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MAPS ON D_1 AND D_2 SPACES

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Abstract

A space X is said to be D_1 provided each closed set has a countable basis for the open sets containing it. It is said to be D_2 provided there is a countable base $\{U_n\}$ such that each closed set has a countable base for the open sets containing it, which is a subfamily of $\{U_n\}$. In this paper, we give a separation theorem for D_1 spaces, and provide a characterization of D_1 and D_2 spaces in terms of maps.

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Introduction

Aull introduced the D_1 and D_2 spaces in [1]. In this paper, we give a separation theorem for D_1 spaces, and provide a characterization of D_1 and D_2 spaces in terms of maps. Some of these results have appeared in [3], namely Theorems 1 and 6; we include them here for completeness.

DEFINITION 1. A topological space X is said to be D_1 provided each closed set has a countable basis for the open sets containing it.

DEFINITION 2. A topological space X is said to be D_2 provided there is a countable base $\{U_n\}$ such that each closed set has a countable base for the open sets containing it which is a subfamily of $\{U_n\}$.

DEFINITION 3. A map is a continuous function.

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LEMMA 1. *Let X be D_1, T_1. Then X is first countable.*

THEOREM 1. *Let X be D_1, T_2. Then X is regular.*

This is a result in [3].

In [1], Aull presents a number of results about D_1, T_3 spaces. These can be strengthened slightly by replacing T_3 with T_2. Moreover, this is about the best separation result possible in this direction. If the D_1 property in Theorem 1 is replaced by first countability, it is well-known that the result is false. Let X be a countable set with the finite complement topology. Then X is D_1, T_1 (in fact, D_2), yet fails to be regular.

THEOREM 2. *Let X be a D_1 space. Let f be an open map from X onto a space Y. Then Y is a D_1 space.*

PROOF. Let A be a closed subset of Y. Then $f^{-1}[A]$ is closed in X, hence has a countable basis for its neighborhood system. Let $\{V_i \mid i \in \mathbf{Z}^+\}$ be such a basis. For each integer i, $f[V_i]$ is an open subset of Y, and $f[V_i] \supset A$. Further, if W is open in Y, and $W \supset A$, then $f^{-1}[W]$ is open in X and contains $f^{-1}[A]$, whence there is an integer i such that $f^{-1}[W] \supset V_i \supset f^{-1}[A]$. Then $W \supset f[V_i] \supset A$. This suffices to show that $\{f[V_i] \mid i \in \mathbf{Z}^+\}$ is a basis for the neighborhood system of A.

THEOREM 3. *Let X be a D_2-space. Let f be an open map from X onto a space Y. Then Y is a D_2 space.*

This is an easy modification of Theorem 2.

THEOREM 4. *Let X be a D_1 space. Let f be a closed map from X onto a space Y. Then Y is a D_1 space.*

PROOF. Let A be a closed subset of Y. Then $f^{-1}[A]$ is a closed subset of X, hence has a countable basis for its neighborhood system. Let $\{V_i \mid i \in \mathbf{Z}^+\}$ be such a basis. Then $\{Y - f[X - V_i] \mid i \in \mathbf{Z}^+\}$ is a collection of open subsets of Y. We show that this collection is a basis for the neighborhood system of A. Suppose W is open in Y, and $W \supset A$. Then $f^{-1}[W] \supset f^{-1}[A]$, and $f^{-1}[W]$ is open in X, whence there is an integer i such that $f^{-1}[A] \subset V_i \subset f^{-1}[W]$. But this implies that $A \subset Y - f[X - V_i] \subset W$, so Y is seen to be a D_1 space.

THEOREM 5. *Let X be a D_2 space. Let f be a closed map from X onto a space Y. Then Y is a D_2 space.*

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This is an easy modification of Theorem 4.

THEOREM 6. *Let X be a T_1 space. Then X is a D_1 space if and only if each closed continuous image of X is first countable.*

This is a result of [3].

A natural conjecture is that if X is T_1, then X is D_2 if and only if each closed continuous image of X is second countable. Only half this proposition is true. If X is D_2, T_1, then by Theorem 5, each closed continuous image of X is D_2 and T_1, hence second countable. To see that the converse is not true, let X be a countably infinite set with the discrete topology. X is easily seen to be D_1 and T_1 hence each closed continuous image of X is D_1 and T_1, hence first countable. Further, each closed map defined on X has countable range. Thus, any closed continuous image of X is countable and first countable, hence second countable. But X is not D_2, for as Aull shows in Theorem 13, of [1], D_2 metric space are compact.

There is a result in this direction however.

THEOREM 7. *Let X be a T_2 space with at most finitely many isolated points. Suppose that each closed continuous image of X is second countable. Then X is D_2.*

PROOF. If each closed continuous image of X is second countable, then in particular, each is first countable, so that X is D_1. The D_1, T_2 properties imply that X is regular. Further, the identity map is closed and continuous, X itself is second countable, hence metrizable. Now Corollary 12, of [1], shows that the D_1 property characterizes the metric spaces which are the union of a compact set and isolated points. Since X has at most finitely many isolated points, X is compact metric, hence by Theorem 13, of [1], must be a D_2 space.

Product spaces

The first countability property is productive up to a countable number of non-trivial factor spaces. A natural conjecture is that the D_1 property, an analogue of first countability, is also productive. This is false, even for two factor spaces. Let $\mathbf{Z}^+$ be the positive integers with the discrete topology and let I be the interval $[-1, 1]$ with the usual topology. Both of these spaces are D_1, but their product is not. to see this, note first that $A = \{(n, 0) \mid n \in \mathbf{Z}+\}$ is a closed subset of $\mathbf{Z}^+ \times I$. Suppose $\{V_n \mid n \in \mathbf{Z}^+\}$ is a countable collection of open sets containing A. For each integer n, $V_n \cap (\{n\} \times I)$ is an open subset of $\{n\} \times I$, and

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contains the point $(n, 0)$. Thus, for each integer n, there is a set W_n, open in $\{n\} \times I$, which contains $(n, 0)$ and which is properly contained in $V_n \cap (\{n\} \times I)$. But then $W = \bigcup_{n=1}^{\infty} W_n$ is an open subset of $\mathbf{Z}^+ \times I$ and contains A. Further, W was constructed in such a fashion that it contains no set V_n. Thus, no countable collection of open sets can be a basis for A, whence $\mathbf{Z}^+ \times I$ is not a D_1 space.

It would be interesting to know if the presence of isolated points in one of the factor spaces is what causes the product to fail to have the D_1 property. Specifically: if X, Y are D_1 spaces, each with at most a finite number of isolated points, is the product a D_1 space?

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