 is defined by*

$$\mathcal{D}_{\underline{w}_0}^q := \left\{ (d_\beta)_{\beta \in \Delta_+} \in \mathbb{R}_{\geq 0}^{\Delta_+} \mid \forall i < j, d_{\beta_i} + d_{\beta_j} > \sum_{k=i+1}^{j-1} n_k d_{\beta_k} \text{ if } c(n_{i+1}, \dots, n_{j-1}) \neq 0 \text{ in (3)} \right\}.$$

We denote the set

$$\mathcal{D}^q := \bigcup_{\underline{w}_0 \in R(w_0)} \mathcal{D}_{\underline{w}_0}^q \subset \mathbb{R}_{\geq 0}^{\Delta_+}.$$

Let $\mathcal{D} \subset \mathbb{R}_{\geq 0}^{\Delta_+}$ be the classical degree cone. Specializing the quantum parameter q to 1 proves the following lemma:

LEMMA 3.3. *We have $\mathcal{D}^q \subset \mathcal{D}$.*

REMARK 3.4. Except for small rank cases $\mathfrak{g} = \mathfrak{sl}_2, \mathfrak{sl}_3$ (see Example 4.1), the inclusion in Lemma 3.3 is strict. For example, the constant function $\mathbf{1}$ taking value 1 on each positive root is in the classical degree cone \mathcal{D} , but for $\mathfrak{g} \neq \mathfrak{sl}_2, \mathfrak{sl}_3$, there is no reduced decomposition \underline{w}_0 such that $\mathbf{1} \notin \mathcal{D}_{\underline{w}_0}^q$. See for example [7, Section 2.4] and Example 4.2 for type C_2 , Section 4.3 for type G_2 .

Let $\mathbf{d} = (d_\beta)_{\beta \in \Delta_+} \in S(\mathcal{D}_{\underline{w}_0}^q) =: \mathcal{D}_{\underline{w}_0}^q \cap \mathbb{N}^{\Delta_+}$. For a monomial $F^{\mathbf{t}}$ where $\mathbf{t} = (t_1, \dots, t_N)$, we define its \mathbf{d} -degree $\text{deg}_{\mathbf{d}}$ by

$$\text{deg}_{\mathbf{d}}(F^{\mathbf{t}}) := t_1 d_{\beta_1} + t_2 d_{\beta_2} + \dots + t_N d_{\beta_N}.$$

THEOREM 3.5. *The set $\mathcal{D}_{\underline{w}_0}^q$ is a non-empty open polyhedral cone.*

Proof. By definition, $\mathcal{D}_{\underline{w}_0}^q$ is an open polyhedral cone. We describe an inductive procedure to construct an element $\mathbf{d} = (d_{\beta_1}, \dots, d_{\beta_N}) \in \mathcal{D}_{\underline{w}_0}^q$.

We set $d_{\beta_1} = 1$. Suppose that $d_{\beta_1}, \dots, d_{\beta_k}$ are chosen such that they satisfy the inequalities in the definition of $\mathcal{D}_{\underline{w}_0}^q$.

Let $M_{k+1}^q := \bigcup_{s=1}^k M_{s,k+1}^q$. Since M_{k+1}^q is a finite set, we set

$$d_{\beta_{k+1}} = 1 + \max_{F^{\mathbf{t}} \in M_{k+1}^q} (\text{deg}_{\mathbf{d}}(F^{\mathbf{t}})).$$

Since $F^{\mathbf{t}} \in M_{k+1}^q$ is a monomial on $\{F_{\beta_1}, \dots, F_{\beta_k}\}$, the degree is well-defined. By definition, for any $1 \leq s \leq k$ and any $F^{\mathbf{t}} \in M_{s,k+1}^q$, $d_{\beta_s} + d_{\beta_{k+1}} > \text{deg}_{\mathbf{d}}(F^{\mathbf{t}})$. This terminates the proof. \square

For $\mathbf{d} \in S(\mathcal{D}_{\underline{w}_0}^q)$, we define a filtration $\mathcal{F}_{\bullet}^{\mathbf{d}} = (\mathcal{F}_0^{\mathbf{d}} \subset \mathcal{F}_1^{\mathbf{d}} \subset \dots \subset \mathcal{F}_n^{\mathbf{d}} \subset \dots)$ on $U_q(\mathfrak{n}^-)$ by

$$\mathcal{F}_k^{\mathbf{d}} U_q(\mathfrak{n}^-) := \text{span}\{F^{\mathbf{t}} \mid \text{deg}_{\mathbf{d}}(F^{\mathbf{t}}) \leq k\}.$$

Let $S_q(\mathfrak{n}^-)$ be the algebra generated by x_1, x_2, \dots, x_N , subjects to the following relations: For $1 \leq i < j \leq N$,

$$x_i x_j = q^{(\beta_i, \beta_j)} x_j x_i.$$

The following proposition is clear from the L–S formula (3).

PROPOSITION 3.6.

- (1) *The filtration \mathcal{F}_\bullet endows $U_q(\mathfrak{n}^-)$ with a filtered algebra structure.*
- (2) *The associated graded algebra $\text{gr}_{\mathcal{F}} U_q(\mathfrak{n}^-)$ is isomorphic to $S_q(\mathfrak{n}^-)$.*

For $\lambda \in \mathcal{P}_+$, the above filtration on $U_q(\mathfrak{n}^-)$ induces a filtration on $V_q(\lambda)$ by letting

$$\mathcal{F}_k^{\mathbf{d}} V_q(\lambda) := \mathcal{F}_k^{\mathbf{d}} U_q(\mathfrak{n}^-) \cdot \mathbf{v}_\lambda.$$

We let $V_q^{\mathbf{d}}(\lambda)$ denote the associated graded vector space: It is a cyclic $S_q(\mathfrak{n}^-)$ -module. Let $\mathbf{v}_\lambda^{\mathbf{d}}$ be the cyclic vector corresponding to \mathbf{v}_λ .

4. Examples of rank 2 and properties of quantum degree cones.

4.1. Examples of rank 2. Before studying properties of these cones, we examine some small rank examples.

EXAMPLE 4.1. Let $\mathfrak{g} = \mathfrak{sl}_3$ be the Lie algebra of type A_2 . For $\mathbf{d} \in \mathcal{D}$, let $d_{i,j} = \mathbf{d}(\alpha_{i,j})$.

We fix a sequence of positive roots $(\alpha_{1,1}, \alpha_{1,2}, \alpha_{2,2})$. The classical degree cone \mathcal{D} is given by

$$\mathcal{D} = \{(d_{1,1}, d_{1,2}, d_{2,2}) \in \mathbb{R}_{\geq 0}^{\Delta_+} \mid d_{1,1} + d_{2,2} > d_{1,2}\}.$$

We consider the quantum degree cones: $R(w_0) = \{s_1 s_2 s_1, s_2 s_1 s_2\}$. For the reduced decomposition $\underline{w}_0 = s_1 s_2 s_1$, let $F_{1,1}, F_{1,2}, F_{2,2}$ be the corresponding quantum PBW root vectors. The formula (3) reads

$$F_{1,1} F_{2,2} = q^{-1} F_{2,2} F_{1,1} - q^{-1} F_{1,2},$$

implying $\mathcal{D}_{\underline{w}_0}^q = \mathcal{D}$. For $\underline{w}'_0 = s_2 s_1 s_2$, the same computation shows that $\mathcal{D}_{\underline{w}'_0}^q = \mathcal{D}$.

EXAMPLE 4.2. Let $\mathfrak{g} = \mathfrak{sp}_4$ be the Lie algebra of type C_2 .

For $\mathbf{d} \in \mathcal{D}$, let $d_{i,j} := \mathbf{d}(\alpha_{i,j})$ and $d_{i,\bar{j}} := \mathbf{d}(\alpha_{i,\bar{j}})$. The classical degree cone \mathcal{D} is given by the following inequalities in $\mathbb{R}_{\geq 0}^{\Delta_+}$:

$$d_{1,1} + d_{2,2} > d_{1,2}, \quad d_{1,1} + d_{1,2} > d_{1,\bar{1}}.$$

Fix a reduced decomposition $\underline{w}_0 = s_1 s_2 s_1 s_2$ of the longest element w_0 in the Weyl group of \mathfrak{g} . Let

$$F_{1,1}, F_{1,\bar{1}}, F_{1,2}, F_{2,2}$$

be the corresponding quantum PBW root vectors, their commutation relations are

$$\begin{aligned} F_{1,1} F_{1,\bar{1}} &= q^2 F_{1,\bar{1}} F_{1,1}, & F_{1,1} F_{1,2} &= F_{1,2} F_{1,1} - (q + q^{-1}) F_{1,\bar{1}}, \\ F_{1,1} F_{2,2} &= q^{-2} F_{2,2} F_{1,1} - q^{-2} F_{1,2}, \end{aligned}$$

$$F_{1,\bar{1}}F_{1,2} = q^2F_{1,2}F_{1,\bar{1}}, \quad F_{1,\bar{1}}F_{2,2} = F_{2,2}F_{1,\bar{1}} + (1 - q^{-2})F_{1,2}^{(2)}, \quad F_{1,2}F_{2,2} = q^2F_{2,2}F_{1,2}.$$

The quantum degree cone $\mathcal{D}_{\underline{w}_0}^q \subset \mathcal{D}$ is given by

$$d_{1,1} + d_{2,2} > d_{1,2}, \quad d_{1,1} + d_{1,2} > d_{1,\bar{1}}, \quad d_{2,2} + d_{1,\bar{1}} > 2d_{1,2}. \tag{6}$$

The same construction with the reduced decomposition $\underline{w}'_0 = s_2s_1s_2s_1$ shows that $\mathcal{D}_{\underline{w}_0}^q = \mathcal{D}_{\underline{w}'_0}^q$.

EXAMPLE 4.3. Let \mathfrak{g} be the Lie algebra of type G_2 with positive roots

$$\Delta_+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}.$$

For $\mathbf{d} \in \mathbb{R}_{\geq 0}^{\Delta_+}$, we write $d_1 = \mathbf{d}(\alpha_1)$, $d_2 = \mathbf{d}(\alpha_2)$, $d_{12} = \mathbf{d}(\alpha_1 + \alpha_2)$, $d_{112} = \mathbf{d}(2\alpha_1 + \alpha_2)$, $d_{1112} = \mathbf{d}(3\alpha_1 + \alpha_2)$ and $d_{11122} = \mathbf{d}(3\alpha_1 + 2\alpha_2)$. The classical degree cone $\mathcal{D} \subset \mathbb{R}_{\geq 0}^{\Delta_+}$ is determined by the following inequalities:

$$d_1 + d_2 > d_{12}, \quad d_1 + d_{12} > d_{112}, \quad d_1 + d_{112} > d_{1112},$$

$$d_2 + d_{1112} > d_{11122}, \quad d_{112} + d_{12} > d_{11122}.$$

For example, $(d_1, d_{1112}, d_{112}, d_{11122}, d_{12}, d_2) = (2, 1, 3, 1, 3, 2) \in \mathcal{D}$.

We fix a reduced decomposition $\underline{w}_0 = s_1s_2s_1s_2s_1s_2 \in R(w_0)$. Let

$$F_1, F_{1112}, F_{112}, F_{11122}, F_{12}, F_2$$

be the corresponding quantum PBW root vectors. The quantum degree cone $\mathcal{D}_{\underline{w}_0}^q$ in \mathcal{D} is given by the following inequalities:

$$\begin{aligned} d_1 + d_{11122} &> 2d_{112}, & d_{1112} + d_{11122} &> 3d_{112}, & d_{1112} + d_{12} &> 2d_{112}, \\ d_{1112} + d_2 &> d_{112} + d_{12}, & d_{112} + d_2 &> 2d_{12}, & d_{11122} + d_2 &> 3d_{12}. \end{aligned} \tag{7}$$

These inequalities are the same for the other reduced decomposition $\underline{w}'_0 = s_2s_1s_2s_1s_2s_1 \in R(w_0)$. It is clear that $\mathbf{d} \notin \mathcal{D}_{\underline{w}_0}^q$ for any $\underline{w}_0 \in R(w_0)$, we see once again that the quantum degree cone is different from the classical degree cone.

In the rank 2 case, the quantum degree cone does not depend on the reduced decomposition.

PROPOSITION 4.4. *Let \mathfrak{g} be a simple Lie algebra of rank no more than 2. For any $\underline{w}_0 \in R(w_0)$, we have $\mathcal{D}^q = \mathcal{D}_{\underline{w}_0}^q$.*

Proof. The proof follows from the preceding examples. □

4.2. Properties of quantum degree cones. The first property of the quantum degree cones we will prove is the following:

THEOREM 4.5. *Let \mathfrak{g} be a simple Lie algebra of rank $n \geq 3$, then*

$$\bigcap_{\underline{w}_0 \in R(w_0)} \mathcal{D}_{\underline{w}_0}^q = \emptyset.$$

Proof. We show that there exist two reduced decompositions $\underline{w}_0^1, \underline{w}_0^2 \in R(w_0)$ such that $\mathcal{D}_{\underline{w}_0^1}^q \cap \mathcal{D}_{\underline{w}_0^2}^q = \emptyset$. When \mathfrak{g} is a simple Lie algebra \mathfrak{g} of rank 3, this is proved in Section 6.1–6.3 by explicit constructions.

Let \mathfrak{g} be a simple Lie algebra of rank > 3 . There exists a Lie sub-algebra $\mathfrak{g}' \subset \mathfrak{g}$ of rank 3 such that \mathfrak{g}' is a simple Lie algebra, we denote it by X_3 . The set of positive roots of X_3 is denoted by Δ'_+ . We take \underline{w}_L and \underline{w}'_L as in the example of X_3 in Section 6.1–6.3, such that $\mathcal{D}_{\underline{w}_L}^q \cap \mathcal{D}_{\underline{w}'_L}^q = \emptyset$. Let $\underline{w}_0^1 = \underline{w}_L \underline{w}_R$ and $\underline{w}_0^2 = \underline{w}'_L \underline{w}'_R \in R(w_0)$. We claim that $\mathcal{D}_{\underline{w}_0^1}^q \cap \mathcal{D}_{\underline{w}_0^2}^q = \emptyset$. Let $p : \mathcal{D}_{\underline{w}_0^1}^q \rightarrow \mathbb{R}_{\geq 0}^{\Delta'_+}$ be the restriction of functions. Then by definition,

$$p(\mathcal{D}_{\underline{w}_0^1}^q) = \mathcal{D}_{\underline{w}_L}^q, \quad p(\mathcal{D}_{\underline{w}_0^2}^q) = \mathcal{D}_{\underline{w}'_L}^q.$$

This terminates the proof. □

This theorem implies that there is no degree function working for all reduced decompositions. In general, to study the relations between the cones associated to \underline{w}_0 and $\underline{w}'_0 \in R(w_0)$ is a difficult task. But in some cases, the cone remain the same.

Two reflections s_p and s_q in W with $p \neq q$ are said to be *orthogonal* if $s_p s_q = s_q s_p$. Two reduced decompositions $\underline{w}_0, \underline{w}'_0 \in R(w_0)$ are said to be *related by orthogonal reflections* if one can be obtained from the other by using only orthogonal reflections.

PROPOSITION 4.6. *Let $\underline{w}_0, \underline{w}'_0 \in R(w_0)$ such that they are related by orthogonal reflections. Then, $\mathcal{D}_{\underline{w}_0}^q = \mathcal{D}_{\underline{w}'_0}^q$.*

Proof. By definition, it suffices to consider the case where

$$\underline{w}_0 = \underline{w}_L s_p s_q \underline{w}_R, \quad \underline{w}'_0 = \underline{w}_L s_q s_p \underline{w}_R$$

with $1 \leq p, q \leq n$ such that $s_p s_q = s_q s_p$. In this case, Lemma 3.1 can be applied to finish the proof. □

5. Local and global monomial sets.

5.1. Local monomial set for type A_n . Let $\mathfrak{g} = \mathfrak{sl}_{n+1}$ be the Lie algebra of type A_n .

The following lemma gives an easy criterion to determine whether a degree is in the local monomial set.

PROPOSITION 5.1. *Let $\mathbf{d} \in S(\mathcal{D})$. The following statements are equivalent:*

- (1) *For any four different positive roots $\alpha, \beta, \gamma, \delta$ satisfying $\alpha + \beta = \gamma + \delta$, $d_\alpha + d_\beta \neq d_\gamma + d_\delta$;*
- (2) $\mathbf{d} \in \mathcal{S}_{\text{lm}}$.

Proof. (1) \Rightarrow (2): Since in the A_n case, all fundamental representations are minuscule. The proof of [7, Proposition 2] can be applied to show the validity of the hypothesis of Corollary 2.8.

(2) \Rightarrow (1): Since $\mathfrak{g} = \mathfrak{sl}_{n+1}$, we can suppose that $\alpha = \alpha_{i,j}, \beta = \alpha_{k,l}, \gamma = \alpha_{i,l}$ and $\delta = \alpha_{j,k}$ with $i \leq k \leq j \leq l$. We consider the fundamental representation $V(\varpi_l)$: In $I(\varpi_l)$, there is a relation $f_{i,j} f_{k,l} v_{\varpi_l} \pm f_{i,l} f_{k,j} v_{\varpi_l} = 0$. Since $I^{\mathbf{d}}(\varpi_l)$ is monomial, either $f_{i,j} f_{k,l}$ or $f_{i,l} f_{k,j}$ is in $I^{\mathbf{d}}(\varpi_l)$, which forbids the case $d_{\alpha_{i,j}} + d_{\alpha_{k,l}} = d_{\alpha_{i,l}} + d_{\alpha_{j,k}}$. □

EXAMPLE 5.2. In general, the inclusions $\mathcal{D}_{\underline{w}_0}^q \subset \mathcal{S}_{\text{lm}}$ and $\mathcal{S}_{\text{lm}} \subset \mathcal{D}^q$ do not hold. Let \mathfrak{g} be of type A_3 . The reduced decomposition $\underline{w}_0 = s_1 s_2 s_3 s_2 s_1 s_2 \in R(w_0)$ induces the convex order on Δ_+ :

$$\alpha_{1,1} < \alpha_{1,2} < \alpha_{1,3} < \alpha_{3,3} < \alpha_{2,3} < \alpha_{2,2}.$$

We fix this sequence and identify \mathbb{N}^{Δ_+} with \mathbb{N}^6 . Let $\mathbf{d} = (1, 1, 1, 1, 1, 1)$, $\mathbf{d}' = (2, 2, 1, 1, 1, 1)$ and $\mathbf{d}'' = (1, 1, 1, 1, 1, 2)$. By Proposition 5.1, $\mathbf{d} \notin \mathcal{S}_{\text{lm}}$, $\mathbf{d}', \mathbf{d}'' \in \mathcal{S}_{\text{lm}}$, but $\mathbf{d}, \mathbf{d}' \in \mathcal{D}_{\underline{w}_0}^q$, $\mathbf{d}'' \notin \mathcal{D}^q$.

We show in the following example that \mathcal{S}_{gm} is in general a proper subset of \mathcal{S}_{lm} .

EXAMPLE 5.3. Let \mathfrak{g} be of type A_n and let \mathbf{d} be defined by $d_{\alpha_{ij}} = 2^{(n-1)-(j-i)}$. It is clear that $\mathbf{d} \in \mathcal{D}$. If $\alpha_{i,j}, \alpha_{k,l}, \alpha_{i,l}, \alpha_{k,j}$ are four different positive roots in Δ_+ such that $\alpha_{i,j} + \alpha_{k,l} = \alpha_{i,l} + \alpha_{k,j}$, the indices must satisfy $1 \leq i < k \leq l < j \leq n$. In this case, we have $d_{\alpha_{i,j}} + d_{\alpha_{k,l}} > d_{\alpha_{i,l}} + d_{\alpha_{k,j}}$. Hence, by Proposition 5.1, we have $\mathbf{d} \in \mathcal{S}_{\text{lm}}$.

For arbitrary $1 \leq i \leq n$, let $P(\varpi_i)$ be the polytope obtained in [1], such that its lattice points $S(\varpi_i) := P(\varpi_i) \cap \mathbb{N}^{\Delta_+}$ parameterizes a basis of $V(\varpi_i)$. Furthermore, by the choice of \mathbf{d} , we have

$$S(\varpi_i) = \{ \mathbf{s} \in \mathbb{N}^{\Delta_+} \mid f^{\mathbf{s}} \cdot v_{\varpi_i}^{\mathbf{d}} \neq 0 \text{ in } V^{\mathbf{d}}(\varpi_i) \}.$$

But in general, for $\lambda = m_1 \varpi_1 + \dots + m_n \varpi_n \in \mathcal{P}_+$, the Minkowski sum of lattice points $S(\varpi_1)^{+m_1} + \dots + S(\varpi_n)^{+m_n}$ may not parameterize a basis of $V(\lambda)$. For instance, let \mathfrak{g} be of type A_4 , we have [loc.cit]: $\#(S(\varpi_1) + S(\varpi_2) + S(\varpi_3) + S(\varpi_4)) = 1023$ but $\dim V(\varpi_1 + \varpi_2 + \varpi_3 + \varpi_4) = 1024$. Hence, in general, $\mathbf{d} \notin \mathcal{S}_{\text{gm}}$.

5.2. FFLV polytopes. We start with recalling the Dyck paths and FFLV polytopes [8], [9].

A sequence $\mathbf{b} = (\delta_1, \dots, \delta_r)$ of positive roots is called a *Dyck path of type A_n* if $\delta_1 = \alpha_{i,i}$ and $\delta_r = \alpha_{j,j}$ for $i \leq j$ are simple roots, and if $\delta_m = \alpha_{p,q}$, then $\delta_{m+1} = \alpha_{p+1,q}$ or $\delta_{m+1} = \alpha_{p,q+1}$.

Let $A = \{1, 2, \dots, n, \overline{n-1}, \dots, \overline{1}\}$ be the totally ordered index set $1 < 2 < \dots < n < \overline{n-1} < \dots < \overline{1}$. A *symplectic Dyck path* is a sequence $\mathbf{b} = (\delta_1, \dots, \delta_r)$ of positive roots (of \mathfrak{sp}_{2n}) such that the first root is a simple root, $\beta_1 = \alpha_{i,i}$; the last root is either a simple root $\beta_r = \alpha_{j,j}$ or $\beta_r = \alpha_{j\bar{j}}$ ($i \leq j \leq n$); if $\beta_m = \alpha_{r,q}$ with $r, q \in A$, then β_{m+1} is either $\alpha_{r,q+1}$ or $\alpha_{r+1,q}$, where $x+1$ denotes the smallest element in A which is bigger than x .

For a dominant integral weight $\lambda = \lambda_1 \varpi_1 + \lambda_2 \varpi_2 + \dots + \lambda_n \varpi_n$ in the corresponding weight lattice, the FFLV polytopes $P_{A_n}(\lambda)$ and $P_{C_n}(\lambda)$ are defined by

$$P_{A_n}(\lambda) = \left\{ \mathbf{m} \in \mathbb{R}_{\geq 0}^{\Delta_+} \mid \begin{array}{l} \text{for any } i = 1, \dots, n \text{ and any Dyck paths } \mathbf{b} = (\delta_1, \dots, \delta_r) \\ \text{starting in } \alpha_{i,i}, \text{ ending in } \alpha_{j,j} : \sum_{\ell=1}^r m_{\delta_\ell} \leq \lambda_i + \dots + \lambda_j \end{array} \right\},$$

$$P_{C_n}(\lambda) = \left\{ \mathbf{m} \in \mathbb{R}_{\geq 0}^{\Delta_+} \mid \begin{array}{l} \text{for any } i = 1, \dots, n \text{ and any symplectic Dyck paths } \mathbf{b} = (\delta_1, \dots, \delta_r) \\ \text{starting in } \alpha_{i,i}, \text{ ending in } \alpha_{j,j} : \sum_{\ell=1}^r m_{\delta_\ell} \leq \lambda_i + \dots + \lambda_j; \\ \text{for any } i = 1, \dots, n \text{ and any symplectic Dyck paths } \mathbf{b} = (\delta_1, \dots, \delta_r) \\ \text{starting in } \alpha_{i,i}, \text{ ending in } \alpha_{j\bar{j}} : \sum_{\ell=1}^r m_{\delta_\ell} \leq \lambda_i + \dots + \lambda_n \end{array} \right\}.$$

Let $S_{A_n}(\lambda)$ and $S_{C_n}(\lambda)$ denote the set of lattice points in the corresponding polytopes. It has been shown in [8] and [9] that the polytopes satisfy for all $\lambda = \lambda_1 + \lambda_2$:

$$S_{A_n}(\lambda) = S_{A_n}(\lambda_1) + S_{A_n}(\lambda_2) \text{ and } S_{C_n}(\lambda) = S_{C_n}(\lambda_1) + S_{C_n}(\lambda_2). \tag{8}$$

5.3. Global monomial sets: A_n . For A_n , consider $\mathbf{d} \in \mathcal{D}$ defined by $d_{\alpha_{ij}} = (j - i + 1)(n - j + 1)$, then the following theorem has been proved in [7]:

THEOREM 5.4.

- (1) We have $\mathbf{d} \in S_{\mathfrak{g}_{\text{m}}}$. Moreover, let $\underline{w}_0 = (s_n \dots s_1)(s_n \dots s_2) \dots (s_n s_{n-1})s_n$, then $\mathbf{d} \in \mathcal{D}_{\underline{w}_0}^q$.
- (2) The set $\{f^{\mathbf{a}} \cdot v_{\lambda}^{\mathbf{d}} \mid \mathbf{a} \in S_{A_n}(\lambda)\}$ forms a monomial basis of $V^{\mathbf{d}}(\lambda)$.

5.4. Global monomial sets: C_n . Let us consider the C_n case and $\mathbf{d} \in \mathcal{D}$ defined by

$$d_{i,j} := d_{\alpha_{ij}} := (2n - j)(j - i + 1), \quad d_{i,\bar{j}} := d_{\alpha_{i,\bar{j}}} := j(2n - i - j + 1).$$

This degree arises from an embedding of \mathfrak{g} into a Lie algebra of type A_{2n-1} . We will show that $\mathbf{d} \in S_{\mathfrak{g}_{\text{m}}}$ and moreover, the monomial basis of $V^{\mathbf{d}}(\lambda)$ is parameterized by $S_{C_n}(\lambda)$. For this, we need an explicit description of the monomials associated to $S_{C_n}(\varpi_k)$ from [9]:

$$\{f_{i_1, j_{\ell-1}} \cdots f_{i_{\ell}, j_1} \mid 1 < i_1 < \dots < i_{\ell} \leq k \leq j_1 < \dots < j_{\ell}\}.$$

LEMMA 5.5. *The degree function $\mathbf{d} \in S_{\text{lm}}$ and for any fundamental weight ϖ_k ,*

$$\{f^{\mathbf{a}} \cdot v_{\varpi_k}^{\mathbf{d}} \mid \mathbf{a} \in S_{C_n}(\varpi_k)\} \text{ is a basis of } V^{\mathbf{d}}(\varpi_k).$$

Proof. We need to show that the annihilating ideal of $V^{\mathbf{d}}(\varpi_k)$ is monomial for all ϖ_k . We start with the natural representation, namely the vector space \mathbb{C}^{2n} with basis $\{e_1, \dots, e_n, e_{\bar{n}}, \dots, e_{\bar{1}}\}$ and operation $f_{i,j}e_{\ell} = \delta_{i,\ell}c_{i,j}e_{j+1}$ for some $c_{i,j} \in \mathbb{C}^*$ (when $j = \bar{p}$, we set $j + 1 := \bar{p} - 1$). We will further use that we can identify $V(\varpi_k)$ uniquely with a submodule in $\bigwedge^k \mathbb{C}^{2n}$.

First of all, since $\mathbf{d} \in \mathcal{D}$, we see that we can restrict ourselves to the nilpotent radical of the fundamental weight ϖ_k (since all other root vectors are acting by 0 on $v_{\varpi_k} \in V(\varpi_k)$ and hence on $V^{\mathbf{d}}(\varpi_k)$), e.g. we have to consider monomials in $M_k := \{f_{i,j} \mid i \leq k \leq j\}$ only. We will prove the lemma in two steps:

- (1) For any $i \leq k < j$, there exists a unique monomial \mathbf{m} in the variables from M_k with minimal degree such that $\mathbf{m} \cdot e_i = e_j$.
- (2) For any $j_1 < j_2 < \dots < j_k$ with $e_{j_1} \wedge \dots \wedge e_{j_k} \in V(\varpi_k)$, there exists a unique monomial \mathbf{m} in the variables from M_k with minimal degree such that $\mathbf{m} \cdot e_1 \wedge e_2 \wedge \dots \wedge e_k = e_{j_1} \wedge e_{j_2} \wedge \dots \wedge e_{j_k}$.

Then, the second step implies the Lemma.

We start with proving (1). Suppose $i \leq k \leq j \leq \bar{k}$, for weight reasons, there exists a unique monomial \mathbf{m} in the variables from M_k such that $\mathbf{m} \cdot e_i = e_j$, namely $\mathbf{m} = f_{\alpha_{i,j-1}}$. Suppose $i \leq k \leq n < \bar{k} \leq \bar{p}$, and for simplicity, we assume that $i \leq p$ (the $p \leq i$ case is

similar). Let \mathbf{m} be a monic monomial in the variables from M_k such that $\mathbf{m} \cdot e_i = e_{\overline{p-1}}$, then for weight reasons \mathbf{m} is in one of the following sets:

$$\{f_{i,q-1}f_{p,\overline{q}}, f_{i,\overline{q}}f_{p,q-1} \mid q = k + 1, \dots, n\} \cup \{f_{i,n}f_{p,n}\} \cup \{f_{i,\overline{p}}\}. \tag{9}$$

We will see that among these monomials, $f_{i,\overline{p}}$ is the unique monomial of minimal degree, namely of degree $p(2n - i - p + 1)$.

Let $i \leq q \leq n$ and denote

$$\begin{aligned} \text{for } p < q, & \quad Y(q) := q(2n - i - q + 1) + (2n - (q - 1))(q - p); \\ \text{for } p \geq q, & \quad Y(q) := q(2n - i - q + 1) + (2n - p)(p - q + 2); \\ \text{for any } p, & \quad X(q) := (2n - (q - 1))((q - 1) - i + 1) + q(2n - p - q + 1). \end{aligned}$$

We have

$$\begin{aligned} \text{if } p \leq q, & \text{ then } \deg_{\mathbf{d}}(f_{i,q-1}f_{p,\overline{q}}) = X(q); \\ \text{if } p > q, & \text{ then } \deg_{\mathbf{d}}(f_{i,q-1}f_{q,\overline{p}}) = X(q); \\ \text{if } p < q, & \text{ then } \deg_{\mathbf{d}}(f_{i,\overline{q}}f_{p,q-1}) = Y(q); \\ \text{if } p \geq q, & \text{ then } \deg_{\mathbf{d}}(f_{i,\overline{q}}f_{q-1,p}) = Y(q). \end{aligned}$$

Now, it is straightforward to see that for $q > i$:

$$X(q) > X(q - 1) \text{ and moreover } X(i) > p(2n - i - p + 1),$$

as well as

$$Y(q) > Y(q + 1) \text{ and } Y(n) = X(n).$$

Combining both gives

$$Y(i) > \dots > Y(n) = X(n) > \dots > X(i) > p(2n - i - p + 1) = \deg_{\mathbf{d}}f_{i,\overline{p}}.$$

Moreover,

$$\deg_{\mathbf{d}}(f_{i,n}f_{p,n}) = n(2n - i - p + 2) > p(2n - i - p + 1) = p(2n - i - p + 1) = \deg_{\mathbf{d}}f_{i,\overline{p}}.$$

This implies, that $f_{i,\overline{p}}$ is the unique monomial of minimal degree among all monomials in (9) and the first step is done.

We are left with step (2). Let $e_{i_1} \wedge \dots \wedge e_{i_k} \in V(\varpi_k) \subset \bigwedge^k \mathbb{C}^{2n}$, with $i_1 < \dots < i_k$ and $i_j \in \{1, \dots, \overline{1}\}$. Let \mathbf{m} be a monomial of minimal degree in the variables from M_k such that

$$\mathbf{m} \cdot e_1 \wedge \dots \wedge e_k = e_{i_1} \wedge \dots \wedge e_{i_k} + \text{rest}.$$

Due to the operation on the tensor product, there exists a factorization $\mathbf{m} = \prod_{\ell=1}^k \mathbf{m}_{\ell}$ and a permutation $\sigma \in \mathfrak{S}_k$, such that $\mathbf{m}_{\ell} \cdot e_{\ell} = e_{i_{\sigma(\ell)}}$. Since \mathbf{m} is in the variables from M_k only, we see that if $\ell \in \{i_1, \dots, i_k\} \cap \{1, \dots, k\}$, then $\mathbf{m}_{\ell} = 1$ and hence $i_{\sigma(\ell)} = \ell$. So without loss of generality, we may assume that $k < i_1 < i_2 < \dots < i_k$.

Suppose now there exist $\ell < j$ with $\sigma(\ell) < \sigma(j)$. We have

$$\mathbf{m}_{\ell} \mathbf{m}_j \cdot e_{\ell} \wedge e_j = e_{i_{\sigma(\ell)}} \wedge e_{i_{\sigma(j)}} + \text{rest}.$$

From step (1), we deduce that if \mathbf{m} is of minimal degree, then

$$\mathbf{m}_\ell = f_{\ell, i_{\sigma(\ell)-1}} , \mathbf{m}_j = f_{j, i_{\sigma(j)-1}}.$$

Similarly to the A_n considerations (recall that the C_n -degree is a obtain from an A_{2n-1} -degree), we see that

$$\deg_{\mathbf{d}}(f_{\ell, i_{\sigma(\ell)-1}} f_{j, i_{\sigma(\ell)-1}}) < \deg_{\mathbf{d}}(f_{\ell, i_{\sigma(\ell)-1}} f_{j, i_{\sigma(j)-1}}).$$

We denote $\mathbf{m}' := f_{\ell, i_{\sigma(\ell)-1}} f_{j, i_{\sigma(\ell)-1}} \left(\prod_{i \neq j, \ell} \mathbf{m}_i \right)$, then

$$\deg_{\mathbf{d}} \mathbf{m} > \deg_{\mathbf{d}} \mathbf{m}'.$$

But by construction,

$$\mathbf{m}' \cdot e_1 \wedge \dots \wedge e_k = e_{i_1} \wedge \dots \wedge e_{i_k} + \text{rest},$$

we have a contradiction to the minimality of the degree of \mathbf{m} and hence $\sigma(\ell) > \sigma(j)$ for all $\ell < j$.

Let $\{i_1, \dots, i_k\} = \{p_1 < \dots < p_s\} \cup \{\ell_1 < \dots < \ell_{k-s}\}$, where $\ell_{k-s} \leq k < p_1$ and $\{q_1 < \dots < q_s\}$ be the complement of $\{\ell_1 < \dots < \ell_{k-s}\}$ in $\{1, \dots, k\}$, then the monomial of minimal degree to obtain $e_{i_1} \wedge \dots \wedge e_{i_k}$ is

$$f_{q_1, p_s-1} \cdots f_{q_s, p_1-1}.$$

This proves that $\mathbf{d} \in S_{\text{lm}}$ and moreover these are precisely the monomials associated to $S_{C_n}(\varpi_k)$. □

From here, we can deduce by using (8) and Theorem 2.9:

THEOREM 5.6.

- (1) For the degree function, $\mathbf{d} \in S_{\text{gm}}$.
- (2) The set $\{f^{\mathbf{a}} \cdot v_{\lambda}^{\mathbf{d}} \mid \mathbf{a} \in S_{C_n}(\lambda)\}$ forms a monomial basis of $V^{\mathbf{d}}(\lambda)$.
- (3) If $\mathbf{d} \in S_{\text{gm}}$ and the corresponding monomial basis is associated to $S_{C_n}(\lambda)$, then $\mathbf{d} \notin \mathcal{D}^q$.

Proof. Part (1) and (2) are deduced from the lemma, by using (8) and Theorem 2.9. It is left to prove the part (3), i.e. we assume that $\mathbf{d} \in \mathcal{D}$ satisfies (1) and (2) and we want to show $\mathbf{d} \notin \mathcal{D}^q$. We consider the simple Lie subalgebra \mathfrak{g}_2 of type C_2 in \mathfrak{g} with positive roots $\alpha_{n-1, n-1}, \alpha_{n-1, \overline{n-1}}, \alpha_{n-1, n}$ and $\alpha_{n, n}$. In the subalgebra $U_q(\mathfrak{g}_2) \subset U_q(\mathfrak{g})$, we have the following relation, independent of the chosen reduced expression (see Example 4.2),

$$F_{n-1, \overline{n-1}} F_{n, n} = F_{n, n} F_{n-1, \overline{n-1}} + (1 - q^{-2}) F_{n-1, n}^{(2)},$$

implying that every $\mathbf{d}' \in \mathcal{D}^q$ satisfies $\mathbf{d}'_{n-1, \overline{n-1}} + \mathbf{d}'_{n, n} > 2\mathbf{d}'_{n-1, n}$.

Since \mathbf{d} satisfies (1) and (2), in $V^{\mathbf{d}}(\varpi_n)$, we have $f_{n-1, \overline{n-1}} f_{n, n} \cdot v_{\varpi_n}^{\mathbf{d}} \neq 0$ and $f_{n-1, n}^2 \cdot v_{\varpi_n}^{\mathbf{d}} = 0$ which implies $\mathbf{d}_{n-1, \overline{n-1}} + \mathbf{d}_{n, n} < 2\mathbf{d}_{n-1, n}$. Hence, $\mathbf{d} \notin \mathcal{D}^q$. □

REMARK 5.7. If $S_{\text{gm}} \neq \emptyset$, then there exists an \mathbb{N} -filtration arising from $\mathbf{d} \in S_{\text{gm}}$ such that for any $\lambda \in \mathcal{P}_+$, $V^{\mathbf{d}}(\lambda)$ has a unique monomial basis.

If $\mathbf{d} \in \mathcal{D}_{w_0}^q \cap S_{\text{gm}}$, then, by the argument in [7, Theorem 5], there exists an \mathbb{N} -filtration on $U_q(\mathfrak{n}^-)$ arising from \mathbf{d} such that for any $\lambda \in \mathcal{P}_+$, $V_q^{\mathbf{d}}(\lambda)$ has a unique monomial basis in $S_q(\mathfrak{n}^-)$.

5.5. Global monomial set: C_2 . Consider the quantum degree cone $\mathcal{D}_{w_0}^q$ defined in (6). We pick a solution such that the sum $a_1 + a_2 + a_3 + a_4$ takes its minimal value:

$$\mathbf{d} = (d_{1,1}, d_{1,\bar{1}}, d_{1,2}, d_{2,2}) = (1, 1, 1, 2).$$

Since $\mathbf{d} \in \mathcal{D}$, we consider the induced degree on the enveloping algebra with PBW root vectors $f_{1,1}, f_{1,\bar{1}}, f_{1,2}$ and $f_{2,2}$.

We turn to study whether \mathbf{d} is in the global monomial set \mathcal{S}_{gm} .

Let $\mathbf{SP}_4(m_1, m_2) \subset \mathbb{R}^4$ be the polytope defined by the following inequalities:

$$x_1, x_2, x_3, x_4 \geq 0, \quad x_1 \leq m_1, \quad x_4 \leq m_2,$$

$$2x_1 + x_2 + 2x_3 + 2x_4 \leq 2(m_1 + m_2), \quad x_1 + x_2 + x_3 + 2x_4 \leq m_1 + 2m_2.$$

Let $S(m_1, m_2)$ denote the lattice points in $\mathbf{SP}_4(m_1, m_2)$.

THEOREM 5.8. *For any $\lambda = m_1\varpi_1 + m_2\varpi_2 \in \mathcal{P}_+$, the following statements hold:*

- (1) *The set $\{f^{\mathbf{p}}v_\lambda^{\mathbf{d}} \mid \mathbf{p} \in S(m_1, m_2)\}$ forms a basis of $V^{\mathbf{d}}(\lambda)$, hence a basis of $V(\lambda)$.*
- (2) *We have $\mathbf{d} \in \mathcal{S}_{gm}$, i.e. the defining ideal $I^{\mathbf{d}}(\lambda)$ is monomial.*

The proof that $\mathbf{d} \in \mathcal{S}_{gm}$ can be deduced with the help of Corollary 2.8 (1). The rest of this paragraph will be devoted to prove this theorem.

PROPOSITION 5.9. *For any $m_1, m_2, m'_1, m'_2 \in \mathbb{N}$,*

$$S(m_1, m_2) + S(m'_1, m'_2) = S(m_1 + m'_1, m_2 + m'_2).$$

Proof. It suffices to prove that for $m_1 > 0$ and $m_2 \geq 0$,

$$S(m_1 - 1, m_2) + S(1, 0) = S(m_1, m_2) \text{ and } S(0, m_2 - 1) + S(0, 1) = S(0, m_2).$$

First, suppose that $m_1 \neq 0$ and pick $\mathbf{s} = (a_1, a_2, a_3, a_4) \in S(m_1, m_2)$.

- (1) If $a_1 \neq 0$, we set $\mathbf{t}_1 = (a_1 - 1, a_2, a_3, a_4)$ and $\mathbf{t}_2 = (1, 0, 0, 0)$; then $\mathbf{t}_2 \in S(1, 0)$. Since $\mathbf{s} \in S(m_1, m_2)$, $2a_1 + a_2 + 2a_3 + 2a_4 \leq 2(m_1 + m_2)$ implies that $2(a_1 - 1) + a_2 + 2a_3 + 2a_4 \leq 2(m_1 - 1 + m_2)$; $a_1 + a_2 + a_3 + 2a_4 \leq m_1 + 2m_2$ implies that $a_1 - 1 + a_2 + a_3 + 2a_4 \leq (m_1 - 1) + 2m_2$. Combining them together, we get $\mathbf{t}_1 \in S(m_1 - 1, m_2)$.
- (2) If $a_1 = 0$ and $a_3 \neq 0$, the very similar argument with $\mathbf{t}_2 = (0, 0, 1, 0)$ implies again $\mathbf{t}_1 = \mathbf{s} - \mathbf{t}_2 \in S(m_1 - 1, m_2)$.
- (3) Suppose that $a_1 = 0, a_3 = 0$ but $a_2 \neq 0$. The inequalities for $\mathbf{s} \in S(m_1, m_2)$ are reduced to $a_2 + 2a_4 \leq m_1 + 2m_2$. We see that $\mathbf{s} = (0, a_2 - 1, 0, a_4) + (0, 1, 0, 0)$ gives a decomposition in $S(m_1 - 1, m_2) + S(1, 0)$.
- (4) When $a_1 = a_2 = a_3 = 0$ but $a_4 \neq 0$, the decomposition is obvious.

Suppose now $m_1 = 0$ and pick $\mathbf{s} = (a_1, a_2, a_3, a_4) \in S(0, m_2)$. Then, $a_1 = 0$ and the inequality $a_2 + a_3 + 2a_4 \leq 2m_2$ is redundant.

- (1) Suppose $a_3 \neq 0$, then we decompose $\mathbf{s} = (0, a_2 - 1, a_3, a_4) + (0, 1, 0, 0)$: It is clear $(0, 1, 0, 0) \in S(0, 1)$; since $a_2 + 2a_3 + 2a_4 \leq 2m_2$, we get $a_2 + 2(a_3 - 1) + 2a_4 \leq 2(m_2 - 1)$, it implies $(0, a_2 - 1, a_3, a_4) \in S(0, m_2 - 1)$.
- (2) The case $a_1 = a_3 = 0$ but $a_4 \neq 0$ can be dealt in a similar way.

(3) We are left with the case where $0 \neq a_2 \leq 2m_2$. If $a_2 \leq 2$, there is nothing to be shown; if $a_2 > 2$, we decompose it as $(0, a_2 - 2, 0, 0) + (0, 2, 0, 0)$.

Repeating this procedure shows that any element in $S(m_1, m_2)$ can be decomposed as the sum of elements in $S(m_1 - k, m_2 - \ell)$ and in $S(k, \ell)$. □

To apply Theorem 2.9 to terminate the proof of Theorem 5.8, it suffices to count the number of lattice points in $\mathbf{SP}_4(m_1, m_2)$.

For any integers $a, b \in \mathbb{N}$, we define a polytope $\mathbf{P}(a, b) \subset \mathbb{R}^2$ by the following inequalities:

$$x \geq 0, \quad y \geq 0, \quad x + 2y \leq a, \quad x + y \leq b.$$

Let $N(a, b)$ denote the number of lattice points in $\mathbf{P}(a, b)$.

LEMMA 5.10. *The number of lattice points $N(a, b)$ has the following expression:*

$$(1) \quad N(a, a) = \begin{cases} l(l + 1) & \text{if } a = 2l - 1; \\ (l + 1)^2 & \text{if } a = 2l. \end{cases}$$

$$(2) \quad N(a, b) = \begin{cases} N(a, a), & \text{if } b \geq a; \\ \frac{1}{2}(b + 1)(b + 2), & \text{if } a \geq 2b; \\ -l^2 + 2lb - \frac{1}{2}b^2 + \frac{1}{2}b + l + 1, & \text{if } 2b > a > b \text{ and } a = 2l; \\ -l^2 + 2lb - \frac{1}{2}b^2 + \frac{3}{2}b + 1, & \text{if } 2b > a > b \text{ and } a = 2l + 1. \end{cases}$$

Proof. It amounts to count the integral points in the closed region cutting by the lines $x + 2y = a$, $x + y = b$ and the two axes in \mathbb{R}^2 which depends on the position of the intersection of these two lines. □

PROPOSITION 5.11. *The number of lattice points in $\mathbf{SP}_4(m_1, m_2)$ is*

$$\frac{1}{6}(m_1 + 1)(m_2 + 1)(m_1 + m_2 + 2)(m_1 + 2m_2 + 3).$$

Proof. Let H be the intersection of hyperplanes $x_1 = \alpha$ and $x_4 = \beta$ in \mathbb{R}^4 with coordinates (x_1, x_2, x_3, x_4) where $\alpha, \beta \geq 0$. By definition,

$$H \cap \mathbf{SP}_4(m_1, m_2) = \mathbf{P}(2m_1 + 2m_2 - 2\alpha - 2\beta, m_1 + 2m_2 - \alpha - 2\beta).$$

Therefore, by Lemma 5.10, the number of integral points in $\mathbf{SP}_4(m_1, m_2)$ equals

$$\sum_{\alpha=0}^{m_1} \sum_{\beta=0}^{m_2} N(2m_1 + 2m_2 - 2\alpha - 2\beta, m_1 + 2m_2 - \alpha - 2\beta). \tag{10}$$

Since $\alpha \leq m_1$ and $\beta \leq m_2$, it falls into the third case in Lemma 5.10 (2) and (10) reads (where $l = m_1 + m_2 - \alpha - \beta$ and $b = m_1 + 2m_2 - \alpha - 2\beta$):

$$\sum_{\alpha=0}^{m_1} \sum_{\beta=0}^{m_2} \frac{1}{2}\alpha^2 + 2\alpha\beta + \beta^2 - \left(m_1 + 2m_2 + \frac{3}{2}\right)\alpha - 2(m_1 + m_2 + 1)\beta + \left(\frac{1}{2}m_1^2 + 2m_1m_2 + m_2^2 + \frac{3}{2}m_1 + 2m_2 + 1\right).$$

An easy summation provides the number in the statement. □

By Weyl character formula, for $\lambda = m_1\varpi_1 + m_2\varpi_2 \in \mathcal{P}_+$, $\dim V(\lambda)$ coincides with the number of lattice points in $\mathbf{SP}_4(m_1, m_2)$. This terminates the proof of Theorem 5.8.

REMARK 5.12. By permuting the second and the third coordinates, the polytope $\mathbf{SP}_4(m_1, m_2)$ coincides with the one in Proposition 4.1 of [15] (see also [16]), which is unimodularly equivalent to the Newton–Okounkov body of some valuation arising from inclusions of (translated) Schubert varieties.

There are several other known polytopes parameterizing bases of a finite-dimensional irreducible representation $V(\lambda)$ of \mathfrak{sp}_4 . For example, the Gelfand–Tsetlin polytope $P_1(\lambda)$ [3]; the FFLV polytope $P_2(\lambda)$ [9]; the string polytope $P_3(\lambda)$ associated to the reduced decomposition $\underline{w}_0 = s_1s_2s_1s_2$ [17]; the string polytope $P_4(\lambda)$ associated to the reduced decomposition $\underline{w}_0 = s_2s_1s_2s_1$ [loc.cit.]; when $\lambda = m_1\varpi_1 + m_2\varpi_2$, the polytope $\mathbf{SP}_4(m_1, m_2)$.

With the help of Polymake [11], one can verify that the polytopes $P_1(\lambda)$, $P_2(\lambda)$ and $P_4(\lambda)$ are unimodular equivalent; but the polytope $P_3(\lambda)$ and $\mathbf{SP}_4(m_1, m_2)$ are not unimodular equivalent to any other polytopes.

REMARK 5.13. Using the polyhedral cones associated to these polytopes, the construction in [6] can be applied to produce three non-isomorphic toric degenerations of the spherical varieties associated to the symplectic group Sp_4 , see for example [6, Sections 10, 12, 13].

5.6. Global monomial set: D_4 . We prove that the global monomial set for D_4 is non-empty. We refer to Section 6.4 for details on the cones and the enumeration of positive roots. Let $\mathbf{d} = (5, 5, 1, 2, 4, 1, 1, 2, 6, 10, 12, 20)$. It is shown in Section 6.4 that there exists a $\underline{w}_0 \in R(w_0)$ such that $\mathbf{d} \in \mathcal{D}_{\underline{w}_0}^q$. We will freely use the notations in Section 6.4.

Let $P_{D_4}(\lambda)$ be the polytope defined in [13, Section 3] and $S_{D_4}(\lambda)$ be the set of lattice points in $P_{D_4}(\lambda)$.

THEOREM 5.14.

- (1) We have $\mathbf{d} \in S_{\mathrm{gm}}$.
- (2) The set $\{f^{\mathbf{s}} \cdot v_{\lambda}^{\mathbf{d}} \mid \mathbf{s} \in S_{D_4}(\lambda)\}$ forms a monomial basis of $V^{\mathbf{d}}(\lambda)$.
- (3) Let $\underline{w}_0 = s_2s_1s_2s_3s_2s_4s_2s_1s_2s_3s_2s_4 \in R(w_0)$, then $\mathbf{d} \in \mathcal{D}_{\underline{w}_0}^q$.

Proof. The part of the proof that gives $\mathbf{d} \in S_{\mathrm{im}}$ is straightforward with Corollary 2.8. By a straightforward comparison, we obtain

$$S_{D_4}(\varpi_i) = \{\mathbf{s} \in \mathbb{N}^{\Delta^+} \mid f^{\mathbf{s}} \cdot v_{\varpi_i}^{\mathbf{d}} \neq 0 \text{ in } V^{\mathbf{d}}(\varpi_i)\}, \quad i = 1, 2, 3, 4.$$

It is shown in [loc.cit.] that for any $\lambda, \mu \in \mathcal{P}_+$, we have

$$P_{D_4}(\lambda) + P_{D_4}(\mu) = P_{D_4}(\lambda + \mu) \quad \text{and} \quad S_{D_4}(\lambda) + S_{D_4}(\mu) = S_{D_4}(\lambda + \mu)$$

and $\dim V(\lambda) = \#S_{D_4}(\lambda)$.

The statements (1) and (2) follow from Theorem 2.9. The part (3) is shown in Section 6.4. □

5.7. Global monomial set: B_3 . Let \mathfrak{g} be of type B_3 . For $\lambda \in \mathcal{P}_+$, let $P_{B_3}(\lambda)$ denote the polytope defined in [2, Section 5] and $S_{B_3}(\lambda)$ be the set of lattice points in $P_{B_3}(\lambda)$.

Let $f_{i,j}$ and $f_{i,\bar{j}}$ be the PBW root vectors associated to the positive roots $\alpha_{i,j}$ and $\alpha_{i,\bar{j}}$, respectively. For $\mathbf{d} \in \mathbb{R}_{\geq 0}^{\Delta_+}$, we write $d_{i,j} = \mathbf{d}(\alpha_{i,j})$ and $d_{i,\bar{j}} = \mathbf{d}(\alpha_{i,\bar{j}})$. We consider the element $\mathbf{d} \in \mathbb{R}_{\geq 0}^{\Delta_+}$ defined by

$$d_{1,1} = 4, \quad d_{1,2} = 3, \quad d_{2,2} = 3, \quad d_{1,3} = 3, \quad d_{1,\bar{2}} = 1, \quad d_{1,\bar{3}} = 1, \quad d_{2,3} = 4, \quad d_{2,\bar{3}} = 3, \quad d_{3,3} = 2.$$

We will show in Section 6.2 that $\mathbf{d} \in \mathcal{D}$.

THEOREM 5.15.

- (1) We have $\mathbf{d} \in \mathcal{S}_{\text{gm}}$, and the set $\{f^{\mathbf{s}} \cdot v_{\lambda}^{\mathbf{d}} \mid \mathbf{s} \in \mathcal{S}_{\mathbb{B}_3}(\lambda)\}$ forms a monomial basis of $V^{\mathbf{d}}(\lambda)$.
- (2) For any $\mathbf{e} \in \mathcal{D}$ satisfying (1), we have $\mathbf{e} \notin \mathcal{D}^q$.

Proof.

- (1) As before by computing each weight space in $V^{\mathbf{d}}(\varpi_i)$, $i = 1, 2, 3$, we obtain $\mathbf{d} \in \mathcal{S}_{\text{lm}}$. By comparing the basis arising from the monomiality of the defining ideals $I^{\mathbf{d}}(\varpi_i)$ with the basis obtain in [loc.cit.], we get for $i = 1, 2, 3$:

$$\mathcal{S}_{\mathbb{B}_3}(\varpi_i) = \{\mathbf{s} \in \mathbb{N}^{\Delta_+} \mid f^{\mathbf{s}} \cdot v_{\varpi_i}^{\mathbf{d}} \neq 0 \text{ in } V^{\mathbf{d}}(\varpi_i)\}.$$

For any $\lambda, \mu \in \mathcal{P}_+$, we have

$$P_{\mathbb{B}_3}(\lambda) + P_{\mathbb{B}_3}(\mu) = P_{\mathbb{B}_3}(\lambda + \mu), \quad \mathcal{S}_{\mathbb{B}_3}(\lambda) + \mathcal{S}_{\mathbb{B}_3}(\mu) = \mathcal{S}_{\mathbb{B}_3}(\lambda + \mu)$$

and $\dim V(\lambda) = \#\mathcal{S}_{\mathbb{B}_3}(\lambda)$. By Theorem 2.9, $\mathbf{d} \in \mathcal{S}_{\text{gm}}$.

- (2) Let $\mathbf{e} \in \mathcal{S}_{\text{gm}}$. From reading the lattice points in $P_{\mathbb{B}_3}(\varpi_2)$, we get $f_{1,2}f_{1,\bar{3}} \cdot v_{\varpi_2}^{\mathbf{e}} \neq 0$ in $V^{\mathbf{e}}(\varpi_2)$. Since the corresponding weight space is one-dimensional and $f_{1,3}^2$ has the same weight, $f_{1,3}^2 \cdot v_{\varpi_2}^{\mathbf{e}} = 0$.

Assume $\underline{w}_0 \in R(w_0)$ such that $\mathbf{e} \in \mathcal{D}_{\underline{w}_0}^q$. Let $<$ be the induced convex order on Δ_+ . Without loss of generality, we suppose that $\alpha_{1,2} < \alpha_{1,\bar{3}}$.

CASE 1. Assume $\alpha_{1,2} < \alpha_{1,3} < \alpha_{1,\bar{3}}$, for the quantum degree cone $\mathcal{D}_{\underline{w}_0}^q$, by computing the L–S formula explicitly, this would imply the following inequality: $d_{1,2} + d_{1,\bar{3}} > 2d_{1,3}$. This implies, turning to the classical case, $f_{1,3}^2 \cdot v_{\varpi_2}^{\mathbf{e}} \neq 0$, which is a contradiction.

CASE 2. Assume $\alpha_{1,3} < \alpha_{1,2} < \alpha_{1,\bar{3}}$. Consider the root $\alpha_{3,3}$: By the convexity, it must be simultaneously larger than $\alpha_{1,\bar{3}}$ and smaller than $\alpha_{1,2}$. This is a contradiction.

CASE 3. Assume $\alpha_{1,2} < \alpha_{1,\bar{3}} < \alpha_{1,3}$, with similar arguments as in Case 2 we get a contradiction.

As a conclusion, for any $\underline{w}_0 \in R(w_0)$, $\mathbf{e} \notin \mathcal{D}_{\underline{w}_0}^q$. □

5.8. Global monomial set: G_2 . Let \mathfrak{g} be of type G_2 . We use the notations in Section 4.3. Consider the following $\mathbf{d} \in \mathcal{D}$:

$$d_1 = 2, \quad d_{1112} = 1, \quad d_{112} = 3, \quad d_{11122} = 1, \quad d_{12} = 3, \quad d_2 = 2.$$

It is clear that $\mathbf{d} \in \mathcal{D}$.

Let $P_{G_2}(\lambda)$ be the polytope defined in [12, Section 1], the set of its lattice points will be denoted by $\mathcal{S}_{G_2}(\lambda)$. With similar arguments and calculations as before, we

obtain the first statement of the following theorem. The second statement follows from Section 4.3, there we show, for each $\mathbf{e} \in \mathcal{D}^q$, there exist a unique monomial basis of $V^{\mathbf{e}}(\varpi_i)$, $i = 1, 2$, i.e. $\mathbf{e} \in \mathcal{S}_{\text{lm}}$, which does not coincide with the basis of the following theorem.

THEOREM 5.16. *We have $\mathbf{d} \in \mathcal{S}_{\text{gm}}$, and the set $\{f^{\mathbf{s}} \cdot v_{\lambda}^{\mathbf{d}} \mid \mathbf{s} \in S_{G_2}(\lambda)\}$ forms a monomial basis of $V^{\mathbf{d}}(\lambda)$.*

REMARK 5.17. There is no $\mathbf{d} \in \mathcal{D}^q$, such that $\{f^{\mathbf{s}} \cdot v_{\lambda}^{\mathbf{d}} \mid \mathbf{s} \in S_{G_2}(\lambda)\}$ is a basis of $V^{\mathbf{d}}(\lambda)$.

5.9. Local monomial sets: G_2 . Let \mathfrak{g} be of type G_2 . By Proposition 4.4, the quantum degree cone $\mathcal{D}_{\underline{w}_0}^q$ does not depend on the choice of $\underline{w}_0 \in R(w_0)$. Let

$$\mathbf{d} = (d_1, d_{1112}, d_{112}, d_{11122}, d_{12}, d_2) = (2, 2, 1, 2, 2, 5).$$

We will show in Section 4.3 that $\mathbf{d} \in \mathcal{D}_{\underline{w}_0}^q$. Let $f_1, f_{1112}, f_{112}, f_{11122}, f_{12}, f_2$ be the PBW root vectors in \mathfrak{n}^- .

PROPOSITION 5.18. *We have $\mathbf{d} \in \mathcal{S}_{\text{lm}}$, i.e. the defining ideals $I^{\mathbf{d}}(\varpi_1)$ and $I^{\mathbf{d}}(\varpi_2)$ are monomial.*

We omit the proof as before.

We want to examine that this degree function is quite interesting, due to the fact that the induced semigroup is not saturated as we will explain.

Let $S(\varpi_2) = \{\mathbf{s} \in \mathbb{N}^{\Delta_+} \mid f^{\mathbf{s}} \cdot v_{\varpi_2}^{\mathbf{d}} \neq 0\}$. We have by construction $\#S(\varpi_2) = \dim V(\varpi_2) = 14$, but there are 16 lattice points in the convex hull $P = \text{conv}(S(\varpi_2))$.

We fix the sequence of positive roots $(\alpha_1, \alpha_{1112}, \alpha_{112}, \alpha_{11122}, \alpha_{12}, \alpha_2)$ to identify \mathbb{R}^{Δ_+} and \mathbb{R}^6 . Let $\mathbf{G}_2^{\varpi_1}(m_1) \subset \mathbb{R}^6$ be the polytope defined by the inequalities:

$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0, \quad x_1 \leq m_1, \quad x_6 \leq 0, \quad 2x_1 + 2x_2 + x_3 + 2x_4 + 2x_5 \leq 2m_1.$$

Let $\mathbf{G}_2^{\varpi_2}(m_2) \subset \mathbb{R}^6$ be the polytope defined by the inequalities:

$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0, \quad x_1 \leq 0, \quad 2x_2 + x_3 + x_4 + x_5 + 2x_6 \leq 2m_2.$$

Let $\{e_1, e_2, \dots, e_6\}$ be the standard basis of \mathbb{R}^6 .

Let $\lambda = m_1\varpi_1 + m_2\varpi_2 \in \mathcal{P}_+$. We conjecture that the number of lattice points in the Minkowski sum

$$m_1\mathbf{G}_2^{\varpi_1}(1) + m_2(\mathbf{G}_2^{\varpi_2}(1) \cup \{3e_3, 3e_5\})$$

coincides with $\dim V(m_1\varpi_1 + m_2\varpi_2)$.

REMARK 5.19. Note that the proof of Lemma 5.18 does not depend on the choice of $\mathbf{d} \in \mathcal{D}_{\underline{w}_0}^q$. Further we have $\mathcal{D}^q = \mathcal{D}_{\underline{w}_0}^q$ (see Proposition 4.4). This implies the inclusion $\mathcal{D}^q \subset \mathcal{S}_{\text{lm}}$.

REMARK 5.20. Let G be the complex algebraic group of type G_2 and U (resp. U^-) be the maximal unipotent subgroup of G having \mathfrak{n}^+ (resp. \mathfrak{n}^-) as Lie algebra.

Let $S = (\alpha_1, \alpha_{1112}, \alpha_{112}, \alpha_{11122}, \alpha_{12}, \alpha_2)$ be a birational sequence for U^- (see [6, Section 3] for definition). Using S , we identify \mathbb{N}^{Δ_+} and \mathbb{N}^6 . We fix the integral weight function $\Psi : \Delta_+ \rightarrow \mathbb{N}$ by $\Psi = \mathbf{d}$ and the lexicographic order on \mathbb{N}^6 . Let $>$ be the total order on \mathbb{N}^6 defined in [6, Section 5] by combining Ψ and the lexicographic

order. In [loc.cit], a monoid $\Gamma = \Gamma(S, >) \subset \mathcal{P}_{\mathbb{R}} \times \mathbb{N}^6$ is attached to $G//U$ to study its toric degenerations. Let $\pi_1 : \mathcal{P}_{\mathbb{R}} \times \mathbb{N}^6 \rightarrow \mathcal{P}_{\mathbb{R}}$ and $\pi_2 : \mathcal{P}_{\mathbb{R}} \times \mathbb{N}^6 \rightarrow \mathbb{N}^6$ be the canonical projections.

We claim that Γ is not saturated: First notice that $\pi_2 \circ \pi_1^{-1}(\varpi_2) = S(\varpi_2)$. Pick a lattice point $\mathbf{q} \in P$ which is not in $S(\varpi_2)$. Since $q \in \text{conv}(S(\varpi_2))$, there exists $s_1, \dots, s_m \in \mathbb{Q}$ and $\mathbf{p}_1, \dots, \mathbf{p}_m \in S(\varpi_2)$ such that $s_1 + \dots + s_m = 1$ and $\mathbf{q} = s_1\mathbf{p}_1 + \dots + s_m\mathbf{p}_m$. Multiplying both sides by the least common multiple M of the denominators of s_1, \dots, s_m , we know that $(M\varpi_2, M\mathbf{q}) \in \Gamma$ as Γ is a monoid. If Γ were saturated, $(M\varpi_2, M\mathbf{q}) \in \Gamma$ will imply $(\varpi_2, \mathbf{q}) \in \Gamma$, contradicts to $\pi_2 \circ \pi_1^{-1}(\varpi_2) = S(\varpi_2)$.

This example explains that the saturated assumption in [6] is necessary. We thank Peter Littelmann for pointing out this application.

6. Higher rank examples of quantum degree cones.

6.1. Lie algebra A_3 . Let \mathfrak{g} be of type A_3 . For $\mathbf{d} \in \mathbb{R}_{\geq 0}^{\Delta_+}$, we write $d_{ij} = \mathbf{d}(\alpha_{ij})$. The classical degree cone $\mathcal{D} \subset \mathbb{R}_{\geq 0}^{\Delta_+}$ is defined by the following inequalities:

$$d_{1,1} + d_{2,2} > d_{1,2}, \quad d_{2,2} + d_{3,3} > d_{2,3}, \quad d_{1,1} + d_{2,3} > d_{1,3}, \quad d_{1,2} + d_{3,3} > d_{1,3},$$

Let $\underline{w}_0^1 = s_1s_2s_1s_3s_2s_1$ and $\underline{w}_0^2 = s_1s_3s_2s_3s_1s_2 \in R(w_0)$. We claim that the corresponding quantum degree cones satisfy $\mathcal{D}_{\underline{w}_0^1}^q \cap \mathcal{D}_{\underline{w}_0^2}^q = \emptyset$.

Let F_{ij} (resp. F'_{ij}) denote the quantum PBW root vector associated to \underline{w}_0^1 (resp. \underline{w}_0^2) and root α_{ij} . We have the following commutation relations between the quantum PBW root vectors:

$$F_{1,2}F_{2,3} = F_{2,3}F_{1,2} + (q - q^{-1})F_{2,2}F_{1,3}, \quad F'_{1,3}F'_{2,2} = F'_{2,2}F'_{1,3} + (q - q^{-1})F'_{1,2}F'_{2,3},$$

giving two contradicting inequalities in the corresponding quantum degree cones:

$$d_{1,2} + d_{2,3} > d_{2,2} + d_{1,3}, \quad d_{1,3} + d_{2,2} > d_{1,2} + d_{2,3}.$$

Projecting to the corresponding coordinates proves the claim.

6.2. Lie algebra B_3 . Let \mathfrak{g} be of type B_3 . The set of positive roots

$$\Delta_+ = \{\alpha_{1,1}, \alpha_{1,2}, \alpha_{1,3}, \alpha_{2,2}, \alpha_{2,3}, \alpha_{3,3}, \alpha_{1,\bar{3}}, \alpha_{2,\bar{3}}, \alpha_{1,\bar{2}}\}.$$

For $\mathbf{d} \in \mathbb{R}_{\geq 0}^{\Delta_+}$, we write $d_{ij} = \mathbf{d}(\alpha_{ij})$ and $d_{i\bar{j}} = \mathbf{d}(\alpha_{i\bar{j}})$. The classical degree cone \mathcal{D} is determined by

$$d_{1,1} + d_{2,2} > d_{1,2}, \quad d_{1,1} + d_{2,3} > d_{1,3}, \quad d_{1,1} + d_{2,\bar{3}} > d_{1,\bar{3}}, \quad d_{1,2} + d_{3,3} > d_{1,3},$$

$$d_{1,2} + d_{2,\bar{3}} > d_{1,\bar{2}}, \quad d_{2,2} + d_{3,3} > d_{2,3}, \quad d_{2,2} + d_{1,\bar{3}} > d_{1,\bar{2}},$$

$$d_{1,3} + d_{2,3} > d_{1,\bar{2}}, \quad d_{1,3} + d_{3,3} > d_{1,\bar{3}}, \quad d_{2,3} + d_{3,3} > d_{2,\bar{3}}.$$

For example, $\mathbf{d} = (4, 3, 3, 3, 1, 1, 4, 3, 2) \in \mathcal{D}$.

We consider

$$\underline{w}_0^1 = s_1s_2s_1s_3s_2s_1s_3s_2s_3 \text{ and } \underline{w}_0^2 = s_1s_3s_2s_3s_2s_1s_2s_3s_2 \in R(w_0).$$

The quantum degree cone $\mathcal{D}_{\underline{w}_0^1}^q$ in \mathcal{D} is defined by the following inequalities:

$$\begin{aligned} d_{1,1} + d_{1,\bar{2}} &> 2d_{1,3}, \quad d_{1,2} + d_{1,\bar{2}} > d_{2,2} + 2d_{1,3}, \quad d_{1,2} + d_{1,\bar{3}} > 2d_{1,3}, \quad d_{1,2} + d_{2,\bar{3}} \\ &> d_{2,2} + d_{1,\bar{3}}, \\ d_{1,2} + d_{2,\bar{3}} &> d_{1,3} + d_{1,2}, \quad d_{2,2} + d_{2,\bar{3}} > 2d_{2,3}, \quad d_{1,3} + d_{2,\bar{3}} > d_{1,\bar{3}} + d_{2,3}, \\ d_{1,\bar{2}} + d_{2,\bar{3}} &> d_{1,\bar{3}} + 2d_{2,3}, \quad d_{1,\bar{2}} + d_{3,3} > d_{1,\bar{3}} + d_{2,3}, \quad d_{1,2} + d_{2,3} > d_{1,3} + d_{2,2}. \end{aligned}$$

The quantum degree cone $\mathcal{D}_{\underline{w}_0^2}^q$ in \mathcal{D} is defined by the following inequalities:

$$\begin{aligned} d_{1,1} + d_{1,\bar{2}} &> 2d_{1,3}, \quad d_{1,1} + d_{1,\bar{2}} > d_{1,2} + d_{1,\bar{3}}, \quad d_{1,2} + d_{1,\bar{3}} > 2d_{1,3}, \quad d_{1,\bar{3}} + d_{2,3} \\ &> d_{1,3} + d_{2,\bar{3}}, \\ d_{1,\bar{3}} + d_{2,2} &> d_{1,3} + d_{2,3}, \quad d_{1,\bar{3}} + d_{2,2} > d_{1,2} + d_{2,\bar{3}}, \quad d_{1,3} + d_{2,3} > d_{1,2} + d_{2,\bar{3}}, \\ d_{2,\bar{3}} + d_{2,2} &> 2d_{2,3}, \quad d_{1,3} + d_{2,2} > d_{1,2} + d_{2,3}. \end{aligned}$$

By the contradiction of the last inequalities, we obtain $\mathcal{D}_{\underline{w}_0^1}^q \cap \mathcal{D}_{\underline{w}_0^2}^q = \emptyset$.

6.3. Lie algebra C_3 . Let \mathfrak{g} be of type C_3 . For $\mathbf{d} \in \mathbb{R}_{\geq 0}^{\Delta_+}$, we write $d_1 = \mathbf{d}(\alpha_{1,1})$, $d_2 = \mathbf{d}(\alpha_{1,2})$, $d_3 = \mathbf{d}(\alpha_{1,\bar{1}})$, $d_4 = \mathbf{d}(\alpha_{1,3})$, $d_5 = \mathbf{d}(\alpha_{1,\bar{2}})$, $d_6 = \mathbf{d}(\alpha_{2,2})$, $d_7 = \mathbf{d}(\alpha_{2,\bar{2}})$, $d_8 = \mathbf{d}(\alpha_{2,3})$ and $d_9 = \mathbf{d}(\alpha_{3,3})$. The classical degree cone $\mathcal{D} \subset \mathbb{R}_{\geq 0}^{\Delta_+}$ is determined by

$$\begin{aligned} d_1 + d_5 &> d_3, \quad d_1 + d_6 > d_2, \quad d_1 + d_7 > d_5, \quad d_1 + d_8 > d_4, \quad d_2 + d_4 > d_3, \\ d_2 + d_8 &> d_5, \quad d_2 + d_9 > d_4, \quad d_4 + d_6 > d_5, \quad d_6 + d_8 > d_7, \quad d_6 + d_9 > d_8. \end{aligned}$$

We consider the reduced decompositions

$$\underline{w}_0^1 = s_1s_2s_3s_2s_1s_2s_3s_2s_3 \text{ and } \underline{w}_0^2 = s_1s_3s_2s_3s_2s_1s_2s_3s_2.$$

Moreover, the inequalities determining the cone $\mathcal{D}_{\underline{w}_0^1}^q$ in \mathcal{D} are

$$\begin{aligned} d_1 + d_5 &> d_2 + d_4, \quad d_3 + d_9 > 2d_4, \quad d_7 + d_9 > 2d_8, \quad d_3 + d_7 > 2d_5, \\ d_1 + d_7 &> d_4 + d_6, \quad d_2 + d_7 > d_5 + d_6, \quad d_2 + d_7 > d_4 + 2d_6, \quad d_3 + d_7 > d_4 + d_5 + d_6, \\ d_3 + d_7 &> 2d_4 + 2d_6, \quad d_3 + d_8 > d_4 + d_5, \quad d_3 + d_8 > 2d_4 + d_6, \quad d_2 + d_8 > d_4 + d_6. \end{aligned}$$

The inequalities determining the cone $\mathcal{D}_{\underline{w}_0^2}^q$ in \mathcal{D} are

$$\begin{aligned} d_1 + d_5 &> d_2 + d_4, \quad d_3 + d_9 > 2d_4, \quad d_7 + d_9 > 2d_8, \quad d_3 + d_7 > 2d_5, \\ d_1 + d_7 &> d_2 + d_8, \quad d_4 + d_7 > d_5 + d_8, \quad d_3 + d_7 > d_2 + d_5 + d_8, \quad d_3 + d_7 > 2d_2 + 2d_8, \\ d_3 + d_6 &> d_2 + d_5, \quad d_3 + d_6 > 2d_2 + d_8, \quad d_4 + d_7 > d_2 + d_8, \quad d_4 + d_6 > d_2 + d_8. \end{aligned}$$

Notice that there is a contradiction in the last inequalities, implying that $\mathcal{D}_{\underline{w}_0^1}^q \cap \mathcal{D}_{\underline{w}_0^2}^q = \emptyset$.

There are four elements in $\mathcal{D}_{\underline{w}_0}^q$, which are minimal regarding the sum over all entries:

$$\mathbf{d}_1 = (2, 1, 1, 1, 1, 1, 4, 4, 5), \mathbf{d}_2 = (3, 2, 2, 1, 1, 1, 3, 3, 4),$$

$$\mathbf{d}_3 = (5, 4, 4, 1, 1, 1, 1, 1, 2), \mathbf{d}_4 = (4, 3, 3, 1, 1, 1, 2, 2, 3).$$

Since $\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3, \mathbf{d}_4 \in \mathcal{D}$, we go back to the classical case. We consider the fundamental module $V(\varpi_2)$ and the weight $\tau = 2\alpha_1 + 3\alpha_2 + \alpha_3$ whose weight space $V(\varpi_2)_{\varpi_2 - \tau}$ is of dimension 1. We have to choose an element with minimal degree from the following set, where we neglect the elements which have obviously a higher degree:

$$\{f_{1,2}f_{1,\bar{2}}, f_{1,\bar{1}}f_{2,2}\}.$$

For each of the above elements in $\mathcal{D}_{\underline{w}_0}^q$, both monomials have the same degree, so we do not obtain a monomial ideal $I^{\mathbf{d}_i}, 1 \leq i \leq 4$.

By taking larger degrees $\mathbf{d} \in \mathcal{D}_{\underline{w}_0}^q$, it is possible to obtain a unique monomial basis of $V^{\mathbf{d}}(\varpi_2)$, where it is possible to obtain a basis with either of both monomials applied to $v_{\varpi_2}^{\mathbf{d}}$. We conclude $\mathcal{D}_{\underline{w}_0}^q \not\subseteq \mathcal{S}_{\text{lm}}$, but $\mathcal{D}_{\underline{w}_0}^q \cap \mathcal{S}_{\text{lm}} \neq \emptyset$. We also see, different elements in $\mathcal{D}_{\underline{w}_0}^q$ can produce different monomial bases. This observation still holds, even if we consider elements where the sum over the entries is the same.

6.4. Lie algebra D_4 . Let \mathfrak{g} be of type D_4 . In the Dynkin diagram, we let 2 be the central node. We consider the following reduced decomposition

$$\underline{w}_0 = s_2s_1s_2s_3s_2s_4s_2s_1s_2s_3s_2s_4 \in R(w_0).$$

For a positive root $a\alpha_1 + b\alpha_2 + c\alpha_3 + d\alpha_4 \in \Delta_+$, we let f_{abcd} denote the corresponding quantum PBW root vector. In the convex order on positive roots given by \underline{w}_0 , they are

$$f_{0100}, f_{1100}, f_{1000}, f_{1110}, f_{0110}, f_{1211}, f_{1101}, f_{1111}, f_{0010}, f_{0111}, f_{0101}, f_{0001}.$$

For $d \in \mathbb{R}_{\geq 0}^{\Delta_+}$, let d_i be the value of \mathbf{d} at the positive root corresponding to the i th quantum PBW root vector above. The quantum degree cone $\mathcal{D}_{\underline{w}_0}^q \subset \mathbb{R}_{\geq 0}^{\Delta_+}$ is defined by

$$\begin{aligned} d_1 + d_3 > d_2, \quad d_1 + d_8 > d_5 + d_7, \quad d_1 + d_8 > d_6, \quad d_1 + d_9 > d_5, \quad d_1 + d_{12} > d_{11} \\ d_2 + d_8 > d_3 + d_5 + d_7, \quad d_2 + d_8 > d_3 + d_6, \quad d_2 + d_8 > d_4 + d_7, \quad d_2 + d_9 > d_3 + d_5, \\ d_2 + d_9 > d_4, \quad d_2 + d_{10} > d_6, \quad d_2 + d_{12} > d_3 + d_{11}, \quad d_2 + d_{12} > d_7 \\ d_3 + d_5 > d_4, \quad d_3 + d_{10} > d_7 + d_9, \quad d_3 + d_{10} > d_8, \quad d_3 + d_{11} > d_7 \\ d_4 + d_{10} > d_5 + d_7 + d_9, \quad d_4 + d_{10} > d_5 + d_8, \quad d_4 + d_{10} > d_6 + d_9 \\ d_4 + d_{11} > d_5 + d_7, \quad d_4 + d_{11} > d_6, \quad d_4 + d_{12} > d_8, \quad d_5 + d_7 > d_6, \\ d_5 + d_{12} > d_9 + d_{11}, \quad d_5 + d_{12} > d_{10}, \quad d_6 + d_{12} > d_7 + d_9 + d_{11}, \quad d_6 + d_{12} > d_7 + d_{10} \\ d_6 + d_{12} > d_8 + d_{11}, \quad d_7 + d_9 > d_8, \quad d_9 + d_{11} > d_{10}. \end{aligned}$$

For example, $\mathbf{d} = (5, 5, 1, 2, 4, 1, 1, 2, 6, 10, 12, 20) \in \mathcal{D}_{\underline{w}_0}^q$.

ACKNOWLEDGEMENTS. X.F. is supported by the Alexander von Humboldt Foundation. T.B. and G.F. are partially supported by the DFG priority programme SPP 1388 “Representation Theory”.

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