J. Aust. Math. Soc. **97** (2014), 180–211
doi:10.1017/S1446788714000263

NONORTHOGONAL GEOMETRIC REALIZATIONS OF COXETER GROUPS

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(Received 8 March 2013; accepted 4 May 2014; first published online 25 July 2014)

Communicated by G. Willis

Abstract

We define in an axiomatic fashion a *Coxeter datum* for an arbitrary Coxeter group W. This Coxeter datum will specify a pair of reflection representations of W in two vector spaces linked only by a bilinear pairing without any integrality or nondegeneracy requirements. These representations are not required to be embeddings of W in the orthogonal group of any vector space, and they give rise to a pair of interrelated root systems generalizing the classical root systems of Coxeter groups. We obtain comparison results between these nonorthogonal root systems and the classical root systems. Further, we study the equivalent of the Tits cone in these nonorthogonal representations.

2010 *Mathematics subject classification*: primary 20F55; secondary 20F10, 20F65.

Keywords and phrases: Coxeter groups, Kac–Moody Lie algebras, root systems, Tits cone.

1. Introduction

The notion of *root systems* is an essential tool in the study of Coxeter groups. For an arbitrary Coxeter system (W, R) in the sense of [1] and [20], its root system Φ is a set of vectors arising from the *Tits representation* of W as a group generated by reflections with respect to some hyperplanes in a real vector space V. Here the representation space V is equipped with a symmetric bilinear form, and it has dimension equal to the cardinality of R, and Φ consists of a set of representative normal vectors for the reflecting hyperplanes. The elements of Φ are called *roots*, and those roots corresponding to elements of R are known as *simple roots*. Following the convention of [22] and [23], we also use the term *root basis* for the set of simple roots. Under this classical construction, the abstract Coxeter group W is embedded in the orthogonal group of the chosen bilinear form on V, and hence the resulting reflection representation may be referred to as *orthogonal*.

The work presented in this paper was completed under the support of the Australian Research Council Discovery Project: *Invariant theory, cellularity and geometry*, No. DP0772870.

In [1, Ch. V, Section 4] and [20, Section 5.3] simple roots are required to form a basis for V, and for each pair of simple roots the value taken by the bilinear form is completely determined by the order of the product of the corresponding reflections. In particular, if the order is infinite, then the bilinear form takes the value -1.

The orthogonal representations studied in [1] and [20] gave a beautiful theory of root systems for finite Weyl groups. However, the requirement that, for each pair of simple roots, the value taken by the bilinear form is completely determined by the order of the product of the corresponding reflections may not be necessarily satisfied in an arbitrary geometric representation of Coxeter groups. Furthermore, since reflection subgroups of Coxeter groups are themselves Coxeter groups, it seems desirable that the set of roots corresponding to a reflection subgroup should itself constitute a root system. However, this requires relaxing the conditions imposed on the simple roots. Specifically, since a proper reflection subgroup may have more Coxeter generators than the over group (as illustrated in [17, Example 5.1]), linear independence of simple roots is not inherited by reflection subsystems. Moreover, the property that the bilinear form takes the value -1 whenever the product of the corresponding reflections has infinite order is not inherited by reflection subsystems either (as illustrated in [17, Example 1.1]).

The notion of a *nonorthogonal* reflection representation for an infinite Coxeter group W was put to use in [25] to study the Weyl groups of Kac–Moody Lie algebras. A notable feature of this approach is the construction of a pair of root systems generalizing the classical notion in two interrelated vector spaces V_1 and V_2. The resulting reflection representations are not embeddings of W in any orthogonal group, thus explaining the term 'nonorthogonal'. In [25], the vector spaces V_1 and V_2 are linked by a nondegenerate bilinear form satisfying an integrality condition, and the resulting root systems are required to form integral lattices in V_1 and V_2 respectively. This approach gives a well-developed theory of root systems for the Weyl groups of Kac–Moody Lie algebras, but it may not apply to general infinite Coxeter groups, as commented in the remark immediately following the definition of root bases in [7, 1.1.1].

In this paper, we give a generalization of the nonorthogonal reflection representations studied in [25]. In particular, we remove the nondegeneracy and integrality condition on the bilinear form. In this situation, the root systems need not form integral lattices. Furthermore, in contrast to [25], the removal of the nondegeneracy condition on the bilinear pairing implies that the two representation spaces V_1 and V_2 may not necessarily be identified as algebraic duals of each other.

This paper also aims to give detailed comparisons between the root systems arising from nonorthogonal representations and their classical counterparts. For general Coxeter groups, in the literature, it appears that little attention has been paid to comparing the root systems arising from nonorthogonal representations to their classical counterparts. This paper presents a collection of basic combinatorial and geometric results in nonorthogonal root systems for general Coxeter groups. Specific to the approach taken in this paper, a number of basic results and their proofs, and in some cases even the statements, involve new ideas. For instance, some of the resulting reflection representations have root systems which inevitably involve potentially infinitely many different (positive) roots corresponding to the same reflection; these appear in neither the Kac–Moody nor orthogonal Coxeter settings.

A number of studies on nonorthogonal representations of Coxeter groups exist in the literature, for example, [5–7, 13–15]. Our approach presented here generalizes [5–7]. In particular, the paired representation spaces in those are all required to be algebraic duals of each other, and in [5, 6] the root bases are required to be linearly independent. In [13], a theory of root systems over totally ordered commutative rings is developed with also the requirement that root bases are linearly independent. As observed in the classical orthogonal case, the removal of the linear independence condition imposed on root bases is essential in developing a unified theory of root systems applicable to reflection subgroups, and consequently, in this paper, we do not impose the linear independence condition on root bases. The techniques used in [13] may be combined with those of the present paper to obtain a theory of root systems for arbitrary Coxeter groups (and their reflection subgroups) over totally ordered commutative rings. Such a theory has been described in [14], and we thank the referee of the present paper for communicating [14] to us.

This paper begins with a section devoted to proving that after the removal of a number of strong requirements of a set of root data in the sense of [25] we can still construct a faithful reflection representation of an abstract Coxeter group (Theorem 2.11). Many fundamental results in this section are also contained in [13] and [14], and for completeness we include our own proofs to some of these results in the present paper. We were unaware of [14] until informed by the referee, and we thank the referee for communicating [13] and [15] to us. Section 3 collects a number of basic facts about nonorthogonal root systems. In particular, the issue with specifying a preferred way of expressing roots as linear combinations of simple roots (ambiguity may arise, since simple roots are no longer required to be linearly independent), and the issue of the possibility of many (positive) roots corresponding to the same reflection are addressed. More interesting facts are then given in Sections 4 and 5, and these are the main results of this paper. In particular, Section 4 provides comparison results (mostly in the form of inequalities) between constants associated to nonorthogonal representations and those for the classical orthogonal representation. These should be of utility in further study of nonorthogonal representations by reduction to known results in the orthogonal setting. Finally, Section 5 studies the equivalent of the *Tits cone* in nonorthogonal representations. In particular, we generalize a well-known and useful result of G. A. Maxwell on the dual of the Tits cone from special elements in the dual of the Tits cone to arbitrary elements (Theorem 5.8). In particular, Theorem 5.8 of the present paper can be applied to the study of the so-called *imaginary cones* of arbitrary Coxeter groups. The concept of the imaginary cone was introduced by Kac in [21] to study the imaginary roots of Kac–Moody Lie algebras, and was later generalized to Coxeter groups in [8, 9, 12, 14]. In [8, 9], the imaginary cone played a significant role in exploring the geometric distribution of roots of arbitrary infinite Coxeter groups. Theorem 5.8 of the present paper may, amongst other things, improve the understanding of a number of geometric properties of the imaginary cones of Coxeter groups.

2. Paired reflection representations

Let S be an arbitrary set in which each unordered pair $\{s, t\}$ of elements is assigned an $m_{st} \in \mathbb{Z} \cup \{\infty\}$, subject to the conditions that $m_{ss} = 1$ (for all s in S), and $m_{st} \geq 2$ (for all distinct s, t in S). Suppose that V_1 and V_2 are vector spaces over the real field $\mathbb{R}$, and suppose that there exist a bilinear map $\langle\, ,\rangle : V_1 \times V_2 \to \mathbb{R}$ and sets $\Pi_1 = \{\alpha_s \mid s \in S\} \subseteq V_1$ and $\Pi_2 = \{\beta_s \mid s \in S\} \subseteq V_2$ such that the following conditions hold:

(C1) $\langle \alpha_s, \beta_s \rangle = 1$, for all $s \in S$;
(C2) $\langle \alpha_s, \beta_t \rangle \leq 0$, for all distinct $s, t \in S$;
(C3) for all $s, t \in S$,

$$\langle \alpha_s, \beta_t \rangle \langle \alpha_t, \beta_s \rangle = \begin{cases} \cos^2(\pi/m_{st}) & \text{if } m_{st} \neq \infty, \\ \gamma_{st}^2, \text{ for some } \gamma_{st} \geq 1 & \text{if } m_{st} = \infty; \end{cases}$$

(C4) $\langle \alpha_s, \beta_t \rangle = 0$ if and only if $\langle \alpha_t, \beta_s \rangle = 0$, for all $s, t \in S$;
(C5) $\sum_{s \in S} \lambda_s \alpha_s = 0$ with $\lambda_s \geq 0$ for all s implies $\lambda_s = 0$ for all s, and $\sum_{s \in S} \lambda_s \beta_s = 0$ with $\lambda_s \geq 0$ for all s implies $\lambda_s = 0$ for all s.

Note that (C3) and (C4) together imply that $\langle \alpha_s, \beta_t \rangle$ and $\langle \alpha_t, \beta_s \rangle$ are zero if and only if $m_{st} = 2$. We can express (C5) more compactly as $0 \notin \mathrm{PLC}(\Pi_1)$ and $0 \notin \mathrm{PLC}(\Pi_2)$, where, for any set A, $\mathrm{PLC}(A)$ (the *positive linear combinations of* A) is defined to be

$$\Big\{ \sum_{a \in A} \lambda_a a \;\Big|\; \lambda_a \geq 0 \text{ for all } a \in A, \text{ and } \lambda_{a'} > 0 \text{ for some } a' \in A \Big\}.$$

DEFINITION 2.1. In the above situation, if conditions (C1) to (C5) are satisfied then we call $\mathscr{C} = (S, V_1, V_2, \Pi_1, \Pi_2, \langle\, ,\rangle)$ a *Coxeter datum*. The m_{st} $(s, t \in S)$ are called the *Coxeter parameters* of $\mathscr{C}$.

Following [7, 1.1.1], we call $\mathscr{C}$ *free* if both Π_1 and Π_2 are linearly independent. Throughout this paper, $\mathscr{C} = (S, V_1, V_2, \Pi_1, \Pi_2, \langle\, ,\rangle)$ will be a fixed Coxeter datum with Coxeter parameters m_{st}, unless otherwise stated. We stress that, in general, $\mathscr{C}$ is not required to be free.

REMARK 2.2. In a Coxeter datum $\mathscr{C}$, if we take $V_2 = \mathrm{Hom}(V_1, \mathbb{R})$ and take $\langle\, ,\rangle$ to be the natural pairing on $V_1 \times \mathrm{Hom}(V_1, \mathbb{R})$, then the conditions (RB1), (RB2) and (RB3) required in the definition of the *root data* of [7, 1.1.1] are automatically satisfied in $\mathscr{C}$ (though here we have scaled each element of Π_1 and Π_2 by a factor of $1/\sqrt{2}$). Thus the root data in the sense of [7] are special cases of a Coxeter datum defined here. Further, if we assume that $\mathscr{C}$ is free then we have a set of root data as given in [13] (for the case where the totally ordered commutative ring is $\mathbb{R}$), whereas if we assume that $\langle\, ,\rangle$ is nondegenerate and $\langle \alpha_s, \beta_t \rangle \in \mathbb{Z}$ for all $s, t \in S$ then we recover a set of root data as given in [25].

DEFINITION 2.3. Given a Coxeter datum $\mathscr{C} = (S, V_1, V_2, \Pi_1, \Pi_2, \langle\ ,\ \rangle)$, for each $s \in S$, let $\rho_{V_1}(s)$ and $\rho_{V_2}(s)$ be the linear transformations on V_1 and V_2 defined by

$$\rho_{V_1}(s)(x) = x - 2\langle x, \beta_s\rangle\alpha_s$$

for all $x \in V_1$, and

$$\rho_{V_2}(s)(y) = y - 2\langle \alpha_s, y\rangle\beta_s$$

for all $y \in V_2$. For each $i \in \{1, 2\}$ let $R_i(\mathscr{C}) = \{\rho_{V_i}(s) \mid s \in S\}$, and let $W_i(\mathscr{C})$ be the subgroup of $\mathrm{GL}(V_i)$ generated by $R_i(\mathscr{C})$.

For each $s \in S$, it is readily checked that $\rho_{V_1}(s)$ and $\rho_{V_2}(s)$ are involutions with $\rho_{V_1}(s)(\alpha_s) = -\alpha_s$ and $\rho_{V_2}(s)(\beta_s) = -\beta_s$. Further, we have the following proposition.

PROPOSITION 2.4. *Let $x \in V_1$ and $y \in V_2$. Then for all $s \in S$,*

$$\langle \rho_{V_1}(s)(x), \rho_{V_2}(s)(y)\rangle = \langle x, y\rangle.$$

Though $\mathscr{C}$ lacks freeness in general, nevertheless, conditions (C1), (C2) and (C5) of the definition of a Coxeter datum together yield the following lemma.

LEMMA 2.5. *For each $s \in S$,*

$$\alpha_s \notin \mathrm{PLC}(\Pi_1 \setminus \{\alpha_s\}) \quad \textit{and} \quad \beta_s \notin \mathrm{PLC}(\Pi_2 \setminus \{\beta_s\}).$$

In particular, for distinct $s, t \in S$, the set $\{\alpha_s, \alpha_t\}$ is linearly independent, and so is $\{\beta_s, \beta_t\}$.

For each $i \in \{1, 2\}$, the above lemma yields that $\rho_{V_i}(s) \neq \rho_{V_i}(t)$ whenever $s, t \in S$ are distinct.

LEMMA 2.6. *Suppose that $s, t \in S$ such that $s \neq t$.*

(i) *If $m_{st} = 2$ then for each $n \in \mathbb{N}$,*

$$(\rho_{V_1}(s)\rho_{V_1}(t))^n(\alpha_s) = (-1)^n\alpha_s = \rho_{V_1}(t)(\rho_{V_1}(s)\rho_{V_1}(t))^n(\alpha_s)$$

and

$$(\rho_{V_2}(s)\rho_{V_2}(t))^n(\beta_s) = (-1)^n\beta_s = \rho_{V_2}(t)(\rho_{V_2}(s)\rho_{V_2}(t))^n(\beta_s).$$

(ii) *If $m_{st} \neq 2$ then for each $n \in \mathbb{N}$,*

$$(\rho_{V_1}(s)\rho_{V_1}(t))^n(\alpha_s) = p_n^{st}\alpha_s + \lambda_{st}q_n^{st}\alpha_t,$$

$$(\rho_{V_2}(s)\rho_{V_2}(t))^n(\beta_s) = p_n^{st}\beta_s + \frac{1}{\lambda_{st}}q_n^{st}\beta_t,$$

$$\rho_{V_1}(t)(\rho_{V_1}(s)\rho_{V_1}(t))^n(\alpha_s) = p_n^{st}\alpha_s + \lambda_{st}q_{n+1}^{st}\alpha_t,$$

and

$$\rho_{V_2}(t)(\rho_{V_2}(s)\rho_{V_2}(t))^n(\beta_s) = p_n^{st}\beta_s + \frac{1}{\lambda_{st}}q_{n+1}^{st}\beta_t,$$

https://doi.org/10.1017/S1446788714000263 Published online by Cambridge University Press

where $\lambda_{st} = (\sqrt{\langle\alpha_s,\beta_t\rangle\langle\alpha_t,\beta_s\rangle}/-\langle\alpha_t,\beta_s\rangle)$, γ_{st} *is as defined in (C3),*

$$p_n^{st} = \begin{cases} \dfrac{\sin(2n+1)\theta}{\sin\theta} & \text{if } m_{st} < \infty, \text{ and } \theta = \dfrac{\pi}{m_{st}}; \\ 2n+1 & \text{if } m_{st} = \infty, \text{ and } \gamma_{st} = 1; \\ \dfrac{\sinh(2n+1)\theta}{\sinh\theta} & \text{if } m_{st} = \infty, \text{ and } \theta = \cosh^{-1}(\gamma_{st}) > 0, \end{cases}$$

and

$$q_n^{st} = \begin{cases} \dfrac{\sin(2n)\theta}{\sin\theta} & \text{if } m_{st} < \infty, \text{ and } \theta = \dfrac{\pi}{m_{st}}; \\ 2n, & \text{if } m_{st} = \infty \text{ and } \gamma_{st} = 1; \\ \dfrac{\sinh(2n)\theta}{\sinh\theta} & \text{if } m_{st} = \infty, \text{ and } \theta = \cosh^{-1}(\gamma_{st}) > 0. \end{cases}$$

PROOF. (i) If $m_{st} = 2$ then $\langle\alpha_s,\beta_t\rangle = \langle\alpha_t,\beta_s\rangle = 0$, and hence the desired result follows from the definition of $\rho_{V_i}(s)$ and $\rho_{V_i}(t)$ (for each $i \in \{1,2\}$).

(ii) First we observe that if

$$(\rho_{V_1}(s)\rho_{V_1}(t))^n(\alpha_s) = p_n^{st}\alpha_s + \lambda_{st}q_n^{st}\alpha_t \tag{2.1}$$

and

$$(\rho_{V_2}(s)\rho_{V_2}(t))^n(\beta_s) = p_n^{st}\beta_s + \frac{1}{\lambda_{st}}q_n^{st}\beta_t \tag{2.2}$$

both hold, then it follows from the linear independence of each of $\{\alpha_s,\alpha_t\}$ and $\{\beta_s,\beta_t\}$ and the fact that applying $\rho_{V_1}(t)$ ($\rho_{V_2}(t)$) only changes the coefficient of α_t (β_t) that

$$\rho_{V_1}(t)(\rho_{V_1}(s)\rho_{V_1}(t))^n(\alpha_s) = p_n^{st}\alpha_s + \lambda_{st}q_{n+1}^{st}\alpha_t$$

and

$$\rho_{V_2}(t)(\rho_{V_2}(s)\rho_{V_2}(t))^n(\beta_s) = p_n^{st}\beta_s + \frac{1}{\lambda_{st}}q_{n+1}^{st}\beta_t.$$

Thus it only remains to prove (2.1) and (2.2).

First we consider the case $m_{st} < \infty$. Since $\langle\alpha_s,\beta_t\rangle\langle\alpha_t,\beta_s\rangle = \cos^2\theta$ in this case, the matrix of $\rho_{V_1}(s)\rho_{V_1}(t)$ in its action on the subspace with basis $\{\alpha_s,\alpha_t\}$ is

$$\begin{pmatrix} -1 & -2\langle\alpha_t,\beta_s\rangle \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -2\langle\alpha_s,\beta_t\rangle & -1 \end{pmatrix} = \begin{pmatrix} 4\cos^2\theta - 1 & 2\langle\alpha_t,\beta_s\rangle \\ -2\langle\alpha_s,\beta_t\rangle & -1 \end{pmatrix}.$$

Observe that

$$\begin{aligned} \begin{pmatrix} 4\cos^2\theta - 1 & 2\langle\alpha_t,\beta_s\rangle \\ -2\langle\alpha_s,\beta_t\rangle & -1 \end{pmatrix} &= \begin{pmatrix} \dfrac{-\cos\theta}{\langle\alpha_s,\beta_t\rangle} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 4\cos^2\theta - 1 & -2\cos\theta \\ 2\cos\theta & -1 \end{pmatrix}\begin{pmatrix} \dfrac{-\cos\theta}{\langle\alpha_t,\beta_s\rangle} & 0 \\ 0 & 1 \end{pmatrix} \\ &= \frac{1}{\sin\theta}\begin{pmatrix} \dfrac{-\cos\theta}{\langle\alpha_s,\beta_t\rangle} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \sin 3\theta & -\sin 2\theta \\ \sin 2\theta & -\sin\theta \end{pmatrix}\begin{pmatrix} \dfrac{-\cos\theta}{\langle\alpha_t,\beta_s\rangle} & 0 \\ 0 & 1 \end{pmatrix}, \end{aligned}$$

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and hence an induction on n yields that, for all $n \in \mathbb{N}$,

$$\begin{pmatrix} 4\cos^2\theta - 1 & 2\langle\alpha_t,\beta_s\rangle \\ -2\langle\alpha_s,\beta_t\rangle & 1 \end{pmatrix}^n = \frac{1}{\sin\theta}\begin{pmatrix} \frac{-\cos\theta}{\langle\alpha_s,\beta_t\rangle} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \sin(2n+1)\theta & -\sin 2n\theta \\ \sin 2n\theta & -\sin(2n-1)\theta \end{pmatrix}\begin{pmatrix} \frac{-\cos\theta}{\langle\alpha_t,\beta_s\rangle} & 0 \\ 0 & 1 \end{pmatrix}.$$

Consequently (2.1) holds when $m_{st} < \infty$ with $p_n^{st} = (\sin(2n+1)\theta)/\sin\theta$, $q_n^{st} = \sin 2n\theta/\sin\theta$, and $\lambda_{st} = \sqrt{\langle\alpha_s,\beta_t\rangle\langle\alpha_t,\beta_s\rangle}/-\langle\alpha_t,\beta_s\rangle$. An entirely similar calculation shows that (2.2) holds in this case with the same p_n^{st}, q_n^{st} and λ_{st}, thus establishing (2.1) and (2.2) in the case where $m_{st} < \infty$.

Next consider the case $m_{st} = \infty$. Since $\langle\alpha_s,\beta_t\rangle\langle\alpha_t,\beta_s\rangle = \gamma_{st}^2$ in this case, the matrix of $\rho_{V_1}(s)\rho_{V_1}(t)$ in its action on the subspace with basis $\{\alpha_s,\alpha_t\}$ is

$$\begin{pmatrix} -1 & -2\langle\alpha_t,\beta_s\rangle \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -2\langle\alpha_s,\beta_t\rangle & -1 \end{pmatrix} = \begin{pmatrix} 4\gamma_{st}^2 - 1 & 2\langle\alpha_t,\beta_s\rangle \\ -2\langle\alpha_s,\beta_t\rangle & -1 \end{pmatrix}.$$

Observe that if $\gamma_{st} = 1$ then

$$\begin{pmatrix} 4\gamma_{st}^2 - 1 & 2\langle\alpha_t,\beta_s\rangle \\ -2\langle\alpha_s,\beta_t\rangle & -1 \end{pmatrix} = \begin{pmatrix} \frac{-1}{\langle\alpha_s,\beta_t\rangle} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} \frac{-1}{\langle\alpha_t,\beta_s\rangle} & 0 \\ 0 & 1 \end{pmatrix},$$

and an induction yields that, for all $n \in \mathbb{N}$,

$$\begin{pmatrix} 4\gamma_{st}^2 - 1 & 2\langle\alpha_t,\beta_s\rangle \\ -2\langle\alpha_s,\beta_t\rangle & -1 \end{pmatrix}^n = \begin{pmatrix} \frac{-1}{\langle\alpha_s,\beta_t\rangle} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2n+1 & -2n \\ 2n & -(2n-1) \end{pmatrix}\begin{pmatrix} \frac{-1}{\langle\alpha_t,\beta_s\rangle} & 0 \\ 0 & 1 \end{pmatrix}.$$

Consequently (2.1) holds when $m_{st} = \infty$ and $\gamma_{st} = 1$ with $p_n^{st} = 2n+1$, $q_n^{st} = 2n$ and $\lambda_{st} = 1/-\langle\alpha_t,\beta_s\rangle$. An entirely similar calculation shows that (2.2) holds in this case with the same p_n^{st}, q_n^{st} and λ_{st}, thus establishing (2.1) and (2.2) in the case $m_{st} = \infty$ and $\gamma_{st} = 1$. Finally, if $\gamma_{st} > 1$, then

$$\begin{aligned} \begin{pmatrix} 4\gamma_{st}^2 - 1 & 2\langle\alpha_t,\beta_s\rangle \\ -2\langle\alpha_s,\beta_t\rangle & -1 \end{pmatrix} &= \begin{pmatrix} \frac{-\gamma_{st}}{\langle\alpha_s,\beta_t\rangle} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 4\gamma_{st}^2 - 1 & -2\gamma_{st} \\ 2\gamma_{st} & -1 \end{pmatrix}\begin{pmatrix} \frac{-\gamma_{st}}{\langle\alpha_t,\beta_s\rangle} & 0 \\ 0 & 1 \end{pmatrix} \\ &= \frac{1}{\sinh\theta}\begin{pmatrix} \frac{-\gamma_{st}}{\langle\alpha_s,\beta_t\rangle} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \sinh 3\theta & -\sinh 2\theta \\ \sinh 2\theta & -\sinh\theta \end{pmatrix}\begin{pmatrix} \frac{-\gamma_{st}}{\langle\alpha_t,\beta_s\rangle} & 0 \\ 0 & 1 \end{pmatrix}, \end{aligned}$$

and hence an induction yields that, for all $n \in \mathbb{N}$,

$$\begin{pmatrix} 4\gamma_{st}^2 - 1 & 2\langle\alpha_t,\beta_s\rangle \\ -2\langle\alpha_s,\beta_t\rangle & -1 \end{pmatrix}^n = \frac{1}{\sinh\theta}\begin{pmatrix} \frac{-\gamma_{st}}{\langle\alpha_s,\beta_t\rangle} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \sinh(2n+1)\theta & -\sinh(2n\theta) \\ \sinh(2n\theta) & -\sinh(2n-1)\theta \end{pmatrix}\begin{pmatrix} \frac{-\gamma_{st}}{\langle\alpha_t,\beta_s\rangle} & 0 \\ 0 & 1 \end{pmatrix}.$$

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Consequently (2.1) holds when $m_{st} = \infty$ and $\gamma_{st} > 1$ with $p_n^{st} = (\sinh(2n+1)\theta)/\sinh\theta$, $q_n^{st} = \sinh 2n\theta/\sinh\theta$, and $\lambda_{st} = \gamma_{st}/-\langle\alpha_t, \beta_s\rangle$. An entirely similar calculation shows that (2.2) holds in this case with the same p_n^{st}, q_n^{st} and λ_{st}, completing the proof. □

Remark 2.7. Lemma 2.6(ii) follows readily from [13]. Using the polynomials P_n and Q_n defined in [13, (1.9)] and setting $a = -2\langle\alpha_t, \beta_s\rangle$, $b = -2\langle\alpha_s, \beta_t\rangle$, and $l = ab$, we see that the formulas in Lemma 2.6(ii) can be obtained from [13, (1.10), (1.15)] by setting $p_n^{st} = P_n(l)$, $q_n^{st} = \sqrt{l}Q_n(l)$ and $\lambda_{st} = \sqrt{b/a}$. We thank the referee for pointing this out to us.

Remark 2.8. Let $\mathrm{ord}(\rho_{V_i}(s)\rho_{V_i}(t))$ denote the order of $\rho_{V_i}(s)\rho_{V_i}(t)$ in the group $\mathrm{GL}(V_i)$, for each $i \in \{1, 2\}$. Then:

(i) Lemma 2.6(ii) yields that $\mathrm{ord}(\rho_{V_i}(s)\rho_{V_i}(t)) \geq m_{st}$ when $m_{st} \neq \infty$. Indeed, in the subspace with basis $\{\alpha_s, \alpha_t\}$ the m_{st} elements

$$\alpha_s, (\rho_{V_1}(s)\rho_{V_1}(t))\alpha_s, (\rho_{V_1}(s)\rho_{V_1}(t))^2\alpha_s, \ldots, (\rho_{V_1}(s)\rho_{V_1}(t))^{m_{st}-1}\alpha_s$$

are all distinct, and the same holds in the subspace with basis $\{\beta_s, \beta_t\}$.

(ii) Similarly, it follows from Lemma 2.6(ii) that $\mathrm{ord}(\rho_{V_i}(s)\rho_{V_i}(t)) = \infty$ whenever $m_{st} = \infty$.

The above observations naturally lead to the following proposition.

Proposition 2.9. *Suppose that $s, t \in S$. Then for each $i \in \{1, 2\}$,*

$$\rho_{V_i}(s)\rho_{V_i}(t) \text{ has order } m_{st} \text{ in } \mathrm{GL}(V_i).$$

Proof. [13, Proposition 1.22], [14, Proposition, page 274], and [15] established the current proposition under the setting where the paired real representation spaces are replaced by modules over a totally ordered commutative ring, with the additional requirement that root bases are linearly independent. For completeness and to account for linearly dependent root bases we include a proof here.

We give a proof that $\rho_{V_1}(s)\rho_{V_1}(t)$ has order m_{st} in $\mathrm{GL}(V_1)$ below, and we stress that the same argument will hold for $\rho_{V_2}(s)\rho_{V_2}(t)$ in $\mathrm{GL}(V_2)$. Observe that we only need to consider the cases where $m_{st} \notin \{1, \infty\}$, for the statement of the proposition follows readily from Remark 2.8(ii) and the fact that each $\rho_{V_1}(s)$ is an involution, for all $s \in S$. Next let $\alpha \in V_1$ be arbitrary. Then

$$\begin{aligned}(\rho_{V_1}(s)\rho_{V_1}(t))(\alpha) &= \rho_{V_1}(s)(\alpha - 2\langle\alpha, \beta_t\rangle\alpha_t)\\ &= \alpha - 2\langle\alpha, \beta_s\rangle\alpha_s - 2\langle\alpha, \beta_t\rangle(\alpha_t - 2\langle\alpha_t, \beta_s\rangle\alpha_s)\\ &= \alpha + (4\langle\alpha, \beta_t\rangle\langle\alpha_t, \beta_s\rangle - 2\langle\alpha, \beta_s\rangle)\alpha_s - 2\langle\alpha, \beta_t\rangle\alpha_t. \qquad (2.3)\end{aligned}$$

If $m_{st} = 2$, then (2.3) yields that

$$(\rho_{V_1}(s)\rho_{V_1}(t))(\alpha) = (\rho_{V_1}(t)\rho_{V_1}(s))(\alpha) = \alpha - 2\langle\alpha, \beta_s\rangle\alpha_s - 2\langle\alpha, \beta_t\rangle\alpha_t,$$

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so that $\rho_{V_1}(s)$ and $\rho_{V_1}(t)$ commute. Thus $\operatorname{ord}(\rho_{V_1}(s)\rho_{V_1}(t)) = 2$ when $m_{st} = 2$, and it remains to check the case where $m_{st} > 2$. Observe that if $\alpha = \alpha_s$ and $\alpha = \alpha_t$, then (2.3) yields that

$$(\rho_{V_1}(s)\rho_{V_1}(t))(\alpha_s) = (4\cos^2(\pi/m_{st}) - 1)\alpha_s - 2\langle\alpha_s, \beta_t\rangle\alpha_t$$

and

$$(\rho_{V_1}(s)\rho_{V_1}(t))(\alpha_t) = 2\langle\alpha_t, \beta_s\rangle\alpha_s - \alpha_t.$$

Therefore for those $\alpha \notin \mathbb{R}\{\alpha_s, \alpha_t\}$, the action of $\rho_{V_1}(s)\rho_{V_1}(t)$ on the subspace with ordered basis $\{\alpha_s, \alpha_t, \alpha\}$ may be represented by the matrix

$$M = \begin{pmatrix} 4\cos^2(\pi/m_{st}) - 1 & 2\langle\alpha_t, \beta_s\rangle & 4\langle\alpha, \beta_t\rangle\langle\alpha_t, \beta_s\rangle - 2\langle\alpha, \beta_s\rangle \\ -2\langle\alpha_s, \beta_t\rangle & -1 & -2\langle\alpha, \beta_t\rangle \\ 0 & 0 & 1 \end{pmatrix}.$$

It is readily checked that M has distinct eigenvalues $e^{i(2\pi/m_{st})}$, $e^{-i(2\pi/m_{st})}$ and 1. Hence M has order m_{st}, and so $(\rho_{V_1}(s)\rho_{V_1}(t))^{m_{st}} = 1$ in $\mathrm{GL}(V_1)$. Finally, in view of Remark 2.8(i), it follows that $\operatorname{ord}(\rho_{V_1}(s)\rho_{V_1}(t))$ is precisely m_{st}. □

Remark 2.10. Given a Coxeter datum $\mathscr{C} = (S, V_1, V_2, \Pi_1, \Pi_2, \langle\,,\,\rangle)$ with Coxeter parameters m_{st} (where $s, t \in S$), let (W, R) be a Coxeter system in the sense of [16] or [20] with $R = \{r_s \mid s \in S\}$ being a set of involutions generating W subject only to the condition that the order of the product $r_s r_t$ is m_{st} whenever s, t are in S with $m_{st} \neq \infty$. Then Proposition 2.9 yields that there are group homomorphisms $f_1\colon W \to W_1(\mathscr{C})$ and $f_2\colon W \to W_2(\mathscr{C})$ satisfying $f_1(r_s) = \rho_{V_1}(s)$ and $f_2(r_s) = \rho_{V_2}(s)$ for all $s \in S$.

The principal result of this section is the following theorem.

Theorem 2.11. *Let (W, R), f_1 and f_2 be as in Remark 2.10 above. Then f_1 and f_2 are isomorphisms; that is, $(W_1(\mathscr{C}), R_1(\mathscr{C}))$ and $(W_2(\mathscr{C}), R_2(\mathscr{C}))$ are both Coxeter systems isomorphic to (W, R).*

Remark 2.12. [13, 2.13 (b)], [14, Proposition, page 274], and [15] established the above theorem for free modules over totally ordered commutative rings in which the root bases are bases. For completeness and to account for linearly dependent root bases, we include a proof to Theorem 2.11 in the present paper.

For each $i \in \{1, 2\}$, since $W_i(\mathscr{C})$ is generated by the elements of $R_i(\mathscr{C})$, it follows readily that each f_i is surjective. Thus only the injectivity of f_i needs to be checked. Before we can do so, a few elementary results are needed. First we have a result easily obtained from the formulas in Lemma 2.6.

Lemma 2.13. *Suppose that $s, t \in S$ such that $m_{st} \neq 2$, and let n be an integer such that $0 \leq n < m_{st}$. Let λ_n, μ_n, λ'_n and μ'_n be constants such that*

$$\underbrace{\ldots\rho_{V_1}(t)\rho_{V_1}(s)\rho_{V_1}(t)}_{n\ factors}(\alpha_s) = \lambda_n\alpha_s + \mu_n\alpha_t$$

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and

$$\underbrace{\ldots\rho_{V_1}(s)\rho_{V_1}(t)\rho_{V_1}(s)}_{n\ \textit{factors}}(\alpha_t) = \lambda'_n\alpha_s + \mu'_n\alpha_t.$$

Then all four constants λ_n, μ_n, λ'_n and μ'_n are nonnegative.

REMARK 2.14. The same argument applies equally well if we replace $\rho_{V_1}(s)$, $\rho_{V_1}(t)$, α_s and α_t in the above lemma by $\rho_{V_2}(s)$, $\rho_{V_2}(t)$, β_s and β_t, respectively.

NOTATION 2.15. Let W, f_1 and f_2 be as in Remark 2.10. Then f_1 and f_2 give rise to W-actions on V_1 and V_2 in the following way: $wx = (f_1(w))(x)$ for all $w \in W$ and $x \in V_1$, and $wy = (f_2(w))(y)$ for all $w \in W$ and $y \in V_2$.

DEFINITION 2.16. Given W and R as in Remark 2.10, let $\ell\colon W \to \mathbb{N}$ be the *length* function of W with respect to R. For $w \in W$, we say that an expression of the form $w = r_{s_1}\cdots r_{s_l}$ (where $s_1, \ldots s_l \in S$) is *reduced* if $\ell(w) = l$.

For any $w \in W$, an easy induction on $\ell(w)$ yields the following extension to Proposition 2.4.

LEMMA 2.17. *Given a Coxeter datum $\mathscr{C} = (S, V_1, V_2, \Pi_1, \Pi_2, \langle\,,\rangle)$, and let W be as in Remark 2.10. Then $\langle x, y\rangle = \langle wx, wy\rangle$ for all $w \in W$, $x \in V_1$, and $y \in V_2$.*

The following is the key step in proving Theorem 2.11. It is an adaptation of the well-known result in the orthogonal setting, and for completeness we include a proof here.

PROPOSITION 2.18. *Let W be as above, and let $w \in W$ and $s \in S$. If $\ell(wr_s) \geq \ell(w)$ then $w\alpha_s \in \mathrm{PLC}(\Pi_1)$.*

PROOF. Choose $w \in W$ of minimal length such that the assertion fails for some $\alpha_s \in \Pi_1$, and fix such an α_s. Certainly $w \neq 1$, since $1\alpha_s = \alpha_s$ is trivially a positive linear combination of Π_1. Thus $\ell(w) \geq 1$, and we may choose $t \in S$ such that $w_1 = wr_t$ has length $\ell(w) - 1$. If $\ell(w_1 r_s) \geq \ell(w_1)$, then $\ell(w_1 r) \geq \ell(w_1)$ for both $r = r_s$ and $r = r_t$. Alternatively, if $\ell(w_1 r_s) < \ell(w_1)$, we define $w_2 = w_1 r_s$, and note that $\ell(w_2 r) \geq \ell(w_2)$ will hold for $r = r_s$ and $r = r_t$ if $\ell(w_2 r_t) \geq \ell(w_2)$. If this latter condition is not satisfied then we define $w_3 = w_2 r_t$. Continuing in this way we find, for some positive integer k, a sequence of elements $w_0 = w, w_1, w_2, \ldots, w_k$ with $\ell(w_i) = \ell(w) - i$ for all $i = 0, 1, 2, \ldots, k$, and, when $i < k$,

$$w_{i+1} = \begin{cases} w_i r_s & \text{if } i \text{ is odd;} \\ w_i r_t & \text{if } i \text{ is even.} \end{cases}$$

Now since $0 \leq \ell(w_k) = \ell(w) - k$, we conclude that $\ell(w)$ is an upper bound for the possible values of k. Choosing k to be as large as possible, we deduce that $\ell(w_k r) \geq \ell(w_k)$ for both $r = r_s$ and $r = r_t$, for otherwise the process described above would allow a w_{k+1} to be found, contrary to the definition of k. By the minimality of our original counterexample it follows that $w_k\alpha_s$ and $w_k\alpha_t$ are both in $\mathrm{PLC}(\Pi_1)$.

We have $w = w_k v$, where v is an alternating product of r_ss and r_ts, ending in r_t, and with k factors altogether. This means that $\ell(v) \leq k$. But $w = w_k v$ gives $\ell(w) \leq \ell(w_k) + \ell(v)$, so it follows that $\ell(v) \geq \ell(w) - \ell(w_k) = k$, and hence $\ell(v) = k$. Further, in view of the hypothesis that $\ell(wr_s) \geq \ell(w)$, and since $w_k v r_s = wr_s$, we have

$$\ell(w_k) + \ell(vr_s) \geq \ell(wr_s) \geq \ell(w) = \ell(w_k) + k = \ell(w_k) + \ell(v),$$

and hence $\ell(vr_s) \geq \ell(v)$. In particular, v cannot have a reduced expression in which the final factor is r_s, for if so then vr_s would have a strictly shorter expression.

Since r_s and r_t satisfy the defining relations of the dihedral group of order $2m_{st}$, it follows that every element of the subgroup generated by r_s and r_t has an expression of length less than $m_{st} + 1$ as an alternating product of r_s and r_t. Thus $\ell(v) \leq m_{st}$. Moreover, if m_{st} is finite then the two alternating products of length m_{st} define the same element; so $\ell(v)$ cannot equal m_{st}, as v has no reduced expression ending with r_s. Thus Lemma 2.13 above yields $v\alpha_s = \lambda_1\alpha_s + \mu_1\alpha_t$ for some nonnegative coefficients λ_1 and μ_1. Hence

$$w\alpha_s = w_k v\alpha_s = w_k(\lambda_1\alpha_s + \mu_1\alpha_t) = \lambda_1 w_k\alpha_s + \mu_1 w_k\alpha_t \in \mathrm{PLC}(\Pi_1),$$

since $w_k\alpha_s, w_k\alpha_t \in \mathrm{PLC}(\Pi_1)$. This contradicts our original choice of w and α_s as a counterexample to the statement of the proposition, completing the proof. □

Now we are ready to complete the proof of Theorem 2.11.

Proof of Theorem 2.11. Suppose, for a contradiction, that the kernel of f_1 is nontrivial, and choose w in the kernel of f_1 with $w \neq 1$. Then $\ell(w) > 0$, and we may write $w = w'r_s$ for some $s \in S$ and $w' \in W$ with $\ell(w') = \ell(w) - 1$. Since $\ell(w'r_s) > \ell(w')$, Proposition 2.18 yields $w'\alpha_s \in \mathrm{PLC}(\Pi_1)$. But then

$$\alpha_s = w\alpha_s = (w'r_s)\alpha_s = w'(r_s\alpha_s) = w'(-\alpha_s) = -w'\alpha_s,$$

and hence $0 = \alpha_s + w'\alpha_s \in \mathrm{PLC}(\Pi_1)$, contradicting condition (C5) of a Coxeter datum. In an entirely similar way it can be shown that the kernel of f_2 is trivial. □

Definition 2.19. Let W and R be as in Remark 2.10. We call W the *abstract Coxeter group* determined by the Coxeter parameters of the Coxeter datum $\mathscr{C}$, and we call (W, R) the *abstract Coxeter system* associated with $\mathscr{C}$.

Observe that Theorem 2.11 yields that the W-actions on V_1 and V_2 induced by the isomorphisms f_1 and f_2 are faithful.

3. Root systems and canonical coefficients

Definition 3.1. Suppose that $\mathscr{C} = (S, V_1, V_2, \Pi_1, \Pi_2, \langle\ ,\ \rangle)$ is a Coxeter datum, and suppose that (W, R) is the associated abstract Coxeter system.

(i) Define $\Phi_1(\mathscr{C}) = W\Pi_1 = \{w\alpha_s \mid w \in W, s \in S\}$, and similarly define $\Phi_2(\mathscr{C}) = W\Pi_2 = \{w\beta_s \mid w \in W, s \in S\}$. For each $i \in \{1, 2\}$, we call $\Phi_i(\mathscr{C})$ the *root system* of W in V_i, and its elements the *roots* of W in V_i. We call Π_i the set of *simple roots* in $\Phi_i(\mathscr{C})$, and we say that Π_i forms a *root basis* for $\Phi_i(\mathscr{C})$.

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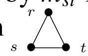
(ii) Set $\Phi_i^+(\mathscr{C}) = \Phi_i(\mathscr{C}) \cap \mathrm{PLC}(\Pi_i)$, and $\Phi_i^-(\mathscr{C}) = -\Phi_i^+(\mathscr{C})$ for each $i \in \{1, 2\}$. We call $\Phi_i^+(\mathscr{C})$ the set of *positive roots* in $\Phi_i(\mathscr{C})$, and $\Phi_i^-(\mathscr{C})$ the set of *negative roots* in $\Phi_i(\mathscr{C})$.

Given the above notation, for each $i \in \{1, 2\}$, we adopt the traditional diagrammatic description of simple roots Π_i: draw a graph that has one vertex for each $s \in S$, and join the vertices corresponding to $s, t \in S$ by an edge labelled by m_{st} if $m_{st} > 2$. The label m_{st} is often omitted in the case $m_{st} = 3$. Thus the diagram 

corresponds to $\Pi_1 = \{\alpha_r, \alpha_s, \alpha_t\}$ and $\Pi_2 = \{\beta_r, \beta_s, \beta_t\}$. Suppose that

$$\langle \alpha_r, \beta_s\rangle = -1/4, \quad \langle \alpha_s, \beta_t\rangle = -1/6, \quad \langle \alpha_t, \beta_r\rangle = -1/10,$$
$$\langle \alpha_s, \beta_r\rangle = -1, \quad \langle \alpha_t, \beta_s\rangle = -3/2, \quad \langle \alpha_r, \beta_t\rangle = -5/2.$$

Then

$$\begin{aligned}
r_s\alpha_r &= \alpha_r - 2\langle \alpha_r, \beta_s\rangle \alpha_s = \alpha_r + \tfrac{1}{2}\alpha_s,\\
(r_r r_s)\alpha_r &= r_r(\alpha_r + \tfrac{1}{2}\alpha_s) = \tfrac{1}{2}\alpha_s,\\
(r_t r_r r_s)\alpha_r &= r_t(\tfrac{1}{2}\alpha_s) = \tfrac{1}{2}\alpha_s + \tfrac{1}{6}\alpha_t,\\
(r_s r_t r_r r_s)\alpha_r &= r_s(\tfrac{1}{2}\alpha_s + \tfrac{1}{6}\alpha_t) = \tfrac{1}{6}\alpha_t,\\
(r_r r_s r_t r_r r_s)\alpha_r &= r_r(\tfrac{1}{6}\alpha_t) = \tfrac{1}{6}\alpha_t + \tfrac{1}{30}\alpha_r,\\
(r_t r_r r_s r_t r_r r_s)\alpha_r &= r_t(\tfrac{1}{6}\alpha_t + \tfrac{1}{30}\alpha_r) = \tfrac{1}{30}\alpha_r.
\end{aligned}$$

In particular, we notice from the above example that it is possible for a nontrivial positive scalar multiple of a root to also be a root, lying in the same W-orbit as the root itself. Clearly if $w\alpha = \lambda\alpha$ where $\alpha \in \Phi_1(\mathscr{C})$ then $w^n\alpha = \lambda^n\alpha$ for all $n \in \mathbb{N}$. Since it is quite possible that $\lambda \neq \pm 1$, it follows that there could well be infinitely many nontrivial scalar multiples of α in $\Phi_1(\mathscr{C})$. Further, all roots in the W-orbit of α will possess this same property. Of course, the same situation could arise in $\Phi_2(\mathscr{C})$ as well. This is one of the features setting $\Phi_1(\mathscr{C})$ and $\Phi_2(\mathscr{C})$ apart from the classical root systems studied in [1, Ch. V], [16] or [20, Ch. 5].

Observe that as in [13, 2.9], Proposition 2.18 above has the following consequences:

Lemma 3.2. *Suppose that $\mathscr{C} = (S, V_1, V_2, \Pi_1, \Pi_2, \langle\ ,\ \rangle)$ is a Coxeter datum, and suppose that (W, R) is the associated abstract Coxeter system.*

(i) $\Phi_i(\mathscr{C}) = \Phi_i^+(\mathscr{C}) \uplus \Phi_i^-(\mathscr{C})$, *for each* $i \in \{1, 2\}$, *where* $\uplus$ *denotes disjoint union.*

(ii) *If* $w \in W$ *and* $s \in S$, *then*

$$\ell(wr_s) = \begin{cases} \ell(w) + 1 & \text{if } w\alpha_s \in \Phi_1^+(\mathscr{C}), \text{ and } w\beta_s \in \Phi_2^+(\mathscr{C}),\\ \ell(w) - 1 & \text{if } w\alpha_s \in \Phi_1^-(\mathscr{C}), \text{ and } w\beta_s \in \Phi_2^-(\mathscr{C}). \end{cases}$$ □

Remark 3.3. Part (ii) of the above lemma implies that for $w \in W$ and $s \in S$, $w\alpha_s \in \Phi_1^+(\mathscr{C})$ if and only if $w\beta_s \in \Phi_2^+(\mathscr{C})$, and $w\alpha_s \in \Phi_1^-(\mathscr{C})$ if and only if $w\beta_s \in \Phi_2^-(\mathscr{C})$.

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Remark 3.4. Since $\mathscr{C}$ is not assumed to be free, although Lemma 3.2(i) makes it possible to assert that each root in $\Phi_i(\mathscr{C})$ (for each $i \in \{1,2\}$) is expressible as a linear combination of simple roots from Π_i with coefficients all being of the same sign, these expressions need not be unique. Thus the concept of *the coefficient of a simple root in a given root* is potentially ambiguous. To obtain a canonical way of expressing roots in terms of simple roots, we employ a construction similar to those of [11, 19, 25]. We define a free Coxeter datum $\mathscr{C}'$ on the same set of Coxeter parameters as those of $\mathscr{C}$. Then both $\mathscr{C}$ and the free Coxeter datum $\mathscr{C}'$ correspond to the same abstract Coxeter system (W, R). It turns out that for each $i \in \{1,2\}$, there exists a canonical W-equivariant bijection $\pi_i \colon \Phi_i(\mathscr{C}) \leftrightarrow \Phi_i(\mathscr{C}')$ which maps simple roots to simple roots. Since each $\Phi_i(\mathscr{C}')$ is free, there is no ambiguity of the coefficient of a simple root in any given root of $\Phi(\mathscr{C}')$, and this will provide a canonical expression of roots in $\Phi_i(\mathscr{C})$ in terms of Π_i via π_i. However, since $\mathscr{C}$ lacks the integrality condition assumed in [25], the proof used in [25] will not apply here. Further, since it is no longer true that there exists a bijection between $\Phi_i(\mathscr{C})^+$ $(i = 1,2)$ and the reflections in W, the proofs used in [11] and [19] will not apply either.

Let V_1' be a vector space over $\mathbb{R}$ with basis $\Pi_1' = \{\alpha_s' \mid s \in S\}$ in bijective correspondence with S, and let V_2' be a vector space over $\mathbb{R}$ with basis $\Pi_2' = \{\beta_s' \mid s \in S\}$, also in bijective correspondence with S. Define linear maps $\pi_1 \colon V_1' \to V_1$ and $\pi_2 \colon V_2' \to V_2$ by requiring that

$$\pi_1\Big(\sum_{s\in S} \lambda_s \alpha_s'\Big) = \sum_{s\in S} \lambda_s \alpha_s \quad \text{and} \quad \pi_2\Big(\sum_{s\in S} \mu_s \beta_s'\Big) = \sum_{s\in S} \mu_s \beta_s,$$

for all $\lambda_s, \mu_s \in \mathbb{R}$, and define a bilinear map $\langle\, ,\rangle' \colon V_1' \times V_2' \to \mathbb{R}$ by requiring that $\langle \alpha_s', \beta_t' \rangle' = \langle \alpha_s, \beta_t \rangle$ for all $s, t \in S$. Then $\langle x', y' \rangle' = \langle \pi_1(x'), \pi_2(y') \rangle$ for all $x' \in V_1'$ and $y' \in V_2'$.

Thus $\mathscr{C}' = (S, V_1', V_2', \Pi_1', \Pi_2', \langle\, ,\rangle')$ is a free Coxeter datum with the same parameters as $\mathscr{C}$, and therefore is associated to the same abstract Coxeter system (W, R). Applying Theorems 2.11 to $\mathscr{C}'$ then yields isomorphisms $f_1' \colon W \to W_1(\mathscr{C}')$ and $f_2' \colon W \to W_2(\mathscr{C}')$. These isomorphisms induce W-actions on V_1' and V_2' via $wx' = (f_1'(w))(x')$ and $uy' = (f_2'(u))(y')$, for all $w, u \in W$, $x' \in V_1'$ and $y' \in V_2'$. Note that for each $i \in \{1,2\}$ and each $s \in S$, it follows that $f_i'(r_s) = \rho_{V_i'}(s)$, and this in turn yields that $\pi_i f_i'(r_s) = f_i(r_s)\pi_i$, where f_1 and f_2 are as in Theorem 2.11. Since W is generated by $\{r_s \mid s \in S\}$, it follows that $\pi_i f_i'(w) = f_i(w)\pi_i$ for all $w \in W$. Summing up, we have

$$\pi_i(wz') = w\pi_i(z') \quad \text{for all } w \in W \text{ and } z' \in V_i',$$

and hence each π_i is a W-module homomorphism.

Proposition 3.5. *For each $i \in \{1,2\}$, the restriction of π_i defines a W-equivariant bijection $\Phi_i(\mathscr{C}') \to \Phi_i(\mathscr{C})$.*

[14, Proposition, page 274] and [15] established Proposition 3.5 under a similar setting. For completeness we include a proof in the present paper, but before we can do so we need a few elementary results and some further notation.

Definition 3.6. For each $i \in \{1, 2\}$, define an equivalence relation $\sim_i$ on $\Phi_i(\mathscr{C})$ as follows: if z_1 and $z_2 \in \Phi_i(\mathscr{C})$, then $z_1 \sim_i z_2$ if and only if z_1 and z_2 are (nonzero) scalar multiples of each other. For each $z \in \Phi_i(\mathscr{C})$, write $\widehat{z}$ for the equivalence class containing z, and set $\widehat{\Phi_i(\mathscr{C})} = \{\widehat{z} \mid z \in \Phi_i(\mathscr{C})\}$.

Observe that the action of W on $\Phi_i(\mathscr{C})$ (for $i = 1, 2$) gives rise to a well-defined action of W on $\widehat{\Phi_i(\mathscr{C})}$ satisfying $w\widehat{z} = \widehat{wz}$ for all $w \in W$, and all $z \in \Phi_i(\mathscr{C})$.

Definition 3.7. For $i \in \{1, 2\}$, and for each $w \in W$, define

$$N_i(w) = \{\widehat{\gamma} \mid \gamma \in \Phi_i^+(\mathscr{C}) \text{ and } w\gamma \in \Phi_i^-(\mathscr{C})\}.$$

Note that for $w \in W$, the set $N_i(w)$ $(i = 1, 2)$ can be alternatively characterized as $\{\widehat{\gamma} \mid \gamma \in \Phi_i^-(\mathscr{C}) \text{ and } w\gamma \in \Phi_i^+(\mathscr{C})\}$. Hence $\widehat{z} \in N_i(w)$ if and only if precisely one element of the set $\{z, wz\}$ is in $\Phi_i^+(\mathscr{C})$. A mild generalization of the techniques used in [20, Section 5.6] then yields the following result.

Lemma 3.8.

(i) *If $s \in S$ then $N_1(r_s) = \{\widehat{\alpha_s}\}$ and $N_2(r_s) = \{\widehat{\beta_s}\}$.*
(ii) *Let $w \in W$. Then $N_1(w)$ and $N_2(w)$ both have cardinality $\ell(w)$.*
(iii) *Let w_1, $w_2 \in W$ and let $\dotplus$ denote set symmetric difference. Then $N_i(w_1w_2) = w_2^{-1}N_i(w_1) \dotplus N_i(w_2)$ for each $i \in \{1, 2\}$.*
(iv) *Let w_1, $w_2 \in W$. Then for each $i \in \{1, 2\}$,*

$$\ell(w_1w_2) = \ell(w_1) + \ell(w_2) \quad \textit{if and only if } N_i(w_2) \subseteq N_i(w_1w_2).$$

If $w_1, w_2 \in W$ with $\ell(w_1w_2) = \ell(w_1) + \ell(w_2)$ then we call w_2 a *right-hand segment* of w_1w_2. If $w = r_{s_1}r_{s_2} \cdots r_{s_l}$ (where $w \in W$ and $s_1, \ldots, s_l \in S$) with $\ell(w) = l$, then the above lemma yields that

$$N_1(w) = \{\widehat{\alpha_{s_l}}, r_{s_l}\widehat{\alpha_{s_{l-1}}}, r_{s_l}r_{s_{l-1}}\widehat{\alpha_{s_{l-2}}}, \ldots, r_{s_l}r_{s_{l-1}} \ldots r_{s_2}\widehat{\alpha_{s_1}}\} \tag{3.1}$$

and

$$N_2(w) = \{\widehat{\beta_{s_l}}, r_{s_l}\widehat{\beta_{s_{l-1}}}, r_{s_l}r_{s_{l-1}}\widehat{\beta_{s_{l-2}}}, \ldots, r_{s_l}r_{s_{l-1}} \cdots r_{s_2}\widehat{\beta_{s_1}}\}.$$

The following lemma is an adaptation of [13, 2.24] into the settings of the present paper.

Lemma 3.9. *W is finite if and only if $\widehat{\Phi_i(\mathscr{C})}$ is finite (for $i = 1, 2$).*

Proof. It is clear that the finiteness of W implies the finiteness of $\widehat{\Phi_i(\mathscr{C})}$ (for $i = 1, 2$). Conversely, for each $i \in \{1, 2\}$, assume that $|\widehat{\Phi_i(\mathscr{C})}| < \infty$, and define an equivalence relation $\approx_i$ on $\Phi_i(\mathscr{C})$ as follows: for $z_1, z_2 \in \Phi_i(\mathscr{C})$, write $z_1 \approx z_2$ if there is a positive λ such that $z_1 = \lambda z_2$. We write $\widetilde{z}$ for the equivalence class containing $z \in \Phi_i(\mathscr{C})$, and set $\widetilde{\Phi_i(\mathscr{C})} := \{\widetilde{z} \mid z \in \Phi_i(\mathscr{C})\}$. Observe that $|\widetilde{\Phi_i(\mathscr{C})}| = 2|\widehat{\Phi_i(\mathscr{C})}| < \infty$. The action of W on $\Phi_i(\mathscr{C})$ naturally induces a well-defined action of W on $\widetilde{\Phi_i(\mathscr{C})}$ satisfying $w\widetilde{z} := \widetilde{wz}$. Now for each $w \in W$ define a map $\sigma_w \colon \widetilde{\Phi_i(\mathscr{C})} \to \widetilde{\Phi_i(\mathscr{C})}$ by $\sigma_w(\widetilde{z}) := \widetilde{wz}$ for all $\widetilde{z} \in \widetilde{\Phi_i(\mathscr{C})}$.

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Then σ_w is a permutation of $\widetilde{\Phi_i(\mathscr{C})}$, and furthermore, $w \mapsto \sigma_w$ is a homomorphism $\sigma : W \to \mathrm{Sym}(\widetilde{\Phi_i(\mathscr{C})})$ (the symmetric group on $\widetilde{\Phi_i(\mathscr{C})}$). Now if w is in the kernel of σ then $w\widetilde{z} = \widetilde{z}$ for all $z \in \Phi_i(\mathscr{C})$, and in particular, $w\widetilde{z} = \widetilde{z}$ for all $z \in \Pi_i$. But by Lemma 3.2(ii) this means that $\ell(wr_s) > \ell(w)$ for all $s \in S$, and therefore $w = 1$. Thus σ is injective, and hence $|W| \leq |\mathrm{Sym}(\widetilde{\Phi_i(\mathscr{C})})| = |\widetilde{\Phi_i(\mathscr{C})}|! < \infty$, as required. □

Remark 3.10. For each $K \subseteq S$, let V_{1K} denote the real vector space spanned by $\Pi_{1K} = \{\alpha_s \mid s \in K\}$, and let V_{2K} denote the real vector space spanned by $\Pi_{2K} = \{\beta_s \mid s \in K\}$. Furthermore, let $\langle\, , \rangle_K$ denote the restriction of $\langle\, , \rangle$ to $V_{1K} \times V_{2K}$. Then clearly $\mathscr{C}_K = (K, V_{1K}, V_{2K}, \Pi_{1K}, \Pi_{2K}, \langle\, , \rangle_K)$ is a Coxeter datum with parameters $\{m_{st} \mid s, t \in K\}$. Next we write $W_K = \langle\{r'_s \mid s \in K\}\rangle$ for the corresponding abstract Coxeter group, and let $\eta\colon W_K \to W$ be the homomorphism defined by $r'_s \mapsto r_s$ for all $s \in K$. It follows immediately from the formulas for the actions of W on V_1 and W_K on V_{1K} that $r'_s v = r_s v$ for all $s \in K$ and $v \in V_{1K}$, and therefore $wv = \eta(w)v$ for all $w \in W_K$ and $v \in V_{1K}$. Since the action of W_K on V_{1K} is faithful, it follows that η is injective. Thus W_K can be identified with the *standard parabolic subgroup* of W generated by the set $\{r_s \mid s \in K\}$.

Remark 3.11. Given $K \subseteq S$, and the Coxeter datum $\mathscr{C}_K$ (as defined in Remark 3.10), it is a particular case of Lemma 3.9 that W_K is finite if and only if $\widehat{\Phi_i(\mathscr{C}_K)}$ is finite, for each $i \in \{1, 2\}$.

Remark 3.12. It follows easily from Lemma 3.2(ii) that if W_K is finite then there exists a (unique) longest element $w_K \in W_K$ satisfying the condition $\widehat{\Phi_1(\mathscr{C}_K)} \subseteq N_1(w_K)$. For present purposes we require this only in the special case where K has cardinality 2.

Notation 3.13. For $r, s \in S$, if $m_{rs} < \infty$ then $\langle\{r_r, r_s\}\rangle$ is finite, and its longest element, denoted by $w_{\{r,s\}}$, is $r_r r_s r_r \cdots = r_s r_r r_s \ldots$, where there are m_{rs} alternating factors on each side.

Lemma 3.14. *Suppose that $s, t \in S$ and $w \in W$ such that $w\alpha_s = \lambda\alpha_t$ for some positive constant λ. Let $r \in S$ be such that $\ell(wr_r) < \ell(w)$. Then*

(i) *$\langle\{r_r, r_s\}\rangle$ is finite;*
(ii) *$N_1(w_{\{r,s\}} r_s) = \widehat{\Phi_1(\mathscr{C}_{\{r,s\}})} \setminus \{\widehat{\alpha_s}\}$;*
(iii) *$\ell(wr_s w_{\{r,s\}}^{-1}) = \ell(w) - \ell(w_{\{r,s\}} r_s)$.*

Proof. (i) Since $\ell(wr_r) < \ell(w)$, Lemma 3.2(ii) yields that $w\alpha_r \in \Phi_1^-(\mathscr{C})$. Furthermore, $w\widehat{\alpha_r} \neq \widehat{\alpha_t}$, for otherwise $\widehat{\alpha_r} = \widehat{\alpha_s}$, contradicting Lemma 2.5. Observe that then

$$wr_s\alpha_r = w(\alpha_r - 2\langle\alpha_r, \beta_s\rangle\alpha_s) = \underbrace{w\alpha_r}_{\in \Phi_1^-(\mathscr{C}) \setminus \mathbb{R}\{\alpha_t\}} - \underbrace{2\lambda\langle\alpha_r, \beta_s\rangle\alpha_t}_{\text{a scalar multiple of } \alpha_t}.$$

Thus it can be checked that $wr_s\alpha_r \in \Phi_1^-(\mathscr{C})$. Now the fact that both $wr_s\alpha_r$ and $wr_s\alpha_s$ are negative implies that $wr_s(\lambda'\alpha_r + \mu'\alpha_s)$ is a negative linear combination of Π_1 whenever $\lambda', \mu' \geq 0$. This means that $\widehat{\Phi_1(\mathscr{C}_{\{r,s\}})} \subseteq N_1(wr_s)$. Since $N_1(wr_s)$ is a finite set of size $\ell(wr_s)$ by Lemma 3.8(ii), it follows from Remark 3.11 above that $\langle\{r_r, r_s\}\rangle$ must be finite.

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(ii) First, $w_{\{r,s\}}$ exists by part (i). Next let $\mu\alpha_r + \nu\alpha_s \in \Phi_1^+(\mathscr{C})$ (where $\mu, \nu \geq 0$) be arbitrary. By Remark 3.12, $w_{\{r,s\}}r_s(\mu\alpha_r + \nu\alpha_s) \in \Phi_1^+(\mathscr{C})$ if and only if $r_s(\mu\alpha_r + \nu\alpha_s) \in \Phi_1^-(\mathscr{C})$, which by Lemma 3.8(i) happens if and only if $\mu = 0$, and consequently $N_1(w_{\{r,s\}}r_s) = \Phi_1(\widehat{\mathscr{C}_{\{r,s\}}}) \setminus \{\widehat{\alpha_s}\}$, as required.

(iii) By Lemma 3.8(iv), to show that $w_{\{r,s\}}r_s$ is a right-hand segment of w, it is enough to show that $N_1(w_{\{r,s\}}r_s) \subseteq N_1(w)$. Suppose that $\alpha \in \Phi_1^+(\mathscr{C})$ such that $\widehat{\alpha} \in N_1(w_{\{r,s\}}r_s)$. By (ii) above we may write $\alpha = c_1\alpha_r + c_2\alpha_s$ for some $c_1 > 0$ and $c_2 \geq 0$. Then $w\alpha = c_1 w\alpha_r + c_2\lambda\alpha_t$. Suppose for a contradiction that $w\alpha \in \Phi_1^+(\mathscr{C})$. Then $c_2\lambda\alpha_t = w\alpha - c_1 w\alpha_r \in \mathrm{PLC}(\Pi_1)$. Rearranging this equation gives $\lambda'\alpha_t = \sum_{t' \in S\setminus\{t\}} \lambda_{t'}\alpha_{t'}$, where λ' is a constant and $\lambda_{t'} \geq 0$ for all $t' \in S \setminus \{t\}$. Now if $\lambda' > 0$ then we have a contradiction to Lemma 2.5; on the other hand, if $\lambda' \leq 0$ then we have a contradiction to $0 \notin \mathrm{PLC}(\Pi_1)$. □

We are now ready to prove Proposition 3.5.

Proof of Proposition 3.5. Since $\Phi_1(\mathscr{C}') = \{w\alpha'_s \mid w \in W \text{ and } s \in S\}$, to prove that the restriction of π_1 to $\Phi_1(\mathscr{C}')$ is bijective it suffices to show that if $\pi_1(w\alpha'_s) = \pi_1(v\alpha'_t)$, for some $w, v \in W$ and $s, t \in S$, then $w\alpha'_s = v\alpha'_t$. Observe that $\pi_1(w\alpha'_s) = w\alpha_s$, and $\pi_1(v\alpha'_t) = v\alpha_t$. Hence it suffices to prove the following statement: if $w\alpha_s = \lambda\alpha_t$, for some $w \in W$, $s, t \in S$, and $\lambda \neq 0$, then $w\alpha'_s = \lambda\alpha'_t$. We assume that $w\alpha_s = \lambda\alpha_t$, and proceed by an induction on $\ell(w)$. The case $\ell(w) = 0$ reduces to the statement: if $\alpha_s = \lambda\alpha_t$, for some $s, t \in S$, then $\alpha'_s = \lambda\alpha'_t$. Given $\alpha_s = \lambda\alpha_t$, Lemma 2.5 and the requirement that $0 \notin \mathrm{PLC}(\Pi_1)$ together yield that $\lambda = 1$ and $s = t$, and we are done. Thus we may assume that $\ell(w) > 0$, and choose $r \in S$ such that $\ell(wr_r) < \ell(w)$. Lemma 3.14 yields that $\langle\{r_r, r_s\}\rangle$ is a finite dihedral group (hence m_{rs} is finite), and $\ell(w(w_{\{r,s\}}r_s)^{-1}) = \ell(w) - \ell(w_{\{r,s\}}r_s)$. We treat separately the cases m_{rs} even and m_{rs} odd.

If $m_{rs} = 2k$ is even, then $w_{\{r,s\}} = (r_r r_s)^k = (r_s r_r)^k$, and the formulas in Lemma 2.6(ii) yield

$$\begin{aligned}(w_{\{r,s\}}r_s)\alpha_s &= -w_{\{r,s\}}\alpha_s = -(r_s r_r)^k\alpha_s \\ &= -\frac{\sin((m_{rs}+1)\pi/m_{rs})}{\sin(\pi/m_{rs})}\alpha_s - \frac{-\cos(\pi/m_{rs})}{\langle\alpha_r, \beta_s\rangle}\frac{\sin(\pi)}{\sin(\pi/m_{rs})}\alpha_r \\ &= \alpha_s;\end{aligned} \tag{3.2}$$

and by exactly the same calculation in V'_1 we have

$$(w_{\{r,s\}}r_s)\alpha'_s = \alpha'_s. \tag{3.3}$$

Observe that (3.2) yields that

$$\lambda\alpha_t = w\alpha_s = w(w_{\{r,s\}}r_s)^{-1}(w_{\{r,s\}}r_s)\alpha_s = w(w_{\{r,s\}}r_s)^{-1}\alpha_s. \tag{3.4}$$

Now since $\ell(w(w_{\{r,s\}}r_s)^{-1}) < \ell(w)$, the inductive hypothesis combined with (3.4) gives us

$$\lambda\alpha'_t = w(w_{\{r,s\}}r_s)^{-1}\alpha'_s. \tag{3.5}$$

Then it follows from (3.3) and (3.5) that

$$w\alpha'_s = (w(w_{\{r,s\}}r_s)^{-1}(w_{\{r,s\}}r_s))\alpha'_s = w(w_{\{r,s\}}r_s)^{-1}\alpha'_s = \lambda\alpha'_t,$$

and the desired result follows by induction.

Next if $m_{rs} = 2k + 1$ is odd, then

$$w_{\{r,s\}}r_s = \underbrace{\dots r_r r_s r_r}_{(m_{rs}-1)\text{ factors}} = (r_s r_r)^k.$$

Then the formulas in Lemma 2.6(ii) yield that

$$\begin{aligned}(w_{\{r,s\}}r_s)\alpha_s &= (r_s r_r)^k\alpha_s \\ &= \frac{\sin(\pi)}{\sin(\pi/m_{rs})}\alpha_s + \frac{-\cos(\pi/m_{rs})}{\langle\alpha_r,\beta_s\rangle}\frac{\sin(2k\pi/m_{rs})}{\sin(\pi/m_{rs})}\alpha_r \\ &= \frac{-\cos(\pi/m_{rs})}{\langle\alpha_r,\beta_s\rangle}\alpha_r;\end{aligned} \tag{3.6}$$

and by exactly the same calculation in V'_1 we have

$$(w_{\{r,s\}}r_s)\alpha'_s = \frac{-\cos(\pi/m_{rs})}{\langle\alpha_r,\beta_s\rangle}\alpha'_r. \tag{3.7}$$

Observe that (3.6) yields that

$$\lambda\alpha_t = w(w_{\{r,s\}}r_s)^{-1}(w_{\{r,s\}}r_s)\alpha_s = w(w_{\{r,s\}}r_s)^{-1}\Big(\frac{-\cos(\pi/m_{rs})}{\langle\alpha_r,\beta_s\rangle}\alpha_r\Big). \tag{3.8}$$

Since $\ell(w(w_{\{r,s\}}r_s)^{-1}) < \ell(w)$, (3.8) together with the inductive hypothesis yields that

$$\lambda\alpha'_t = w(w_{\{r,s\}}r_s)^{-1}\Big(\frac{-\cos(\pi/m_{rs})}{\langle\alpha_r,\beta_s\rangle}\alpha'_r\Big). \tag{3.9}$$

Finally, it follows from (3.7) and (3.9) that

$$\begin{aligned}w\alpha'_s = w(w_{\{r,s\}}r_s)^{-1}(w_{\{r,s\}}r_s)\alpha'_s &= w(w_{\{r,s\}}r_s)^{-1}\Big(\frac{-\cos(\pi/m_{rs})}{\langle\alpha_r,\beta_s\rangle}\alpha'_r\Big) \\ &= \lambda\alpha'_t,\end{aligned}$$

completing the proof that π_1 restricts to a bijection $\Phi_1(\mathscr{C}') \leftrightarrow \Phi_1(\mathscr{C})$. A similar reasoning also yields that π_2 restricts to a bijection $\Phi_2(\mathscr{C}') \leftrightarrow \Phi_2(\mathscr{C})$. □

Under the freeness condition of $\mathscr{C}'$, each root $\alpha' \in \Phi_1(\mathscr{C}')$ can be written uniquely in the form $\alpha' = \sum_{s\in S}\lambda_s\alpha'_s$; we say that λ_s is the *coefficient* of α'_s in α', and denote it by $\operatorname{coeff}_s(\alpha')$. Similarly each root $\beta' \in \Phi_2(\mathscr{C}')$ can be written uniquely as $\beta' = \sum_{s\in S}\mu_s\beta'_s$; we say that μ_s is the *coefficient* of β'_s in β', and denote it by $\operatorname{coeff}_s(\beta')$.

Definition 3.15.

(i) Let $\alpha \in \Phi_1(\mathscr{C})$ be arbitrary. For each $s \in S$, define the *canonical coefficient* of α_s in the root α, written $\operatorname{coeff}_s(\alpha)$, by requiring that $\operatorname{coeff}_s(\alpha) = \operatorname{coeff}_s(\pi_1^{-1}(\alpha))$, where π_1 is as in Proposition 3.5. The *support* of the root α, written $\operatorname{supp}(\alpha)$, is the set $\{\alpha_s \mid \operatorname{coeff}_s(\alpha) \neq 0\}$.

(ii) Let $\beta \in \Phi_2(\mathscr{C})$ be arbitrary. For each $s \in S$, define the *canonical coefficient* of β_s in the root β, written $\operatorname{coeff}_s(\beta)$, by requiring that $\operatorname{coeff}_s(\beta) = \operatorname{coeff}_s(\pi_2^{-1}(\beta))$, where π_2 is as in Proposition 3.5. The *support* of the root β, written $\operatorname{supp}(\beta)$, is the set $\{\beta_s \mid \operatorname{coeff}_s(\beta) \neq 0\}$.

Remark 3.16. By the above definition, the canonical coefficients arise from the free Coxeter datum $\mathscr{C}'$. Thus when dealing with canonical coefficients, we may, for simplicity and clarity, assume that the Coxeter datum concerned is free.

Proposition 3.17. *Suppose that $s, t \in S$. If $w\alpha_s = \lambda\alpha_t$ for some $w \in W$ and some nonzero constant λ, then $w\beta_s = (1/\lambda)\beta_t$.*

Proof. Since $w\alpha_s = \lambda\alpha_t$, it follows from Remark 3.3 and Lemma 3.8(i) that $w\beta_s = \mu\beta_t$ for some $\mu \neq 0$. Then Lemma 2.17 yields that

$$1 = \langle \alpha_s, \beta_s \rangle = \langle w\alpha_s, w\beta_s \rangle = \langle \lambda\alpha_t, \mu\beta_t \rangle = \lambda\mu,$$

hence $\mu = 1/\lambda$. □

Remark 3.18. Proposition 3.17 is a direct consequence of [13, 2.13(d)], and we thank the referee of the present paper for suggesting to us the above proof which replaces a much longer proof used in an earlier version of the present paper.

In particular, Proposition 3.17 implies the following: if $w_1\alpha_s = w_2\alpha_t$ for some $s, t \in S$ and $w_1, w_2 \in W$, then $w_1\beta_s = w_2\beta_t$. Thus there exists a well-defined map $\phi\colon \Phi_1(\mathscr{C}) \to \Phi_2(\mathscr{C})$ satisfying the requirement that $w\alpha_s \mapsto w\beta_s$ for all $s \in S$ and $w \in W$. This is clearly the unique W-equivariant map $\Phi_1(\mathscr{C}) \to \Phi_2(\mathscr{C})$ satisfying $\alpha_s \mapsto \beta_s$ for all $s \in S$. Similarly, there exists a unique W-equivariant map $\phi'\colon \Phi_2(\mathscr{C}) \to \Phi_1(\mathscr{C})$ satisfying $\beta_s \mapsto \alpha_s$ for all $s \in S$. It is clear that the products $\phi' \circ \phi$ and $\phi \circ \phi'$ are the corresponding identity maps on $\Phi_1(\mathscr{C})$ and $\Phi_2(\mathscr{C})$, respectively. Hence we have the following proposition.

Proposition 3.19. *There exists a unique W-equivariant bijection*

$$\phi\colon \Phi_1(\mathscr{C}) \to \Phi_2(\mathscr{C})$$

satisfying $\phi(\alpha_s) = \beta_s$ for all $s \in S$.

For the rest of this paper, we shall fix ϕ for the W-equivariant bijection in the above proposition. Observe that Remark 3.3 then immediately implies the following corollary.

Corollary 3.20. *Let $\alpha \in \Phi_1(\mathscr{C})$ be arbitrary. Then $\alpha \in \Phi_1^+(\mathscr{C})$ if and only if $\phi(\alpha) \in \Phi_2^+(\mathscr{C})$; and $\alpha \in \Phi_1^-(\mathscr{C})$ if and only if $\phi(\alpha) \in \Phi_2^-(\mathscr{C})$.*

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Note that Proposition 3.17 can be generalized as follows.

Lemma 3.21. *Suppose that* $\alpha \in \Phi_1(\mathscr{C})$, *and suppose that* λ *is a nonzero constant such that* $\lambda\alpha \in \Phi_1(\mathscr{C})$. *Then* $\phi(\lambda\alpha) = (1/\lambda)\phi(\alpha)$.

Proof. Write $\alpha = w\alpha_s$ for some $w \in W$ and $s \in S$. The fact that $\lambda\alpha \in \Phi_1(\mathscr{C})$ implies that $w^{-1}(\lambda\alpha) = \lambda w^{-1}\alpha = \lambda\alpha_s \in \Phi_1(\mathscr{C})$. Then $\phi(\lambda\alpha_s) = (1/\lambda)\phi(\alpha_s)$ by Proposition 3.17. Now it follows from the W-equivariance of ϕ that

$$\phi(\lambda\alpha) = \phi(\lambda w\alpha_s) = \phi(w\lambda\alpha_s) = w\phi(\lambda\alpha_s) = \frac{1}{\lambda}w\phi(\alpha_s) = \frac{1}{\lambda}\phi(\alpha). \qquad \square$$

Definition 3.22. Let $\mathscr{C} = (S, V_1, V_2, \Pi_1, \Pi_2, \langle\ ,\ \rangle)$ be a Coxeter datum.

(i) For each $i \in \{1, 2\}$ and $z \in \Phi_i^+(\mathscr{C})$, define the *depth* of z (written $\mathrm{dp}_{\mathscr{C},i}(z)$) to be

$$\mathrm{dp}_{\mathscr{C},i}(z) = \min\{\ell(w) \mid w \in W \text{ and } wz \in \Phi_i^-(\mathscr{C})\}.$$

(ii) For each $i \in \{1, 2\}$ and $z_1, z_2 \in \Phi_i^+(\mathscr{C})$, write $z_1 \preceq_i z_2$ if there exists $w \in W$ such that $z_2 = wz_1$ and $\mathrm{dp}_{\mathscr{C},i}(z_2) = \mathrm{dp}_{\mathscr{C},i}(z_1) + \ell(w)$. Furthermore, write $z_1 \prec_i z_2$ if $z_1 \preceq_i z_2$ but $z_1 \neq z_2$.

A mild generalization of [2, Lemma 1.7] yields the following lemma:

Lemma 3.23. *Suppose that* $\mathscr{C} = (S, V_1, V_2, \Pi_1, \Pi_2, \langle\ ,\ \rangle)$ *is a Coxeter datum. Let* $s \in S$, $\alpha \in \Phi_1^+(\mathscr{C}) \setminus \mathbb{R}\{\alpha_s\}$, *and* $\beta \in \Phi_2^+(\mathscr{C}) \setminus \mathbb{R}\{\beta_s\}$. *Then*

$$\mathrm{dp}_{\mathscr{C},1}(r_s\alpha) = \begin{cases} \mathrm{dp}_{\mathscr{C},1}(\alpha) - 1 & \text{if } \langle\alpha, \beta_s\rangle > 0, \\ \mathrm{dp}_{\mathscr{C},1}(\alpha) & \text{if } \langle\alpha, \beta_s\rangle = 0, \\ \mathrm{dp}_{\mathscr{C},1}(\alpha) + 1 & \text{if } \langle\alpha, \beta_s\rangle < 0; \end{cases}$$

and

$$\mathrm{dp}_{\mathscr{C},2}(r_s\beta) = \begin{cases} \mathrm{dp}_{\mathscr{C},2}(\beta) - 1 & \text{if } \langle\alpha_s, \beta\rangle > 0, \\ \mathrm{dp}_{\mathscr{C},2}(\beta) & \text{if } \langle\alpha_s, \beta\rangle = 0, \\ \mathrm{dp}_{\mathscr{C},2}(\beta) + 1 & \text{if } \langle\alpha_s, \beta\rangle < 0. \end{cases}$$

Lemma 3.24. *Suppose that* $\mathscr{C} = (S, V_1, V_2, \Pi_1, \Pi_2, \langle\ ,\ \rangle)$ *is a Coxeter datum, and* $\alpha \in \Phi_1^+(\mathscr{C})$. *Then* $\mathrm{dp}_{\mathscr{C},1}(\alpha) = \mathrm{dp}_{\mathscr{C},2}(\phi(\alpha))$.

Proof. Let $w \in W$ be such that $w\alpha \in \Phi_1^-(\mathscr{C})$ and $\mathrm{dp}_{\mathscr{C},1}(\alpha) = \ell(w)$. Corollary 3.20 then yields that $\phi(w\alpha) \in \Phi_2^-(\mathscr{C})$, and it follows from the W-equivariance of ϕ that $w(\phi(\alpha)) \in \Phi_2^-(\mathscr{C})$. Hence

$$\mathrm{dp}_{\mathscr{C},2}(\phi(\alpha)) \leq \ell(w) = \mathrm{dp}_{\mathscr{C},1}(\alpha).$$

By symmetry, $\mathrm{dp}_{\mathscr{C},1}(\alpha) \leq \mathrm{dp}_{\mathscr{C},2}(\phi(\alpha))$, whence equality. $\square$

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LEMMA 3.25. *Suppose that* $\mathscr{C} = (S, V_1, V_2, \Pi_1, \Pi_2, \langle\ ,\ \rangle)$ *is a Coxeter datum. Let* $\alpha \in \Phi_1(\mathscr{C})$ *and* $s \in S$. *Then*

$$\langle \alpha, \beta_s \rangle > 0 \quad \text{if and only if } \langle \alpha_s, \phi(\alpha) \rangle > 0,$$

and

$$\langle \alpha, \beta_s \rangle = 0 \quad \text{if and only if } \langle \alpha_s, \phi(\alpha) \rangle = 0,$$

and

$$\langle \alpha, \beta_s \rangle < 0 \quad \text{if and only if } \langle \alpha_s, \phi(\alpha) \rangle < 0.$$

PROOF. Combine Lemmas 3.23 and 3.24, then the desired result follows. □

In fact, the above gives rise to a more general result, stated in the following corollary.

COROLLARY 3.26. *Suppose that* $\mathscr{C} = (S, V_1, V_2, \Pi_1, \Pi_2, \langle\ ,\ \rangle)$ *is a Coxeter datum, and* $\alpha_1, \alpha_2 \in \Phi_1(\mathscr{C})$. *Then*

$$\langle \alpha_1, \phi(\alpha_2) \rangle > 0 \quad \text{if and only if } \langle \alpha_2, \phi(\alpha_1) \rangle > 0,$$

and

$$\langle \alpha_1, \phi(\alpha_2) \rangle = 0 \quad \text{if and only if } \langle \alpha_2, \phi(\alpha_1) \rangle = 0,$$

and

$$\langle \alpha_1, \phi(\alpha_2) \rangle < 0 \quad \text{if and only if } \langle \alpha_2, \phi(\alpha_1) \rangle < 0.$$

PROOF. Write $\alpha_2 = w\alpha_s$ for some $w \in W$ and $s \in S$. Given the W-invariance of $\langle\ ,\ \rangle$ and the W-equivariance of ϕ, we have

$$\langle \alpha_1, \phi(\alpha_2) \rangle = \langle \alpha_1, w\beta_s \rangle = \langle w^{-1}\alpha_1, \beta_s \rangle.$$

It then follows from Lemma 3.25 that $\langle w^{-1}\alpha_1, \beta_s \rangle > 0$ precisely when

$$\langle \alpha_s, \phi(w^{-1}\alpha_1) \rangle = \langle w\alpha_s, \phi(\alpha_1) \rangle = \langle \alpha_2, \phi(\alpha_1) \rangle > 0.$$

The rest of the desired result follows in a similar way. □

4. Comparison with the standard geometric realization of Coxeter groups

In this section we recover the root systems of Coxeter groups in the sense of [16] or [20] as special cases of root systems arising from a Coxeter datum. Subsequently we give comparison results between such special cases and the more general root systems arising from a Coxeter datum. These comparisons will provide a useful reduction of the nonorthogonal representations studied in the previous sections into those of [16, Ch. V] or [20, Sections 5.3–5.4].

Fix a Coxeter datum $\mathscr{C} = (S, V_1, V_2, \Pi_1, \Pi_2, \langle\ ,\ \rangle)$, and let V be a vector space over $\mathbb{R}$ with a basis $\Pi = \{\gamma_s \mid s \in S\}$ in bijective correspondence with S. Suppose that

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$(\,,\,)\colon V \times V \to \mathbb{R}$ is a symmetric bilinear form satisfying the following conditions:

(C1′) $(\gamma_s, \gamma_s) = 1$, for all $s \in S$;
(C2′) $(\gamma_s, \gamma_t) \leq 0$, for all distinct $s, t \in S$;
(C3′) $(\gamma_s, \gamma_t)^2 = \langle \alpha_s, \beta_t \rangle \langle \alpha_t, \beta_s \rangle$, for all $s, t \in S$.

Then $\mathscr{C}'' = (S, V, V, \Pi, \Pi, (\,,\,))$ is a free Coxeter datum with the same Coxeter parameters m_{st}, $(s, t \in S)$ as those of $\mathscr{C}$, and hence $\mathscr{C}$ and $\mathscr{C}''$ are associated to the same abstract Coxeter system (W, R). It is easily checked that $\Phi_1(\mathscr{C}'') = \Phi_2(\mathscr{C}'')$, allowing us to write Φ in place of $\Phi_1(\mathscr{C}'')$ and $\Phi_2(\mathscr{C}'')$. Furthermore, we write Φ^+ and Φ^- for the corresponding set of positive roots and negative roots, respectively. It is also readily checked that W can be faithfully embedded into the orthogonal group of the bilinear form $(\,,\,)$ on V. If $\mathscr{C}''$ satisfies the further condition that $(\gamma_s, \gamma_t) = 1$ whenever $m_{st} = \infty$, then $\mathscr{C}''$ gives rise to the orthogonal representation of W as defined in [1, Ch. V], [16] and [20, Sections 5.3–5.4]. We refer to such V as the *standard geometric realization (Tits representation)* of W or simply the *standard Tits representation* of W (as in [27]).

It follows from Lemma 2.17 that $(\,,\,)$ is W-invariant. It is well known (and can be readily checked) that if x and λx are both in Φ for some constant λ, then $\lambda = \pm 1$. Since $\mathscr{C}''$ is free, it follows that each $\gamma \in \Phi$ can be written uniquely in the form $\gamma = \sum_{s \in S} \lambda_s \gamma_s$. We say that λ_s is the *coefficient* of γ_s in γ, and denote it by $\mathrm{coeff}_s(\gamma)$. Finally, to simplify notation, for any $\gamma \in \Phi$ we write $\mathrm{dp}(\gamma)$ in place of $\mathrm{dp}_{\mathscr{C}'',1}(\gamma)(= \mathrm{dp}_{\mathscr{C}'',2}(\gamma))$, and for $\gamma_1, \gamma_2 \in \Phi$ we simply write $\gamma_1 \preceq \gamma_2$ when $\gamma_1 \preceq_1 \gamma_2$ (and $\gamma_1 \preceq_2 \gamma_2$).

Similar arguments as for Proposition 3.5 give us the following proposition.

Proposition 4.1. *There are W-equivariant maps $\phi_1\colon \Phi_1(\mathscr{C}) \to \Phi$, and $\phi_2\colon \Phi_2(\mathscr{C}) \to \Phi$, satisfying $\phi_1(\alpha_s) = \gamma_s = \phi_2(\beta_s)$, for all $s \in S$.*

Remark 4.2. We stress that, unlike π_1, π_2 of Proposition 3.5 and ϕ of Proposition 3.19, the new maps ϕ_1 and ϕ_2 are not injective in general, and we shall see more on this fact in Lemma 4.9(ii) below.

Lemma 4.3. *Suppose that $\alpha \in \Phi_1(\mathscr{C})$. Then $\alpha \in \Phi_1^+(\mathscr{C})$ implies that $\phi_1(\alpha) \in \Phi^+$, and $\alpha \in \Phi_1^-(\mathscr{C})$ implies that $\phi_1(\alpha) \in \Phi^-$.*

Proof. We may write $\alpha = w\alpha_r$, where $w \in W$ and $r \in S$. If $\alpha \in \Phi_1^+(\mathscr{C})$, then Lemma 3.2(ii) yields that $\ell(wr_r) = \ell(w) + 1$. Hence Lemma 3.2(ii) applied to the Coxeter datum $\mathscr{C}''$ yields that $\phi_1(\alpha) = \phi_1(w\alpha_r) = w\phi_1(\alpha_r) = w\gamma_r \in \Phi^+$. Likewise we see that $\alpha \in \Phi_1^-(\mathscr{C})$ implies that $\phi_1(\alpha) \in \Phi^-$. □

Using the same argument as in Lemma 3.24 we have the following lemma.

Lemma 4.4. *Suppose that $\alpha \in \Phi_1^+(\mathscr{C})$. Then $\mathrm{dp}_{\mathscr{C},1}(\alpha) = \mathrm{dp}(\phi_1(\alpha))$.*

Corollary 4.5. *Suppose that $\alpha \in \Phi_1(\mathscr{C})$, and $s \in S$. Then*

$$(\phi_1(\alpha), \gamma_s) > 0 \quad \textit{if and only if } \langle \alpha, \beta_s \rangle > 0,$$

and

$$(\phi_1(\alpha), \gamma_s) = 0 \quad \text{if and only if } \langle \alpha, \beta_s \rangle = 0,$$

and

$$(\phi_1(\alpha), \gamma_s) < 0 \quad \text{if and only if } \langle \alpha, \beta_s \rangle < 0.$$

PROOF. Follows from Lemmas 4.4 and 3.23 applied to $\mathscr{C}''$. □

Next we have a well-known result on Coxeter groups (a proof can be found in [4], in the discussion immediately before Lemma 2.1) which we shall use repeatedly in later calculations.

LEMMA 4.6. *Suppose that $I \subseteq S$ and $w \in W$. Let W_I denote the standard parabolic subgroup in W corresponding to I, as defined in Remark 3.10. Choose $w' \in wW_I$ to be of minimal length in the left coset of W_I in W containing w. Then $\ell(w'v) = \ell(w') + \ell(v)$ for all $v \in W_I$.*

PROPOSITION 4.7. *For each $\alpha \in \Phi_1(\mathscr{C})$ and each $r \in S$,*

$$\operatorname{coeff}_r(\alpha)\operatorname{coeff}_r(\phi(\alpha)) \geq (\operatorname{coeff}_r(\phi_1(\alpha)))^2.$$

PROOF. Replacing α by $-\alpha$ if needed, we may assume that $\alpha \in \Phi_1^+(\mathscr{C})$, and we may write $\alpha = w\alpha_s$, where $w \in W$ and $s \in S$ (so $\ell(wr_s) = \ell(w) + 1$). The proof is based on an induction on $\ell(w)$. If $\ell(w) = 0$ then the result clearly holds with equality. Thus we may assume that $\ell(w) \geq 1$, and choose $t \in S$ such that $\ell(wr_t) = \ell(w) - 1$ (in particular, $s \neq t$). Observe that $w = w_1w_2$, where w_1 is of minimal length in the coset $w\langle\{r_s, r_t\}\rangle$, and w_2 is an alternating product of r_s and r_t which cannot end in r_s. This implies, in particular, that $m_{st} \neq 2$. Furthermore, Lemma 4.6 yields that $\ell(w) = \ell(w_1) + \ell(w_2)$, $\ell(w_1r_s) = \ell(w_1) + 1$, and $\ell(w_1r_t) = \ell(w_1) + 1$. Consequently, $w_1\alpha_s \in \Phi_1^+(\mathscr{C})$ and $w_1\alpha_t \in \Phi_1^+(\mathscr{C})$ by Lemma 3.2(ii). The formulas in Lemma 2.6(ii) yield that $w_2\alpha_s = p\alpha_s + \lambda q\alpha_t$ and $w_2\beta_s = p\beta_s + (q/\lambda)\beta_t$, where λ is a positive constant and $pq \geq 0$. If p and q are both negative, then

$$\alpha = w\alpha_s = w_1w_2\alpha_s = w_1(p\alpha_s + \lambda q\alpha_t) = pw_1\alpha_s + \lambda qw_1\alpha_t \in \Phi_1^-(\mathscr{C}),$$

contradicting the assumption that $\alpha \in \Phi_1^+(\mathscr{C})$. Therefore $p, q \geq 0$. By Lemma 2.5, $\{\alpha_s, \alpha_t\}$ is linearly independent, hence $\operatorname{coeff}_s(w_2\alpha_s) = p$, and $\operatorname{coeff}_t(w_2\alpha_s) = \lambda q$. Lemma 2.6(ii) applied to the Coxeter datum $\mathscr{C}''$ yields that $w_2\gamma_s = p\gamma_s + q\gamma_t$. And Lemma 2.5 implies that $\operatorname{coeff}_s(w_2\beta_s) = p$ and $\operatorname{coeff}_t(w_2\beta_s) = q/\lambda$. Next we set

$$x = \operatorname{coeff}_r(\alpha), \quad x' = \operatorname{coeff}_r(\phi(\alpha)), \quad x'' = \operatorname{coeff}_r(\phi_1(\alpha));$$

and

$$y = \operatorname{coeff}_r(w_1\alpha_s), \quad y' = \operatorname{coeff}_r(w_1\beta_s), \quad y'' = \operatorname{coeff}_r(w_1\gamma_s);$$

and

$$z = \operatorname{coeff}_r(w_1\alpha_t), \quad z' = \operatorname{coeff}_r(w_1\beta_t), \quad z'' = \operatorname{coeff}_r(w_1\gamma_t).$$

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Then

$$xx' - x''^2 = (py + \lambda qz)\Big(py' + \frac{1}{\lambda}qz'\Big) - (py'' + qz'')^2$$
$$= p^2(yy' - y''^2) + q^2(zz' - z''^2) + pq\Big(\frac{1}{\lambda}yz' + \lambda zy' - 2y''z''\Big). \qquad (4.1)$$

Since $\ell(w_1) < \ell(w)$, it follows from the inductive hypothesis that

$$yy' \geq (y'')^2 \quad \text{and} \quad zz' \geq (z'')^2,$$

and so the first two summands in the last line of (4.1) are nonnegative. Next apply the geometric mean and arithmetic mean inequality to the terms $(1/\lambda)yz'$ and $\lambda y'z$, and we can conclude that $(1/\lambda)yz' + \lambda y'z \geq 2\sqrt{yy'zz'}$. But the inductive hypothesis yields that $yy'zz' \geq (y''z'')^2$, showing that the third summand in the last line of (4.1) is also nonnegative, whence $xx' - x''^2 \geq 0$, and the desired result follows by induction. □

Remark 4.8. In the case where $|S| = 2$ the inequality in Proposition 4.7 is in fact an equality. Then the group element w_1 in the proof of Proposition 4.7 is the identity, and hence $\alpha = w\alpha_s = w_2\alpha_s = p\alpha_s + \lambda q\alpha_t$, $\phi(\alpha) = w\beta_s = w_2\beta_s = p\beta_s + (q/\lambda)\beta_t$, and $\phi_1(\alpha) = w\gamma_s = w_2\gamma_s = p\gamma_s + q\gamma_t$, for the same constants p, q and λ as in the proof of Proposition 4.7. Consequently, $\mathrm{coeff}_r(\alpha)\,\mathrm{coeff}_r(\phi(\alpha)) = \mathrm{coeff}_r(\phi_1(\alpha))^2$, for each $r \in \{s, t\}$. We thank the referee for pointing this out to us.

Lemma 4.9. *Suppose that* $\alpha \in \Phi_1(\mathscr{C})$.

(i) *Let* $t \in S$. *Then*

$$\begin{cases} \mathrm{coeff}_t(\alpha) = 0 & \textit{if and only if } \mathrm{coeff}_t(\phi_1(\alpha)) = 0, \\ \mathrm{coeff}_t(\alpha) > 0 & \textit{if and only if } \mathrm{coeff}_t(\phi_1(\alpha)) > 0, \\ \mathrm{coeff}_t(\alpha) < 0 & \textit{if and only if } \mathrm{coeff}_t(\phi_1(\alpha)) < 0. \end{cases}$$

(ii) *The situation* $\phi_1(\alpha) = \pm\gamma_s$ $(s \in S)$ *arises if and only if* $\alpha = \lambda\alpha_s$, *for some nonzero constant* λ.

Proof. (i) Recalling Remark 3.16, we may assume that $\mathscr{C}$ is free. Under the freeness of $\mathscr{C}$, Lemmas 3.2(i) and 4.3 together imply that to prove the desired result we only need to show that $\mathrm{coeff}_t(\alpha) = 0$ if and only if $\mathrm{coeff}_t(\phi_1(\alpha)) = 0$. Since $\mathscr{C}$ is free, it follows from [13, (2.15)(b)] that $\mathrm{coeff}_t(\alpha) = 0$ if and only if $w\alpha = \lambda\alpha_s$ for some $w \in W_{S\setminus\{t\}}$, $s \in S \setminus \{t\}$, and $\lambda \neq 0$. Observe that this happens if and only if precisely one of $r_s w\alpha$ and $w\alpha$ is positive, and this, in view of Lemma 4.3, happens if and only if precisely one of $r_s w\phi_1(\alpha)$ and $w\phi_1(\alpha)$ is positive. The last statement happens if and only if $w\phi_1(\alpha) = \pm\gamma_s$, which happens if and only if $\mathrm{coeff}_t(\phi_1(\alpha)) = 0$.

(ii) Follows readily from part (i) above. □

Remark 4.10. Suppose that $\beta \in \Phi_2(\mathscr{C})$, and $t \in S$. Then the same arguments as those used in the proof of the above lemma yield that

$$\begin{cases} \mathrm{coeff}_t(\beta) = 0 & \text{if and only if } \mathrm{coeff}_t(\phi_2(\beta)) = 0, \\ \mathrm{coeff}_t(\beta) > 0 & \text{if and only if } \mathrm{coeff}_t(\phi_2(\beta)) > 0, \\ \mathrm{coeff}_t(\beta) < 0 & \text{if and only if } \mathrm{coeff}_t(\phi_2(\beta)) < 0. \end{cases}$$

The next result is taken from [3].

Lemma 4.11 (Brink [3, Proposition 2.1]). *Suppose that* $\gamma \in \Phi$ *and* $r \in S$. *Then* $\operatorname{coeff}_r(\gamma) > 0$ *implies that* $\operatorname{coeff}_r(\gamma) \geq 1$.

Further, suppose that $0 < \operatorname{coeff}_r(\gamma) < 2$. *Then either* $\operatorname{coeff}_r(\gamma) = 1$ *or* $\operatorname{coeff}_r(\gamma) = 2\cos(\pi/m_{r_1 r_2})$, *where* r_1, $r_2 \in S$ *with* $4 \leq m_{r_1 r_2} < \infty$. □

Combining the results in Proposition 4.7 to Lemma 4.11 (inclusive) and an argument similar to the one used in the proof of Proposition 4.7, we may deduce the following proposition.

Proposition 4.12. *Let* $\alpha \in \Phi_1(\mathscr{C})$ *and* $t \in S$ *be arbitrary. Then*

$$\operatorname{coeff}_t(\alpha) > 0 \quad \textit{if and only if } \operatorname{coeff}_t(\phi(\alpha)) > 0,$$

and in this case $\operatorname{coeff}_t(\alpha)\operatorname{coeff}_t(\phi(\alpha)) \geq 1$. *In particular,*

$$\phi(\operatorname{supp}(\alpha)) = \operatorname{supp}(\phi(\alpha)).$$

Furthermore, suppose that $1 \leq \operatorname{coeff}_t(\alpha)\operatorname{coeff}_t(\phi(\alpha)) < 4$. *Then either*

$$\operatorname{coeff}_t(\alpha)\operatorname{coeff}_t(\phi(\alpha)) = 1, \quad \textit{or else } \operatorname{coeff}_t(\alpha)\operatorname{coeff}_t(\phi(\alpha)) = 4\cos^2\Big(\frac{\pi}{m}\Big),$$

where $m = m_{r_1 r_2}$, *for some* $r_1, r_2 \in S$ *with* $4 \leq m < \infty$. *In particular, in the case* $\alpha \in \Phi_1^+(\mathscr{C})$ *we have* $\operatorname{coeff}_t(\alpha)\operatorname{coeff}_t(\phi(\alpha)) = 1$ *if and only if* $\operatorname{coeff}_t(\phi_1(\alpha)) = 1$. □

Proposition 4.13. *Suppose that* $\alpha_1, \alpha_2 \in \Phi_1(\mathscr{C})$. *Then*

$$\langle \alpha_1, \phi(\alpha_2)\rangle\langle \alpha_2, \phi(\alpha_1)\rangle \geq (\phi_1(\alpha_1), \phi_1(\alpha_2))^2.$$

Proof. Since both $\langle\, ,\rangle$ and $(\, ,)$ are W-invariant, and ϕ, ϕ_1 are W-equivariant, we may replace α_1 and α_2 by $u\alpha_1$ and $u\alpha_2$ for a suitable $u \in W$ such that $\alpha_2 \in \Pi_1$. Furthermore, replacing α_1 by $-\alpha_1$ if need be, we may assume that $\alpha_1 \in \Phi_1^+(\mathscr{C})$. Therefore it is enough to prove that $\langle \alpha_1, \beta_s\rangle\langle \alpha_s, \phi(\alpha_1)\rangle \geq (\phi_1(\alpha_1), \gamma_s)^2$, for all $\alpha_1 \in \Phi_1^+(\mathscr{C})$ and $s \in S$. We proceed with an induction on the depth of α_1.

If $\operatorname{dp}_{\mathscr{C},1}(\alpha_1) = 1$, then $\alpha_1 = \lambda\alpha_r$, for some positive constant λ and some $r \in S$. It follows from Lemmas 3.21 and 4.9(ii) that $\phi(\alpha_1) = (1/\lambda)\beta_r$ and $\phi_1(\alpha_1) = \gamma_r$. Therefore, for all $s \in S$,

$$\begin{aligned}\langle \alpha_1, \beta_s)\rangle\langle \alpha_s, \phi(\alpha_1)\rangle &= \lambda\langle \alpha_r, \beta_s\rangle\frac{1}{\lambda}\langle \alpha_s, \beta_r\rangle \\ &= \langle \alpha_r, \beta_s\rangle\langle \alpha_s, \beta_r\rangle = (\gamma_r, \gamma_s)^2 = (\phi_1(\alpha_1), \gamma_s)^2.\end{aligned}$$

Thus we may assume that $\operatorname{dp}_{\mathscr{C},1}(\alpha_1) > 1$. First, if $s \in S$ such that $\langle \alpha_1, \beta_s\rangle > 0$ then Lemma 3.23 yields that $r_s\alpha_1 \prec_1 \alpha_1$, and hence

$$\begin{aligned}\langle \alpha_1, \beta_s\rangle\langle \alpha_s, \phi(\alpha_1)\rangle &= \langle r_s\alpha_1, r_s\beta_s\rangle\langle r_s\alpha_s, r_s\phi(\alpha_1)\rangle \\ &= (-\langle r_s\alpha_1, \beta_s\rangle)(-\langle \alpha_s, \phi(r_s\alpha_1)\rangle) \\ &\geq (\phi_1(r_s\alpha_1), \gamma_s)^2 \quad \text{(by the inductive hypothesis)} \\ &= (\phi_1(\alpha_1), -\gamma_s)^2 \\ &= (\phi_1(\alpha_1), \gamma_s)^2,\end{aligned}$$

as required.

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Next we consider those $s \in S$ with $\langle \alpha_1, \beta_s \rangle \leq 0$. Fix an arbitrary such s, and let $t \in S$ be such that $r_t \alpha_1 \prec_1 \alpha_1$. Then Lemma 3.23 yields that $\langle \alpha_1, \beta_t \rangle > 0$, and, in particular, $s \neq t$. Then we have the following two possibilities to consider:

(A1) $\alpha_1 \in \mathbb{R}\alpha_s + \mathbb{R}\alpha_t$;
(A2) $\alpha_1 \notin \mathbb{R}\alpha_s + \mathbb{R}\alpha_t$.

In case (A1), we first observe that $m_{st} \neq 2$, for otherwise $\langle \alpha_1, \beta_s \rangle \leq 0$ and $\langle \alpha_1, \beta_t \rangle > 0$ together imply that $\alpha_1 = \lambda \alpha_t$ for some $\lambda > 0$, contradicting the assumption that $\mathrm{dp}_{\mathscr{C},1}(\alpha_1) > 1$. Then either $\alpha_1 = (r_t r_s)^n \alpha_t$ or else $\alpha_1 = r_t (r_s r_t)^n \alpha_s$ for some $n < m_{st}/2$. Then it follows from Lemma 2.6(ii) that either

$$\alpha_1 = p_n^{ts} \alpha_t + \lambda_{ts} q_n^{ts} \alpha_s \quad \text{and} \quad \phi(\alpha_1) = p_n^{ts} \beta_t + \frac{1}{\lambda_{ts}} q_n^{ts} \beta_s, \tag{4.2}$$

or else

$$\alpha_1 = p_n^{st} \alpha_s + \lambda_{st} q_{n+1}^{st} \alpha_t \quad \text{and} \quad \phi(\alpha_1) = p_n^{st} \beta_s + \frac{1}{\lambda_{st}} q_{n+1}^{st} \beta_t. \tag{4.3}$$

Furthermore, Lemma 2.6(ii) applied to $\mathscr{C}''$ yields that either

$$\phi_1(\alpha_1) = p_n^{ts} \gamma_t + q_n^{ts} \gamma_s \quad \text{(if (4.2) is the case)}$$

or else

$$\phi_1(\alpha_1) = p_n^{st} \gamma_s + q_{n+1}^{st} \gamma_t \quad \text{(if (4.3) is the case).}$$

If (4.2) is the case then

$$\begin{aligned}
\langle \alpha_1, \beta_s \rangle \langle \alpha_s, \phi(\alpha_1) \rangle &= \langle p_n^{ts} \alpha_t + \lambda_{ts} q_n^{ts} \alpha_s, \beta_s \rangle \Big\langle \alpha_s, p_n^{ts} \beta_t + \frac{1}{\lambda_{ts}} q_n^{ts} \beta_s \Big\rangle \\
&= (p_n^{ts})^2 \langle \alpha_s, \beta_t \rangle \langle \alpha_t, \beta_s \rangle + \frac{1}{\lambda_{ts}} p_n^{ts} q_n^{ts} \langle \alpha_t, \beta_s \rangle \\
&\quad + \lambda_{ts} p_n^{ts} q_n^{ts} \langle \alpha_s, \beta_t \rangle + (q_n^{ts})^2 \\
&= (p_n^{ts})^2 \langle \alpha_s, \beta_t \rangle \langle \alpha_t, \beta_s \rangle - 2 p_n^{ts} q_n^{ts} \sqrt{\langle \alpha_s, \beta_t \rangle \langle \alpha_t, \beta_s \rangle} + (q_n^{ts})^2 \\
&= (p_n^{ts})^2 (\gamma_s, \gamma_t)^2 + 2 p_n^{ts} q_n^{ts} (\gamma_s, \gamma_t) + (q_n^{ts})^2 \\
&= (\phi_1(\alpha_1), \gamma_s)^2,
\end{aligned}$$

where the second last equality follows from the fact that $\lambda_{ts} = \sqrt{\langle \alpha_s, \beta_t \rangle \langle \alpha_t, \beta_s \rangle} / - \langle \alpha_s, \beta_t \rangle$ (see Lemma 2.6(ii)). On the other hand, if (4.3) is the case then

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$$\begin{aligned}\langle \alpha_1, \beta_s\rangle\langle \alpha_s, \phi(\alpha_1)\rangle &= \langle p_n^{st}\alpha_s + \lambda_{st}q_{n+1}^{st}\alpha_t, \beta_s\rangle\Big\langle \alpha_s, p_n^{st}\beta_s + \frac{1}{\lambda_{st}}q_{n+1}^{st}\beta_t\Big\rangle\\
&= (p_n^{st})^2 + \frac{1}{\lambda_{st}}p_n^{st}q_{n+1}^{st}\langle \alpha_s, \beta_t\rangle + \lambda_{st}p_n^{st}q_{n+1}^{st}\langle \alpha_t, \beta_s\rangle\\
&\quad + (q_{n+1}^{st})^2\langle \alpha_t, \beta_s\rangle\langle \alpha_s, \beta_t\rangle\\
&= (p_n^{st})^2 - 2p_n^{st}q_{n+1}^{st}\sqrt{\langle \alpha_s, \beta_t\rangle\langle \alpha_t, \beta_s\rangle}\\
&\quad + (q_{n+1}^{st})^2\langle \alpha_s, \beta_t\rangle\langle \alpha_t, \beta_s\rangle\\
&= (p_n^{st})^2 + 2p_n^{st}q_{n+1}^{st}(\gamma_s, \gamma_t) + (q_{n+1}^{st})^2(\gamma_s, \gamma_t)^2\\
&= (\phi_1(\alpha_1), \gamma_s))^2,\end{aligned}$$

where the second last equality follows from the fact that $\lambda_{st} = \sqrt{\langle \alpha_s, \beta_t\rangle\langle \alpha_t, \beta_s\rangle}/-\langle \alpha_t, \beta_s\rangle$ (see Lemma 2.6(ii)), completing the proof in the case of (A1).

Finally, let us consider case (A2). We first observe that (3.1) yields that $w\alpha_1 \in \Phi_1^+(\mathscr{C})$ for all w in the dihedral reflection subgroup generated by r_s and r_t. Set $\alpha_{1,1} = r_t\alpha_1$ and $w_1 = r_t$. If $\langle \alpha_{1,1}, \beta_s\rangle \leq 0$ then stop; otherwise set $\alpha_{1,2} = r_sr_t\alpha_1$ and $w_2 = r_sr_t$. If $\langle \alpha_{1,2}, \beta_t\rangle \leq 0$ then stop; otherwise set $\alpha_{1,3} = r_tr_sr_t\alpha_1$ and $w_3 = r_tr_sr_t$, and so on. Then

$$\cdots \prec_1 \alpha_{1,3} \prec_1 \alpha_{1,2} \prec_1 \alpha_{1,1} \prec_1 \alpha_1.$$

Since each $\alpha_{1,i} \in \Phi_1^+(\mathscr{C})$, it follows that the sequence $\alpha_{1,1}, \alpha_{1,2}, \alpha_{1,3}, \ldots$ must terminate at some $\alpha_{1,n}$ where $n < \mathrm{dp}_{\mathscr{C},1}(\alpha_1)$. Observe that this sequence terminates at $\alpha_{1,n}$ only if both

$$\langle w_n\alpha_1, \beta_s\rangle = \langle \alpha_{1,n}, \beta_s\rangle \leq 0 \tag{4.4}$$

and

$$\langle w_n\alpha_1, \beta_t\rangle = \langle \alpha_{1,n}, \beta_t\rangle \leq 0. \tag{4.5}$$

In particular, $w_n \neq 1$. Since $\mathrm{dp}_{\mathscr{C},1}(w_n\alpha_1) < \mathrm{dp}_{\mathscr{C},1}(\alpha_1)$, the inductive hypothesis yields that

$$\langle w_n\alpha_1, \beta_s\rangle\langle \alpha_s, \phi(w_n\alpha_1)\rangle \geq (\gamma_s, \phi_1(w_n\alpha_1))^2 \tag{4.6}$$

and

$$\langle w_n\alpha_1, \beta_t\rangle\langle \alpha_t, \phi(w_n\alpha_1)\rangle \geq (\gamma_t, \phi_1(w_n\alpha_1))^2. \tag{4.7}$$

For any reduced expression $w_n = r_{s_1}\ldots r_{s_n}$ ($s_i \in S$), it is readily checked that $r_{s_n}\alpha_1 \prec_1 \alpha_1$, and so it follows from Lemma 3.23 that $\langle \alpha_1, \beta_{s_n}\rangle > 0$. Therefore $s_n \neq s$, and w_n has no reduced expression ending in r_s. That is, w_n is a product of r_s and r_t with strictly fewer than m_{st} factors. Thus by applying Lemma 2.6(ii) to $\mathscr{C}''$ and $\mathscr{C}$, we may deduce that

$$w_n\gamma_s = p\gamma_s + q\gamma_t, \quad w_n\alpha_s = p\alpha_s + \lambda q\alpha_t \quad \text{and} \quad w_n\beta_s = p\beta_s + \frac{q}{\lambda}\beta_t,$$

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for some nonnegative constants p, q and positive constant λ. Thus

$$\begin{aligned}
&\langle \alpha_1, \beta_s\rangle\langle \alpha_s, \phi(\alpha_1)\rangle - (\gamma_s, \phi_1(\alpha_1))^2\\
&\quad = \langle w_n\alpha_1, w_n\beta_s\rangle\langle w_n\alpha_s, \phi(w_n\alpha_1)\rangle - (w_n\gamma_s, \phi_1(w_n\alpha_1))^2\\
&\qquad \text{(since } \langle\ ,\ \rangle \text{ and } (\ ,\ ) \text{ are } W\text{-invariant, and } \phi \text{ and } \phi_i \text{ are } W\text{-equivariant)}\\
&\quad = \Big\langle w_n\alpha_1, p\beta_s + \frac{q}{\lambda}\beta_t\Big\rangle\langle p\alpha_s + \lambda q\alpha_t, \phi(w_n\alpha_1)\rangle - (\phi_1(w_n\alpha_1), p\gamma_s + q\gamma_t)^2\\
&\quad = \underbrace{p^2(\langle w_n\alpha_1, \beta_s\rangle\langle \alpha_s, \phi(w_n\alpha_1)\rangle - (\phi_1(w_n\alpha_1), \gamma_s)^2)}_{\text{A}}\\
&\qquad + \underbrace{q^2(\langle w_n\alpha_1, \beta_t\rangle\langle \alpha_t, \phi(w_n\alpha_1)\rangle - (\phi_1(w_n\alpha_1), \gamma_t)^2)}_{\text{B}} + C,
\end{aligned}$$

where

$$\begin{aligned}
C = pq\Big(&\frac{1}{\lambda}\langle w_n\alpha_1, \beta_t\rangle\langle \alpha_s, \phi(w_n\alpha_1)\rangle + \lambda\langle w_n\alpha_1, \beta_s\rangle\langle \alpha_t, \phi(w_n\alpha_1)\rangle\\
&- 2(\phi_1(w_n\alpha_1), \gamma_s)(\phi_1(w_n\alpha_1), \gamma_t)\Big).
\end{aligned}$$

It follows from (4.6) and (4.7) that A and B are both nonnegative. It follows from (4.4) and (4.5) that

$$\begin{aligned}
&\frac{1}{\lambda}\langle w_n\alpha_1, \beta_t\rangle\langle \alpha_s, \phi(w_n\alpha_1)\rangle + \lambda\langle w_n\alpha_1, \beta_s\rangle\langle \alpha_t, \phi(w_n\alpha_1)\rangle\\
&\qquad \geq 2\sqrt{\langle w_n\alpha_1, \beta_s\rangle\langle \alpha_s, \phi(w_n\alpha_1)\rangle\langle w_n\alpha_1, \beta_t\rangle\langle \alpha_t, \phi(w_n\alpha_1)\rangle}\\
&\qquad \geq 2(\phi_1(w_n\alpha_1), \gamma_s)(\phi_1(w_n\alpha_1), \gamma_t) \quad \text{(by (4.6) and (4.7))},
\end{aligned}$$

that is, $C \geq 0$ as well. Therefore $\langle \alpha_1, \beta_s\rangle\langle \alpha_s, \phi(\alpha_1)\rangle \geq (\gamma_s, \phi_1(\alpha_1))^2$, and the desired result follows by induction. □

REMARK 4.14. The calculations establishing case (A1) in the proof of Proposition 4.13 can also show that the inequality in Proposition 4.13 is in fact an equality if $\alpha_1, \alpha_2 \in \mathbb{R}\alpha_s + \mathbb{R}\alpha_t$, for some $s, t \in S$. We are very grateful to the referee for pointing this out to us.

5. Tits cones and a nonpositivity result

Let $\mathscr{C} = (S, V_1, V_2, \Pi_1, \Pi_2, \langle\ ,\ \rangle)$ be a fixed Coxeter datum, and let (W, R) be the corresponding Coxeter system. In this section we study a class of cones associated to $\mathscr{C}$ that are analogous to the *Tits cones* in the classical setting (as defined in [26] or [20, Section 5.13]). Furthermore, we investigate certain W-invariant sets in V_1 and V_2 that are closely related to these cones. The key result of this section is a generalization to [24, Proposition 1.2] and [19, Proposition 3.4]. For this section we impose one additional condition on $\mathscr{C}$, namely,

(C6) $V_1 = \operatorname{span}(\Pi_1)$ and $V_2 = \operatorname{span}(\Pi_2)$.

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We retain all other conventions and notation of earlier sections.

NOTATION 5.1. For each $i \in \{1, 2\}$ and $I \subseteq S$, recall the notation of Π_{iI} and V_{iI} introduced in Remark 3.10, and set $P_{iI} = \mathrm{PLC}(\Pi_{iI}) \cup \{0\}$. When $I = S$ we write P_i in place of P_{iS}.

For each $i \in \{1, 2\}$ and subset I of S recall the notation W_I (introduced in Remark 3.10) of the standard parabolic subgroup of W corresponding to I; it is clear that W_I acts (faithfully) on V_{iI}. This allows us to specify a W_I-action on $\mathrm{Hom}(V_{iI}, \mathbb{R})$ as follows: if $w \in W_I$ and $g \in \mathrm{Hom}(V_{iI}, \mathbb{R})$ then $wg \in \mathrm{Hom}(V_{iI}, \mathbb{R})$ is given by $(wg)v = g(w^{-1}v)$ for all $v \in V_{iI}$. Naturally, when $I = S$ the Coxeter group W acts on $\mathrm{Hom}(V_i, \mathbb{R})$ in a similar way.

NOTATION 5.2. For each $i \in \{1, 2\}$ and subset I of S we set

$$P^{\#}_{iI} = \{f \in \mathrm{Hom}(V_{iI}, \mathbb{R}) \mid f(x) \geq 0 \text{ for all } x \in P_{iI}\}$$

and

$$P^{\#\#}_{iI} = \{x \in V_{iI} \mid f(x) \geq 0 \text{ for all } f \in P^{\#}_{iI}\}.$$

Moreover, we set $U_{iI} = \bigcup_{w \in W_I} wP^{\#}_{iI}$, and write

$$U^{\#}_{iI} = \{x \in V_{iI} \mid f(x) \geq 0 \text{ for all } f \in U_{iI}\}.$$

When $I = S$ we write $P^{\#}_i$, $P^{\#\#}_i$, U_i and $U^{\#}_i$ in place of $P^{\#}_{iS}$, $P^{\#\#}_{iS}$, U_{iS} and $U^{\#}_{iS}$, respectively.

We call a (convex) subset of a real vector space a *cone* if it is closed under addition and multiplication by positive scalars. It is clear that P_{iI}, $P^{\#}_{iI}$ and $P^{\#\#}_{iI}$ $(i = 1, 2)$ are cones for each $I \subseteq S$.

LEMMA 5.3. *For $i \in \{1, 2\}$ and each $f \in \mathrm{Hom}(V_i, \mathbb{R})$, set*

$$\mathrm{Neg}(f) = \{\widehat{x} \in \widehat{\Phi_i(\mathscr{C})} \mid x \in \Phi^+_i(\mathscr{C}) \text{ and } f(x) < 0\}.$$

Then $U_i = \{f \in \mathrm{Hom}(V_i, \mathbb{R}) \mid |\mathrm{Neg}(f)| < \infty\}$.

PROOF. It is enough to prove the U_1 case. Let $f \in U_1$ be arbitrary. Then $f = wg$ for some $w \in W$ and $g \in P^{\#}_1$. Let $x \in \Phi^+_1(\mathscr{C})$ be such that $f(x) < 0$. Then $(wg)x = g(w^{-1}x) < 0$. Since $g \in P^{\#}_1$, it follows that $w^{-1}x \in \Phi^-_1(\mathscr{C})$, whence $\widehat{x} \in N_1(w^{-1})$. Because $|N_1(w^{-1})| < \infty$, therefore $U_1 \subseteq \{f \in \mathrm{Hom}(V_1, \mathbb{R}) \mid |\mathrm{Neg}(f)| < \infty\}$. Conversely, suppose that $f \in \mathrm{Hom}(V_1, \mathbb{R})$ with $|\mathrm{Neg}(f)| < \infty$. If $|\mathrm{Neg}(f)| = 0$ then $f \in P^{\#}_1 \subseteq U_1$. Thus we may assume that $|\mathrm{Neg}(f)| > 0$, and proceed with an induction. Observe that if $|\mathrm{Neg}(f)| > 0$ then there exists some $s \in S$ such that $f(\alpha_s) < 0$. It is then readily checked that $|\mathrm{Neg}(r_s f)| = |\mathrm{Neg}(f)| - 1$, and hence it follows from our inductive hypothesis that $r_s f \in U_1$. Since U_1 is clearly W-invariant, it follows that $f \in U_1$, and therefore $\{f \in \mathrm{Hom}(V_1, \mathbb{R}) \mid |\mathrm{Neg}(f)| < \infty\} \subseteq U_1$, as required. □

The above lemma yields that U_1 and U_2 are cones. In fact, for each $I \subseteq S$ we can show that U_{1I} and U_{2I} are cones. These cones generalize the notion of the Tits cones as defined, for example, in [20, Section 5.13].

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DEFINITION 5.4. We call U_1 and U_2 the *Tits cones* of the Coxeter datum $\mathscr{C}$; and for each $I \subseteq S$ we call U_{1I} and U_{2I} the Tits cones of the Coxeter datum $\mathscr{C}_I = (I, V_{1I}, V_{2I}, \Pi_{1I}, \Pi_{2I}, \langle\,,\rangle_I)$ (where $\langle\,,\rangle_I$ is the restriction of $\langle\,,\rangle$ to $V_{1I} \times V_{2I}$).

For $i \in \{1, 2\}$, observe that:

$$\begin{aligned} U_i^{\#} &= \{v \in V_i \mid (wf)v \geq 0 \text{ for all } f \in P_i^{\#} \text{ and } w \in W\} \\ &= \bigcap_{w \in W} \{v \in V_i \mid f(w^{-1}v) \geq 0 \text{ for all } f \in P_i^{\#}\} \\ &= \bigcap_{w \in W} \{wv \in V_i \mid f(v) \geq 0 \text{ for all } f \in P_i^{\#}\} \\ &= \bigcap_{w \in W} wP_i^{\#\#}, \end{aligned}$$

and similarly,

$$U_{iI}^{\#} = \bigcap_{w \in W_I} wP_{iI}^{\#\#} \quad \text{for each } I \subseteq S. \tag{5.1}$$

The following is a well-known result (a proof can be found in [18, notes (b) and (c), page 4]).

LEMMA 5.5. *Let I be a finite subset of S. For $i \in \{1, 2\}$, the condition $0 \notin \mathrm{PLC}(\Pi_{iI})$ is equivalent to the existence of an $f_i \in \mathrm{Hom}(V_{iI}, \mathbb{R})$ such that $f_i(x) > 0$ for all $x \in \Pi_{iI}$. Furthermore, $P_{iI}^{\#\#} = P_{iI}$.*

LEMMA 5.6. *For $i \in \{1, 2\}$, if Π_i is linearly independent then*

$$P_i^{\#\#} \cap V_{iI} \subseteq P_{iI}^{\#\#},$$

for each subset I of S.

PROOF. We prove that $P_1^{\#\#} \cap V_{1I} \subseteq P_{1I}^{\#\#}$, and comment that the other case follows analogously.

Since Π_1 is a basis for V_1, it follows that for $I \subseteq S$, every $f \in P_{1I}^{\#}$ can be extended to an $f' \in P_1^{\#}$ by setting $f'(\alpha_s) = f(\alpha_s)$ for all $s \in I$ and $f'(\alpha_s) = 0$ for all $s \in S \setminus I$. Let $x \in P_1^{\#\#} \cap V_{1I}$ be arbitrary. Then for each $f \in P_{1I}^{\#}$, we have $f(x) = f'(x) \geq 0$ (since $f' \in P_1^{\#}$, and $x \in P_1^{\#\#}$), and consequently $x \in P_{1I}^{\#\#}$. □

Let $x_i \in V_i$ ($i = 1, 2$) be arbitrary. Given condition (C6) of $\mathscr{C}$, it follows that $x_i \in V_{iI}$ for some finite subset I of S. From this fact we may prove the following lemma.

LEMMA 5.7. *For $i \in \{1, 2\}$, suppose that Π_i is linearly independent and $x_i \in U_i^{\#}$. Let I be a subset of S such that $x_i \in V_{iI}$. Then $x_i \in U_{iI}^{\#}$. Furthermore, if I is finite then $x_i \in P_{iI}$.*

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Proof. Observe that for $i \in \{1,2\}$,

$$\begin{aligned} x_i \in U_i^{\#} \cap V_{iI} &\subseteq \bigcap_{w\in W_I} (wP_i^{\#\#} \cap V_{iI}) \\ &= \bigcap_{w\in W_I} w\,(P_i^{\#\#} \cap V_{iI}) \quad \text{(since } W_I \text{ preserves } V_{iI}\text{)} \\ &\subseteq \bigcap_{w\in W_I} wP_{iI}^{\#\#} = U_{iI}^{\#} \quad \text{(by Lemma 5.6).} \end{aligned}$$

Finally, it follows from Lemma 5.5 that $x_i \in \bigcap_{w\in W_I} wP_{iI} \subseteq P_{iI}$ whenever I is finite. □

We are now ready for the central result of this section:

Theorem 5.8. *If Π_1 and Π_2 are linearly independent, then $\langle v_1, v_2\rangle \le 0$, for all $v_1 \in U_1^{\#}$ and $v_2 \in U_2^{\#}$.*

Proof. Suppose for a contradiction that there are $v_1 \in U_1^{\#}$ and $v_2 \in U_2^{\#}$ with $\langle v_1, v_2\rangle > 0$. Replacing v_2 by a positive scalar multiple of itself if necessary, we may assume that $\langle v_1, v_2\rangle = 1$. Let I be a finite subset of S with $v_1 \in V_{1I}$, and let J be a finite subset of S with $v_2 \in V_{2J}$. Next, set $K = I \cup J$. Then Lemma 5.7 yields that $v_1 \in U_{1K}^{\#}$ and $v_2 \in U_{2K}^{\#}$. Since K is finite, it follows from Lemma 5.5 that there are linear functionals $f_1 \in \mathrm{Hom}(V_{1K}, \mathbb{R})$ and $f_2 \in \mathrm{Hom}(V_{2K}, \mathbb{R})$ such that $f_1(\alpha) > 1$ for all $\alpha \in \Pi_{1K}$, and $f_2(\beta) > 1$ for all $\beta \in \Pi_{2K}$. Now set

$$\begin{aligned} \mathscr{A} = \{x \in U_{2K}^{\#} \mid\ & f_2(x) \le f_2(v_2) \text{ and } \langle z, x\rangle \ge 1 \\ & \text{for some } z \in U_{1K}^{\#} \text{ with } f_1(z) \le f_1(v_1)\}. \end{aligned}$$

Observe that $\mathscr{A} \ne \emptyset$, since $v_2 \in \mathscr{A}$.

Next, put $\epsilon = 2/(|K| f_1(v_1))$. We claim that for any given $x \in \mathscr{A}$, there exists $y \in \mathscr{A}$ with $f_2(y) \le f_2(x) - \epsilon$. If we can prove this claim, then starting with $x = v_2$, a finite repetition of this process will produce some $y \in \mathscr{A} \subseteq U_{2K}^{\#} \subseteq P_{2K}$ with $f_2(y)$ being negative, contradicting the fact that $f_2(y) \in f_2(P_{2K}) \subseteq (0, \infty)$. Thus all that remains to do is to prove the above claim. Given arbitrary $x \in \mathscr{A}$, let $z = \sum_{\alpha\in\Pi_{1K}} \lambda_\alpha \alpha \in U_{1K}^{\#}$ be such that $\langle z, x\rangle \ge 1$, $f_1(z) \le f_1(v_1)$, and $\lambda_\alpha \ge 0$ for all $\alpha \in \Pi_{1K}$. Note that these conditions imply that $\sum_{\alpha\in\Pi_{1K}} \lambda_\alpha \langle \alpha, x\rangle \ge 1$, which in turn implies that $\lambda_{\alpha_{s_0}} \langle \alpha_{s_0}, x\rangle \ge 1/|K|$ for some $s_0 \in K$. Then

$$\langle \alpha_{s_0}, x\rangle \ge \frac{1}{\lambda_{\alpha_{s_0}} |K|} \ge \frac{1}{f_1(v_1)|K|} = \frac{\epsilon}{2},$$

since $\lambda_{\alpha_{s_0}} \le f_1(z) \le f_1(v_1)$. Set $y = r_{s_0} x$. Observe that (5.1) indicates that $U_{2K}^{\#}$ is W_K-invariant, and given that $r_{s_0} \in W_K$, we have $y \in U_{2K}^{\#}$. Moreover,

$$f_2(y) = f_2(x) - 2\langle \alpha_{s_0}, x\rangle f_2(\beta_{s_0}) < f_2(x) - \epsilon < f_2(v_2).$$

Thus to establish our claim, we only need to show that $y \in \mathscr{A}$, and this in turn amounts to finding some $t \in U_{1K}^{\#}$ such that $\langle t, y\rangle \ge 1$, and $f_1(t) \le f_1(v_1)$. First, suppose

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that $\langle z, \beta_{s_0}\rangle \geq 0$. Put $t = r_{s_0} z$. Since $U^{\#}_{1K}$ is W_K-invariant and $z \in U^{\#}_{1K}$, it follows that $t \in U^{\#}_{1K}$. Moreover,

$$\langle t, y\rangle = \langle r_{s_0} z, r_{s_0} x\rangle = \langle z, x\rangle \geq 1$$

and

$$f_1(t) = f_1(z) - 2\langle z, \beta_{s_0}\rangle f_1(\alpha_{s_0}) \leq f_1(z) \leq f_1(v_1),$$

thus proving $y \in \mathscr{A}$ in the case $\langle z, \beta_{s_0}\rangle \geq 0$. Next, suppose that $\langle z, \beta_{s_0}\rangle < 0$. Then $t = z$ will suffice; indeed,

$$\begin{aligned}\langle z, y\rangle &= \langle z, x - 2\langle \alpha_{s_0}, x\rangle \beta_{s_0}\rangle \\ &= \langle z, x\rangle - 2\langle \alpha_{s_0}, x\rangle\langle z, \beta_{s_0}\rangle \\ &\geq \langle z, x\rangle \\ &\geq 1,\end{aligned}$$

and by our construction, $z \in U^{\#}_{1K}$ and $f_1(z) \leq f_1(v_1)$, thus proving $y \in \mathscr{A}$ in the case $\langle z, \beta_{s_0}\rangle < 0$ too. This completes the proof of the claim, and hence the theorem follows. □

Acknowledgements

A few results presented in this paper are taken from the author's PhD thesis [10] and the author wishes to thank Professor R. B. Howlett for all his help and encouragement throughout the author's PhD candidature. The author also wishes to thank Professor G. I. Lehrer and Professor R. Zhang for supporting this work. Due gratitude must also be paid to Professor W. A. Casselman and Professor George Willis for their helpful comments and suggestions. We sincerely thank the referee of this paper for correcting a number of mistakes contained in an earlier version. We also thank the referee for communicating the manuscripts [14] and [15] to us. Further, we thank the referee for valuable suggestions which have greatly improved both the mathematical arguments and the clarity of exposition for many results in this paper, most notably, Lemma 2.6 (together with its applications later in the paper) and Propositions 2.9, 3.17, 4.7 and 4.13.

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