Canad. J. Math. Vol. **61** (2), 2009 pp. 241–263

Operator Integrals, Spectral Shift, and Spectral Flow

N. A. Azamov, A. L. Carey, P. G. Dodds, and F. A. Sukochev

Abstract. We present a new and simple approach to the theory of multiple operator integrals that applies to unbounded operators affiliated with general von Neumann algebras. For semifinite von Neumann algebras we give applications to the Fréchet differentiation of operator functions that sharpen existing results, and establish the Birman–Solomyak representation of the spectral shift function of M. G. Krein in terms of an average of spectral measures in the type II setting. We also exhibit a surprising connection between the spectral shift function and spectral flow.

1 Introduction

In the seminal paper of Yu. L. Daletskii and S. G. Krein [16], the theory of multiple operator integrals emerged as an important tool in the differentiation theory of operator functions and in perturbation theory. On the other hand, an important concept in the theory of perturbations is the spectral shift function, which first arose in the work of I. M. Lifshits [28] in solid state theory and was put on a firm mathematical basis by M. G. Krein [25].

An important connection between these two theories was made by M. Sh. Birman and M. Z. Solomyak [4], who showed that the theory of double operator integrals led naturally to a new representation for the spectral shift function as an average of spectral measures.

A principal aim of this paper is to present a new approach to the theory of multiple operator integrals, which provides a coherent path to the theory of differentiation of operator functions, the spectral shift function, and the theory of spectral flow, in the setting of type II von Neumann algebras. Our approach is conceptually simpler than those of [29, 30], although it applies to a somewhat narrower class of functions. On the other hand, since our approach does not depend on the vector-valued integration theory against a finitely additive measure as in [5, 29, 30], it is also suitable for general (non-semifinite) von Neumann algebras (see Section 4). Our approach to the theory of multiple operator integrals is quite different from earlier approaches to be found in [16, 31, 39, 40].

During the final stages in the preparation of this paper, the authors became aware of a preprint by V. V. Peller [32], where a similar approach to the theory of multiple operator integrals is presented in the setting of type I von Neumann algebras. While Peller's approach in the type I case applies to the class of integral projective tensor products, the present paper in the more general type II setting restricts attention to

Received by the editors April 13, 2006.
Research of Carey, Dodds, and Sukochev partially supported by the Australian Research Council.
AMS subject classification: Primary 47A56; secondary 47B49, 47A55, 46L51.

https://doi.org/10.4153/CJM-2009-012-0 Published online by Cambridge University Press

the natural Wiener classes and thus permits us to show Fréchet differentiability rather than Gateaux differentiability. In particular, we strengthen the differentiation results of [29] by showing that Gateaux differentiability can be replaced by Fréchet differentiability, and we show the existence of higher order Fréchet derivatives of operator-valued functions.

Our present approach also allows us to consider perturbations of self-adjoint operators which are affiliated with semifinite von Neumann algebras. This is a substantial difference with [29, 30], which treated the more special case of (so-called) τ-measurable operators [21]. The necessity of avoiding the latter restriction is especially clear in the applications. An important ingredient is the recent extension to the type II setting of the Krein spectral shift function [2]. When combined with our development of multiple operator integration together with the ideas of [4], we establish a (type II) extension of the important Birman–Solomyak formula, concerning spectral averaging (see Section 6).

Perhaps the most surprising connection in our study is with the theory of spectral flow, which is presented in Section 7. While the theory of spectral shift function (see the lectures [26] and the survey [3]) is a part of operator theory, the theory of spectral flow, which originated in the work of M. Atiyah, V. Patodi, and I. M. Singer [1] on a generalization of the Atiyah–Singer index theorem, finds its proper analytic setting in the framework of non-commutative geometry created by A. Connes [14]. One of the main results of the latter theory is the odd local index theorem of A. Connes and H. Moscovici [15] which has recently been developed in the type II setting [11, 12]. This latter work inspired our result (in Section 7) that the theory of the spectral shift function and that of spectral flow coincide in the case of trace class perturbations.

2 Notations and Preliminaries

We denote by $\mathcal{H}$ a separable complex Hilbert space, by $\mathcal{N}$ a von Neumann algebra acting on $\mathcal{H}$, by $\mathcal{B}(\mathcal{H})$ the algebra of all bounded linear operators acting on $\mathcal{H}$, and by Tr the standard trace on $\mathcal{H}$. In the case where $\mathcal{N}$ is semifinite, we denote by τ a faithful normal semifinite trace on $\mathcal{N}$, by $L^1(\mathcal{N},\tau)$ the set of τ-trace class operators affiliated with $\mathcal{N}$, by $\mathcal{L}^1(\mathcal{N},\tau) = L^1(\mathcal{N},\tau) \cap \mathcal{N}$ the set of all bounded τ-trace class operators, and by $\mathcal{K}(\mathcal{N},\tau)$ the set of all τ-compact operators (see [21]) from $\mathcal{N}$. If S is a measure space, we denote by $B(S)$ the set of all bounded measurable complex-valued functions on S. The *so*-topology and *so**-topology denote, respectively, the strong operator topology and the strong* operator topology. We denote the uniform norm on $\mathcal{B}(\mathcal{H})$ by $\|\cdot\|$.

If $\mathcal{E}$ is a $*$-ideal in a von Neumann algebra $\mathcal{N}$ which is complete in some norm $\|\cdot\|_{\mathcal{E}}$, then we will call $\mathcal{E}$ an invariant operator ideal (see [13, Definition 1.8]) if

(1) $\|S\|_{\mathcal{E}} \geqslant \|S\|$ for all $S \in \mathcal{E}$,
(2) $\|S^*\|_{\mathcal{E}} = \|S\|_{\mathcal{E}}$ for all $S \in \mathcal{E}$,
(3) $\|ASB\|_{\mathcal{E}} \leqslant \|A\|\,\|S\|_{\mathcal{E}}\,\|B\|$ for all $S \in \mathcal{E}$ and $A, B \in \mathcal{N}$.

We say that an operator ideal $\mathcal{E}$ has property (F) if, for all nets $\{A_\alpha\} \subset \mathcal{E}$ such that there exist $A \in \mathcal{N}$ for which $A_\alpha \to A$ in the *so**-topology and $\|A_\alpha\|_{\mathcal{E}} \leqslant 1$ for all α, it follows that $A \in \mathcal{E}$ and $\|A\|_{\mathcal{E}} \leqslant 1$.

https://doi.org/10.4153/CJM-2009-012-0 Published online by Cambridge University Press

If $\mathcal{E} = \mathcal{N} \cap E(\mathcal{N}, \tau)$ for some rearrangement invariant Banach function space E (see [19]) with the Fatou property (that is, if $0 \leqslant f_\alpha \uparrow$ is an increasing net in E, $\sup_\alpha \|f_\alpha\|_E < \infty$ then $\sup_\alpha f_\alpha$ exists in E and $\|f_\alpha\|_E \uparrow \|f\|_E$), then [19, Proposition 1.6] together with Lemma 2.5 below show that $\mathcal{E}$ has the property (F).

Every von Neumann algebra with the uniform norm is an invariant operator ideal with property (F). If $\mathcal{N}$ is a semifinite von Neumann algebra with a faithful normal semifinite trace τ, then the spaces $\mathcal{L}^p(\mathcal{N}, \tau)$, $\mathcal{L}^{p,+\infty}(\mathcal{N}, \tau)$ are invariant operator ideals with the property (F) (see [18, 27]).

For any C^1-function $f\colon \mathbb{R} \to \mathbb{C}$, we denote by $f^{[1]}$ the continuous function

$$f^{[1]}(\lambda_0, \lambda_1) = \frac{f(\lambda_1) - f(\lambda_0)}{\lambda_1 - \lambda_0},$$

and for any C^{n+1}-function $f\colon \mathbb{R} \to \mathbb{C}$,

$$f^{[n+1]}(\lambda_0, \dots, \lambda_{n+1}) = \frac{f^{[n]}(\lambda_0, \dots, \lambda_{n-1}, \lambda_{n+1}) - f^{[n]}(\lambda_0, \dots, \lambda_{n-1}, \lambda_n)}{\lambda_{n+1} - \lambda_n}.$$

It is well known that $f^{[n]}$ is a symmetric function.

We denote by $W_n(\mathbb{R})$ the set of functions $f \in C^n(\mathbb{R})$, such that the j-th derivative $f^{(j)}$, $j = 0, \dots, n$, is the Fourier transform of a finite measure m_f on $\mathbb{R}$.

The next lemma introduces a finite measure space that will be used in our definition of multiple operator integrals in Section 4 below.

Lemma 2.1 *If*

$$\Pi^{(n)} = \{(s_0, s_1, \dots, s_n) \in \mathbb{R}^{n+1} \colon\ |s_n| \leqslant \cdots \leqslant |s_1| \leqslant |s_0|, \qquad \operatorname{sign}(s_0) = \cdots = \operatorname{sign}(s_n)\},$$

and if $f \in W_n(\mathbb{R})$, $\nu_f^{(n)}(s_0, \dots, s_n) = \frac{i^n}{\sqrt{2\pi}}\, m_f(ds_0) \cdots ds_n$, then $(\Pi^{(n)}, \nu_f^{(n)})$ is a finite measure space.

Proof The total variation of the measure $\nu_f^{(n)}$ on the set $\Pi^{(n)}$ (up to a constant) is equal to

$$\begin{aligned}\int_{\Pi^{(n)}} \left|m_f(ds_0)\right|\, ds_1 \cdots ds_n &= \int_{\mathbb{R}} \Delta_{s_0} \left|m_f(ds_0)\right| \\ &= \frac{1}{n!} \int_{\mathbb{R}} s_0^n \left|m_f(ds_0)\right| \\ &= \frac{1}{n!} \int_{\mathbb{R}} \left|m_{f^{(n)}}(ds_0)\right| = \frac{1}{n!} \left\|m_{f^{(n)}}\right\|,\end{aligned}$$

where Δ_{s_0} is the volume of the n-dimensional simplex of size s_0. ∎

For simplicity we write $\Pi = \Pi^{(1)}$ and $\nu_f = \nu_f^{(1)}$.

The next two lemmas provide concrete representations (see Section 4) for divided differences $f^{[n]}$ of functions belonging to the class $W_n(\mathbb{R})$ (see Section 4).

https://doi.org/10.4153/CJM-2009-012-0 Published online by Cambridge University Press

Lemma 2.2 *If $f \in W_1(\mathbb{R})$, then*

$$f^{[1]}(\lambda_0,\lambda_1) = \iint_{\Pi} \alpha_0(\lambda_0,\sigma)\alpha_1(\lambda_1,\sigma)\,d\nu_f(\sigma),$$

where $\sigma = (s_0, s_1)$, $\alpha_0(\lambda_0,\sigma) = e^{i(s_0-s_1)\lambda_0}$, $\alpha_1(\lambda_1,\sigma) = e^{is_1\lambda_1}$, $s_0, s_1 \in \mathbb{R}$.

Proof We have

$$\begin{aligned}
&\iint_{\Pi} \alpha_0(\lambda_0,\sigma)\alpha_1(\lambda_1,\sigma)\,d\nu_f(\sigma)\\
&\qquad = \frac{i}{\sqrt{2\pi}} \int_{\mathbb{R}} m_f(ds_0) \int_0^{s_0} e^{is_0\lambda_0 - is_1\lambda_0 + is_1\lambda_1}\,ds_1\\
&\qquad = \frac{1}{(\lambda_0-\lambda_1)\sqrt{2\pi}} \int_{\mathbb{R}} m_f(ds_0)(e^{is_0\lambda_0} - e^{is_0\lambda_1})\\
&\qquad = \frac{1}{\lambda_0-\lambda_1}(f(\lambda_0) - f(\lambda_1)) = f^{[1]}(\lambda_0,\lambda_1),
\end{aligned}$$

where the repeated integral can be replaced by the double integral by Fubini's theorem and Lemma 2.1. ■

Lemma 2.3 *If $f \in W_n(\mathbb{R})$, then, for all $\lambda_0,\dots,\lambda_n \in \mathbb{R}$,*

$$f^{[n]}(\lambda_0,\dots,\lambda_n) = \int_{\Pi^{(n)}} e^{i((s_0-s_1)\lambda_0+\cdots+(s_{n-1}-s_n)\lambda_{n-1}+s_n\lambda_n)}\,d\nu_f^{(n)}(s_0,\dots,s_n).$$

Proof By Lemma 2.2 and induction, we have

$$\begin{aligned}
&\int_{\Pi^{(n+1)}} e^{i((s_0-s_1)\lambda_0+\cdots+(s_n-s_{n+1})\lambda_n+s_{n+1}\lambda_{n+1})}\,d\nu_f^{(n+1)}(s_0,\dots,s_{n+1})\\
&\quad = \int_{\Pi^{(n)}} e^{i((s_0-s_1)\lambda_0+\cdots+s_n\lambda_n)} \left(\int_0^{s_n} ie^{is_{n+1}(\lambda_{n+1}-\lambda_n)}\,ds_{n+1}\right) d\nu_f^{(n)}(s_0,\dots,s_n)\\
&\quad = \frac{1}{\lambda_{n+1}-\lambda_n} \int_{\Pi^{(n)}} e^{i((s_0-s_1)\lambda_0+\cdots+s_n\lambda_n)} \left(e^{is_n(\lambda_{n+1}-\lambda_n)} - 1\right) d\nu_f^{(n)}(s_0,\dots,s_n)\\
&\quad = \frac{1}{\lambda_{n+1}-\lambda_n} \left(f^{[n]}(\lambda_0,\dots,\lambda_{n-1},\lambda_{n+1}) - f^{[n]}(\lambda_0,\dots,\lambda_{n-1},\lambda_n)\right)\\
&\quad = f^{[n+1]}(\lambda_0,\dots,\lambda_{n+1}).
\end{aligned}$$ ■

Lemma 2.4 *If $f \in W_{n+1}(\mathbb{R})$, then, for all $\lambda_0,\dots,\lambda_{n+1} \in \mathbb{R}$,*

$$\begin{aligned}
f^{[n+1]}(\lambda_0,\dots,\lambda_{n+1}) = i \int_{\Pi^{(n)}} \int_0^{s_j-s_{j+1}} & e^{i((s_0-s_1)\lambda_0+\cdots+u\lambda_{n+1}+(s_j-s_{j+1}-u)\lambda_j+\cdots+s_n\lambda_n}\\
&\times\ du\,d\nu_f^{(n)}(s_0,\dots,s_n).
\end{aligned}$$

Proof The right-hand side is equal to

$$i \int_{\Pi^{(n)}} e^{i((s_0-s_1)\lambda_0+\cdots+(s_j-s_{j+1})\lambda_j+\cdots+s_n\lambda_n)} \int_0^{s_j-s_{j+1}} e^{iu(\lambda_{n+1}-\lambda_j)} \times du\, d\nu_f^{(n)}(s_0,\ldots,s_n)$$

$$= \frac{1}{\lambda_{n+1}-\lambda_j} \int_{\Pi^{(n)}} e^{i((s_0-s_1)\lambda_0+\cdots+(s_j-s_{j+1})\lambda_j+\cdots+s_n\lambda_n)} \Big(e^{(s_j-s_{j+1})(\lambda_{n+1}-\lambda_j)}-1\Big)\, d\nu_f^{(n)}(s_0,\ldots,s_n)$$

$$= \frac{1}{\lambda_{n+1}-\lambda_j} \int_{\Pi^{(n)}} \Big(e^{i((s_0-s_1)\lambda_0+\cdots+(s_j-s_{j+1})\lambda_{n+1}+\cdots+s_n\lambda_n)} - e^{i((s_0-s_1)\lambda_0+\cdots+(s_j-s_{j+1})\lambda_j+\cdots+s_n\lambda_n)}\Big)\, d\nu_f^{(n)}(s_0,\ldots,s_n)$$

$$= \frac{1}{\lambda_{n+1}-\lambda_j}\Big(f^{[n]}(\lambda_0,\ldots,\lambda_{j-1},\lambda_{n+1},\lambda_{j+1},\ldots,\lambda_n) - f^{[n]}(\lambda_0,\ldots,\lambda_{j-1},\lambda_j,\lambda_{j+1},\ldots,\lambda_n)\Big)$$

$$= f^{[n+1]}(\lambda_0,\ldots,\lambda_{n+1}).$$ ■

Lemma 2.5 *Let $(\mathcal{N},\tau)$ be a semifinite von Neumann algebra. If $A_\alpha \in \mathcal{N}$, $\alpha \in I$, is a uniformly bounded net converging in the so-topology to an operator $A \in \mathcal{N}$ and if $V \in \mathcal{L}^1(\mathcal{N},\tau)$, then the net $\{A_\alpha V\}_{\alpha\in I}$ converges to AV in $L^1(\mathcal{N},\tau)$.*

Proof Without loss of generality, we can assume that $A = 0$. Since the net $\{A_\alpha\}_{\alpha\in I}$ is uniformly bounded, we have $A_\alpha \to 0$ in the σ-strong operator topology (see *e.g.*, [6, Proposition 2.4.1]). Since the σ-strong topology does not depend on representation [6, Theorem 2.4.23], it can be assumed that $\mathcal{N}$ acts on $L^2(\mathcal{N},\tau)$ in the left regular representation, in particular $\|A_\alpha y\|_2 \to 0$ for every $y \in \mathcal{L}^2(\mathcal{N},\tau)$. Without loss of generality, we may suppose that $V \geqslant 0$. Let $y = V^{1/2} \in \mathcal{L}^2(\mathcal{N},\tau)$. Then

$$\tau\left(|A_\alpha V|\right) = \tau\left(u_\alpha A_\alpha y^2\right) = \tau\left(A_\alpha y (u_\alpha^* y)^*\right) \leqslant \|A_\alpha y\|_2 \cdot \|u_\alpha^* y\|_2 \to 0,$$

where u_α^* is the partial isometry from the polar decomposition of $A_\alpha V$. ■

Lemma 2.6 *Let $A, B \in \mathcal{N}$ and suppose that one of these operators is τ-trace-class. If $T = T^*$ is affiliated with $\mathcal{N}$ and if $T = \int_{\mathbb{R}} \lambda\, dE_\lambda$ if the spectral resolution of T, then the (complex) measure $\mu(a,b) := \tau(AE_{(a,b)}B)$ is countably additive (and has finite variation).*

Proof Since the spectral resolution of a self-adjoint operator is strong operator σ-additive (see *e.g.*, [36, VIII.3]), the assertion of the lemma follows from Lemma 2.5. ■

3 Integration of Operator-Valued Functions

Lemma 3.1 *An invariant operator ideal $\mathcal{E}$ has property (F) if and only if the unit ball of $\mathcal{E}$ endowed with so*-topology is a complete separable metrisable space.*

https://doi.org/10.4153/CJM-2009-012-0 Published online by Cambridge University Press

Proof The "if" part is evident. Since $\mathcal{H}$ is separable, the unit ball $(\mathcal{B}(\mathcal{H})_1, so^*)$ of $\mathcal{B}(\mathcal{H})$ is a metrisable space [17, Proposition I.3.1]. Hence, the unit ball $(\mathcal{E}_1, so^*)$ of $\mathcal{E}$ is also metrisable. Since $\mathcal{H}$ is separable the unit ball $(\mathcal{B}(\mathcal{H})_1, so^*)$ is also separable. Thus, every subset of $(\mathcal{B}(\mathcal{H})_1, so^*)$ is separable [20, I.6.12], and in particular $\mathcal{E}_1$. Since the unit ball $(\mathcal{B}(\mathcal{H})_1, so^*)$ is complete [6, Prop. 2.4.1], the property (F) of $\mathcal{E}$ implies that $(\mathcal{E}_1, so^*)$ is also complete. ∎

Let (S, Σ, ν) be a finite measure space and $\mathcal{E}$ be an invariant operator ideal with property (F). A bounded function $f\colon (S,\nu) \to \mathcal{E}$ will be called

(i) *weakly measurable* if, for any $\eta, \xi \in \mathcal{H}$, the function $\langle f(\cdot)\eta, \xi\rangle$ is measurable;
(ii) $*$- *measurable* if, for all $\eta \in \mathcal{H}$, the functions $f(\cdot)\eta, f(\cdot)^*\eta\colon (S,\nu) \to \mathcal{H}$ are Bochner measurable from S into $\mathcal{H}$;
(iii) so^*-*measurable* if there exist a sequence of simple (finitely-valued) measurable functions $f_n\colon S \to \mathcal{E}$ such that $f_n(\sigma) \to f(\sigma)$ in the so^*-topology for a. e. $\sigma \in S$.

Proposition 3.2 *If $\mathcal{E}$ has property (F), then, for any $\mathcal{E}$-bounded function $f\colon (S,\nu) \to \mathcal{E}$, the following conditions are equivalent.*

(i) *f is weakly measurable,*
(ii) *f is $*$- measurable,*
(iii) *f is so^*-measurable.*

Proof The implications (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) are evident (and do not depend on property (F)). That (i) $\Rightarrow$ (iii) follows from Lemma 3.1 and [41, Propositions 1.9 and 1.10]. ∎

We denote the set of all $\|\cdot\|$-bounded $*$- measurable functions $f\colon S \to \mathcal{E}$ by $\mathcal{L}_\infty^{so^*}(S,\nu;\mathcal{E})$. Examples of such functions are bounded $\|\cdot\|$-Bochner-measurable functions and, in the case that S is a locally compact space, all so^*-continuous bounded functions.

The following lemma is a simple consequence of the previous proposition (cf. [29, Lemmas 5.5 and 5.6]).

Lemma 3.3 [29]

(i) *The set $\mathcal{L}_\infty^{so^*}(S,\nu;\mathcal{E})$ is a $*$-algebra;*
(ii) *if $\phi \in B_{\mathbb{R}}(\mathbb{R})$, $f \in \mathcal{L}_\infty^{so^*}(S,\nu;\mathcal{B}(\mathcal{H})_{sa})$, then $\phi(f) \in \mathcal{L}_\infty^{so^*}(S,\nu)$.*

For any bounded function $f \in \mathcal{L}_\infty^{so^*}(S,\nu;\mathcal{E})$, we define the integral $\int_S f(\sigma)\,d\nu(\sigma)$ by the formula

$$\left(\int_S f(\sigma)\,d\nu(\sigma)\right)\eta = \int_S f(\sigma)\eta\,d\nu(\sigma), \tag{3.1}$$

where the last integral is a Bochner integral. Evidently, such an integral exists and it is a bounded linear operator with (uniform) norm less or equal to $|\nu|\,\|f\|_\infty$.

Lemma 3.4 *If $\mathcal{E}$ has property (F), and if the sequence $f_n \in \mathcal{L}_\infty^{so^*}(S,\nu;\mathcal{E})$, $n = 1, 2, \dots$ is $\mathcal{E}$-bounded and ν-a. e. converges to $f\colon S \to \mathcal{B}(\mathcal{H})$ in the so^*-topology, then $f \in \mathcal{L}_\infty^{so^*}(S,\nu;\mathcal{E})$.*

Proof We have that, for any $\eta \in \mathcal{H}$, the sequence $f_n(\sigma)\eta$ converges to $f(\sigma)\eta$ for ν-a.e. $\sigma \in S$. Since the $\mathcal{H}$-valued functions $f_n(\cdot)\eta$ are Bochner measurable and since the pointwise limit of a sequence of Bochner measurable functions is a Bochner measurable function, we have that $f \in \mathcal{L}_\infty^{so^*}(S,\nu)$. That $f(\sigma) \in \mathcal{E}$ for a. e. $\sigma \in S$ follows from property (F). ∎

Lemma 3.5 *If $\mathcal{E}$ has property (F), $f \in \mathcal{L}_\infty^{so^*}(S,\nu;\mathcal{E})$ and if f is uniformly $\mathcal{E}$-bounded, then $\int_S f\,d\nu \in \mathcal{E}$.*

Proof By Proposition 3.2, we can choose a sequence of simple functions $f_n \in \mathcal{L}_\infty^{so^*}(S,\nu;\mathcal{E})$ converging a. e. in *so**-topology to f. Evidently, $A_n := \int_S f_n\,d\nu \in \mathcal{E}$ for all $n \in \mathbb{N}$. By the definition of operator-valued integral (3.1), the sequence $\{A_n\}_{n=1}^\infty$ converges to $\int_S f\,d\nu$ in the *so**-topology by the Lebesgue Dominated Convergence Theorem for the Bochner integral. That $\int_S f\,d\nu \in \mathcal{E}$ now follows from the property (F) of $\mathcal{E}$. ∎

Corollary 3.6 *Under the assumptions of Lemma* 3.4*, we have*

$$\int_S f_n\,d\nu \to \int_S f\,d\nu$$

in the so*-topology.

Lemma 3.7 *For any $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{L}_\infty^{so^*}(S,\nu;\mathcal{E})$*

$$A\int_S B(\sigma)\,d\nu(\sigma) = \int_S AB(\sigma)\,d\nu(\sigma)$$

The lemma follows directly from [43, Corollary V.5.2].

Lemma 3.8 *If (S_i,Σ_i,ν_i), $i = 1,2$ are two finite measure spaces and if $f \in \mathcal{L}_\infty^{so^*}(S_1\times S_2,\nu_1\times\nu_2)$, then $f(\,\cdot\,,t) \in \mathcal{L}_\infty^{so^*}(S_1,\nu_1)$ for almost all $t \in S_2$ and*

$$\int_{S_2}\int_{S_1} f(s,t)\,d\nu_1(s)\,d\nu_2(t) = \int_{S_1\times S_2} f(s,t)\,d(\nu_1\times\nu_2)(s,t). \tag{3.2}$$

Proof Since $f(\,\cdot\,,\,\cdot\,)$ is integrable, for any $\eta \in \mathcal{H}$ there exists a ν_2-measure zero set $A_\eta \subset S_2$ such that for all $t \notin A_\eta$ the function $f(\,\cdot\,,t)\eta$ is Bochner integrable (see [20, Theorem III.11.13]). If $\{\xi_j\}_{j=1}^\infty$ is an orthonormal basis in $\mathcal{H}$ and $A = \bigcup_{j=1}^\infty A_{\xi_j}$, then $\nu_2(A) = 0$ and, for any $\eta \in \mathcal{H}$ and $t \notin A$, we have

$$f(\,\cdot\,,t)\eta = \sum_{j=1}^\infty c_n f(\,\cdot\,,t)\xi_n,$$

where $\eta = \sum_{j=1}^\infty c_n\xi_n$. Since linear combinations and uniformly bounded pointwise limits of sequences of Bochner integrable functions on the measure space (S,ν) are Bochner integrable (by the Lebesgue Dominated Convergence Theorem), it follows that $f(\,\cdot\,,t)\eta$ is integrable for $t \notin A$. Similarly, there exists a ν_2-measure zero set

A' such that $f(\cdot,t)^*\eta$ is integrable for all $\eta \in \mathcal{H}$ and $t \notin A'$. Hence, $f(\cdot,t)$ is integrable for all $t \notin A \cup A'$ and the operator-valued function $g(t) := \int_{S_1} f(s,t)\, d\nu_1(s)$ is well defined. Now, the integral $g(t)\eta = \int_{S_1} f(s,t)\eta\, d\nu_1(s)$ exists and is equal to $\int_{S_1 \times S_2} f(s,t)\eta\, d(\nu_1 \times \nu_2)(s,t)$ by Fubini's theorem for the Bochner integral of $\mathcal{H}$-valued functions [20, Theorem III.11.13]. The latter means that the equality (3.2) holds. ■

Lemma 3.9 *If* $f \in \mathcal{L}^{so^*}_\infty(S,\nu;\mathcal{N})$, *then*

(i) $X := \int_S f(\sigma)\, d\nu(\sigma)$ *belongs to* $\mathcal{N}$;

(ii) X *as an element of the* W^**-algebra* $\mathcal{N}$ *does not depend on any representation of* $\mathcal{N}$.

Proof (i) Let $A' \in \mathcal{N}'$. Then by Lemma 3.7

$$A'X\eta = \int_S A' f(\sigma)\eta\, d\nu(\sigma) = \int_S f(\sigma)A'\eta\, d\nu(\sigma) = \int_S f(\sigma)\, d\nu(\sigma) A'\eta = XA'\eta$$

for any $\eta \in \mathcal{H}$. Hence, $X \in \mathcal{N}$.
(ii) This follows from the fact that two representations of a von Neumann algebra can be obtained from each other by ampliation, reduction, and spatial isomorphism [17], since for each of these isomorphisms the claim is evident. ■

Lemma 3.10 *If* $(\mathcal{N},\tau)$ *is a semifinite von Neumann algebra, if*

$$f \in \mathcal{L}^{so^*}_\infty(S,\nu;\mathcal{L}^1(\mathcal{N},\tau))$$

and if f *is uniformly* $\mathcal{L}^1(\mathcal{N},\tau)$*-bounded, then* $X := \int_S f(\sigma)\, d\nu(\sigma) \in \mathcal{L}^1(\mathcal{N},\tau)$, *the function* $\tau(f(\cdot))$ *is measurable and*

$$\tau\left(\int_S f(\sigma)\, d\nu(\sigma)\right) = \int_S \tau\left(f(\sigma)\right)\, d\nu(\sigma).$$

Proof Lemma 3.5 implies that $X \in \mathcal{L}^1(\mathcal{N},\tau)$, so that the left hand side of the equality above makes sense.

By [29, Lemma 5.9], the function $\tau(f(\cdot))$ is measurable.

By linearity and by Lemma 3.3(i), we can assume that $f(\cdot) \geqslant 0$. By Lemma 3.9(ii), we can assume that $\mathcal{N}$ acts on $L^2(\mathcal{N},\tau)$ in the left regular representation. Let E be an arbitrary τ-finite projection from $\mathcal{N}$. Then $E \in L^2(\mathcal{N},\tau)$ and by the definition of the operator-valued integral (3.1),

$$XE = \int_S f(\sigma)E\, d\nu(\sigma),$$

where the right hand side is a Bochner integral in $L^2(\mathcal{N},\tau)$. Since E is τ-finite, the convergence in $L^2(\mathcal{N},\tau)$ of the Bochner integral implies convergence in $L^1(\mathcal{N},\tau)$, so that we have

$$\tau(XE) = \int_S \tau(f(\sigma)E)\, d\nu(\sigma).$$

Now, normality of the trace τ and the dominated convergence theorem imply that

$$\tau(X) = \int_S \tau(f(\sigma))\, d\nu(\sigma).$$ ■

4 Multiple Operator Integrals

In this section, we define multiple operator integrals of the form

$$\int_{\mathbb{R}^{n+1}} \phi(\lambda_0, \dots, \lambda_n)\, dE^{B_0}_{\lambda_0} V_1\, dE^{B_1}_{\lambda_1} V_2\, dE^{B_2}_{\lambda_2} \cdots V_n\, dE^{B_n}_{\lambda_n}.$$

We denote by $B(\mathbb{R}^{n+1})$ the set of all bounded Borel functions on $\mathbb{R}^{n+1}$. Throughout this section, we consider the set of those functions $\phi \in B(\mathbb{R}^{n+1})$ which admit a representation of the form

$$\phi(\lambda_0, \lambda_1, \dots, \lambda_n) = \int_S \alpha_0(\lambda_0, \sigma) \cdots \alpha_n(\lambda_n, \sigma)\, d\nu(\sigma), \tag{4.1}$$

where (S, ν) is a finite measure space and $\alpha_0, \dots, \alpha_n$ are bounded Borel functions on $\mathbb{R} \times S$. Similar representations (for the case $n = 1$) were discussed in [29].

Definition 4.1 For arbitrary self-adjoint operators $B_0, \dots, B_n$ on the Hilbert space $\mathcal{H}$, bounded operators $V_1, \dots, V_n$ on $\mathcal{H}$ and any function $\phi \in B(\mathbb{R}^{n+1})$ that admits a representation given by (4.1), the multiple operator integral $T^{B_0,\dots,B_n}_{\phi}(V_1, \dots, V_n)$ is defined as

$$T^{B_0,\dots,B_n}_{\phi}(V_1, \dots, V_n) := \int_S \alpha_0(B_0, \sigma) V_1 \cdots V_n \alpha_n(B_n, \sigma)\, d\nu(\sigma), \tag{4.2}$$

where the integral is taken in the sense of definition (3.1).

Remark 4.2. By [29, Lemma 5.13] and Lemma 3.3(i) applied to $\mathcal{E} = \mathcal{B}(\mathcal{H})$, the function $\sigma \mapsto \alpha_0(B_0, \sigma) V_1 \cdots V_n \alpha_n(B_n, \sigma)$ is $*$- measurable and therefore the integral above exists.

Lemma 4.3 *The multiple operator integral in Definition* 4.1 *is well defined in the sense that it does not depend on the representation* (4.1) *of* ϕ.

Proof We first prove that if the operators $V_1, \dots, V_n$ are all one-dimensional, then the right hand side of (4.2) does not depend on the representation of ϕ given by (4.1).

For $\eta, \xi \in \mathcal{H}$, we denote by $\theta_{\eta,\xi}$ the one-dimensional operator defined by formula $\theta_{\eta,\xi}\zeta = \langle \eta, \zeta \rangle\, \xi,\ \zeta \in \mathcal{H}$. It is clear that $\mathrm{Tr}(\theta_{\eta,\xi}) = \langle \eta, \xi \rangle,\ A\theta_{\eta,\xi} = \theta_{\eta,A\xi}$ for any $A \in \mathcal{B}(\mathcal{H})$ and that $\theta_{\eta_1,\xi_1} \cdots \theta_{\eta_n,\xi_n} = \langle \eta_1, \xi_2 \rangle \cdots \langle \eta_{n-1}, \xi_n \rangle\, \theta_{\eta_n,\xi_1}$.

Let $V_j = \theta_{\eta_j,\xi_j},\ j = 0, \dots, n$. Then

$$\begin{aligned} E := {} & \mathrm{Tr}\left(V_0 \int_S \alpha_0(B_0, \sigma) V_1 \cdots V_n \alpha_n(B_n, \sigma)\, d\nu(\sigma) \right) \\ = {} & \mathrm{Tr} \int_S V_0 \alpha_0(B_0, \sigma) V_1 \cdots V_n \alpha_n(B_n, \sigma)\, d\nu(\sigma) \\ = {} & \mathrm{Tr} \int_S \theta_{\eta_0,\xi_0} \alpha_0(B_0, \sigma) \theta_{\eta_1,\xi_1} \cdots \theta_{\eta_n,\xi_n} \alpha_n(B_n, \sigma)\, d\nu(\sigma) \end{aligned}$$

https://doi.org/10.4153/CJM-2009-012-0 Published online by Cambridge University Press

$$= \int_S \operatorname{Tr}\left(\theta_{\eta_0,\xi_0}\alpha_0(B_0,\sigma)\theta_{\eta_1,\xi_1}\cdots\theta_{\eta_n,\xi_n}\alpha_n(B_n,\sigma)\right)\, d\nu(\sigma)$$

$$= \int_S \operatorname{Tr}\left(\alpha_0(B_0,\sigma)\theta_{\eta_1,\xi_1}\cdots\theta_{\eta_n,\xi_n}\alpha_n(B_n,\sigma)\theta_{\eta_0,\xi_0}\right)\, d\nu(\sigma)$$

$$= \int_S \operatorname{Tr}\left(\theta_{\eta_1,\alpha_0(B_0,\sigma)\xi_1}\cdots\theta_{\eta_n,\alpha_{n-1}(B_{n-1},\sigma)\xi_n}\theta_{\eta_0,\alpha_n(B_n,\sigma)\xi_0}\right)\, d\nu(\sigma)$$

$$= \int_S \langle\eta_0,\alpha_0(B_0,\sigma)\xi_1\rangle\,\langle\eta_1,\alpha_1(B_1,\sigma)\xi_2\rangle\cdots\langle\eta_n,\alpha_n(B_n,\sigma)\xi_0\rangle\, d\nu(\sigma).$$

Now, since $\langle\eta,\alpha(B)\xi\rangle = \int_{\mathbb{R}}\alpha(\lambda)\left\langle\eta, dE^B_\lambda\xi\right\rangle$, we have that

$$E = \int_S\int_{\mathbb{R}}\alpha_0(\lambda_0,\sigma)\left\langle\eta_0, dE^{B_0}_{\lambda_0}\xi_1\right\rangle\cdots\int_{\mathbb{R}}\alpha_n(\lambda_n,\sigma)\left\langle\eta_n, dE^{B_n}_{\lambda_n}\xi_0\right\rangle\, d\nu(\sigma).$$

Since the measure $\langle\eta, dE_\lambda\xi\rangle$ has finite total variation, Fubini's theorem implies

$$E = \int_S\left(\int_{\mathbb{R}^{n+1}}\alpha_0(\lambda_0,\sigma)\cdots\alpha_n(\lambda_n,\sigma)\left\langle\eta_0, dE^{B_0}_{\lambda_0}\xi_1\right\rangle\cdots\left\langle\eta_n, dE^{B_n}_{\lambda_n}\xi_0\right\rangle\right)\, d\nu(\sigma)$$

$$= \int_{\mathbb{R}^{n+1}}\left(\int_S\alpha_0(\lambda_0,\sigma)\cdots\alpha_n(\lambda_n,\sigma)\, d\nu(\sigma)\right)\left\langle\eta_0, dE^{B_0}_{\lambda_0}\xi_1\right\rangle\cdots\left\langle\eta_n, dE^{B_n}_{\lambda_n}\xi_0\right\rangle$$

$$= \int_{\mathbb{R}^{n+1}}\phi(\lambda_0,\ldots,\lambda_n)\left\langle\eta_0, dE^{B_0}_{\lambda_0}\xi_1\right\rangle\cdots\left\langle\eta_n, dE^{B_n}_{\lambda_n}\xi_0\right\rangle.$$

We recall that, if A, B are bounded operators, then $A = B$ if and only if the equality $\operatorname{Tr}(VA) = \operatorname{Tr}(VB)$ holds for all one-dimensional operators V. It now follows immediately that the multiple operator integral does not depend on the representation (4.1) of ϕ in the case that the operators $V_1,\ldots,V_n$ are one-dimensional.

By linearity, it follows that the definition of multiple operator integral does not depend on the representation (4.1) in the case of finite-dimensional operators $V_1,\ldots,V_n$. Since every bounded operator is an *so*-limit of a sequence of finite-dimensional operators, the claim follows from Proposition 4.9. ■

Lemma 4.4 *If $\mathcal{N}$ is a von Neumann algebra, if $B_0,\ldots,B_n$ are self-adjoint operators affiliated with $\mathcal{N}$ and if $V_1,\ldots,V_n\in\mathcal{N}$, then $T^{B_0,\ldots,B_n}_\phi(V_1,\ldots,V_n)\in\mathcal{N}$.*

This follows from Lemma 3.9.

The following observation is a direct consequence of Lemma 2.3 and Definition 4.1.

Lemma 4.5 *If $f\in W_n(\mathbb{R})$, then*

$$T^{B_0,\ldots,B_n}_{f^{[n]}}(V_1,\ldots,V_n) = \int_{\Pi^{(n)}} e^{i(s_0-s_1)B_0}V_1e^{i(s_1-s_2)B_1}V_2\cdots V_ne^{is_nB_n}\, d\nu^{(n)}_f(s_0,\ldots,s_n).$$

https://doi.org/10.4153/CJM-2009-012-0 Published online by Cambridge University Press

Lemma 4.6 *If $\mathcal{E}$ is an invariant operator ideal with property (F) and if one of the operators $V_1, \dots, V_n$ belongs to $\mathcal{E}$, then*

$$T_\phi^{B_0,\dots,B_n}(V_1, \dots, V_n) \in \mathcal{E}.$$

If $n = 2$, this yields

$$\left\| T_\phi^{B_1,B_2} \right\|_{\mathcal{E}\to\mathcal{E}} \leqslant \|\phi\|,$$

where (see [29]*)*

$$\|\phi\| = \inf\Big\{ \int_S \|\alpha(\,\cdot\,,\sigma)\|_\infty \, \|\beta(\,\cdot\,,\sigma)\|_\infty \; d\nu(\sigma) \colon \phi(\lambda,\mu) = \int_S \alpha(\lambda,\sigma)\beta(\mu,\sigma)\, d\nu(\sigma) \Big\}.$$

Proof This follows from Lemmas 3.3(i) and 3.5. ■

Remark 4.7. If $V \in \mathcal{L}^2(\mathcal{N},\tau)$ and if $n = 2$, then the preceding definition coincides with the definition of double operator integral as a spectral integral given in [5] and [29].

Corollary 4.8 *If $V_1, \dots, V_n \in \mathcal{N}$, $V_j \in \mathcal{L}^1(\mathcal{N},\tau)$ for some $j = 1, \dots, n$, $B_0, \dots, B_n$ are self-adjoint operators affiliated with $\mathcal{N}$, $\phi \in B(\mathbb{R}^{n+1})$ and $\phi(\lambda_0, \dots, \lambda_n)$ admits the representation* (4.1)*, then*

$$\tau\left(T_\phi^{B_0,\dots,B_n}(V_1, \dots, V_n) \right) = \int_S \tau\left(\alpha_0(B_0,\sigma)V_1\alpha_1(B_1,\sigma)\cdots V_n\alpha_n(B_n,\sigma)\right) \, d\nu(\sigma).$$

Proof It is enough to note that the operator-valued function

$$\sigma \mapsto \alpha_0(B_0,\sigma)V_1\alpha_1(B_1,\sigma)\cdots V_n\alpha_n(B_n,\sigma)$$

is $*$- measurable by [29, Lemma 5.11] and Lemma 3.3(i), so that we can apply Lemma 3.10. ■

Proposition 4.9 (i) *If a sequence of self-adjoint operators $V_j^{(k_j)} \in \mathcal{B}(\mathcal{H})$, $j = 1, \dots, n$, converges to $V_j \in \mathcal{B}(\mathcal{H})$ in the so-topology (respectively, norm topology) as $k_j \to \infty$, then*

$$T_\phi^{B_0,\dots,B_n}(V_1^{(k_1)}, \dots, V_n^{(k_n)}) \to T_\phi^{B_0,\dots,B_n}(V_1, \dots, V_n)$$

in the so-topology (respectively, norm topology) as $k_1, \dots, k_n \to \infty$.

(ii) *If a sequence of self-adjoint operators $B_j^{(k_j)}$, $j = 0, \dots, n$ resolvent strongly converges to B_j as $k_j \to \infty$ and $V_1, \dots, V_n \in \mathcal{B}(\mathcal{H})$, then*

$$T_\phi^{B_0^{(k_0)},\dots,B_n^{(k_n)}}(V_1, \dots, V_n) \to T_\phi^{B_0,\dots,B_n}(V_1, \dots, V_n)$$

in the so-topology as $k_0, \dots, k_n \to \infty$.

https://doi.org/10.4153/CJM-2009-012-0 Published online by Cambridge University Press

Proof We prove only part (ii); the proof of part (i) is similar (and simpler). Suppose that

$$\phi(\lambda_0, \dots, \lambda_n) = \int_S \alpha_0(\lambda_0, \sigma) \cdots \alpha_n(\lambda_n, \sigma) \, d\nu(\sigma)$$

is a representation of ϕ given by (4.1). Since $\alpha(\,\cdot\,, \sigma)$ is a bounded function for every $\sigma \in S$, the operators $\alpha(B_j^{(k_j)}, \sigma)$ converge to $\alpha(B_j, \sigma)$ in the *so*-topology [36, Theorem VIII.20(b)]. Since multiplication of operators is jointly continuous in the *so*-topology on the unit ball of $\mathcal{N}$ [6, Proposition 2.4.1], the operator

$$\alpha(B_0^{(k_0)}, \sigma) V_1 \cdots V_n \alpha(B_n^{(k_n)}, \sigma)$$

converges in the *so*-topology to $\alpha(B_0, \sigma) V_1 \cdots V_n \alpha(B_n, \sigma)$, $\sigma \in S$. Now, an application of the Dominated Convergence Theorem for the Bochner integral of $\mathcal{H}$-valued functions [20, Corollary III.6.16] completes the proof. ∎

This new definition of multiple operator integral enables us to give a simple proof of the following.

Proposition 4.10 *The multiple operator integral has the properties:*
(i) *if* ϕ_1 *and* ϕ_2 *admit a representation of the type given in* (4.1)*, then so does* $\phi_1 + \phi_2$ *and*

$$T^{B_1, \dots, B_n}_{\phi_1 + \phi_2} = T^{B_1, \dots, B_n}_{\phi_1} + T^{B_1, \dots, B_n}_{\phi_2}; \tag{4.3}$$

(ii) *in the case of double operator integrals, if* ϕ_1 *and* ϕ_2 *admit a representation of the type given in* (4.1)*, then so does* $\phi_1 \phi_2$ *and*

$$T^{B_1, B_2}_{\phi_1 \phi_2} = T^{B_1, B_2}_{\phi_1} T^{B_1, B_2}_{\phi_2}.$$

Proof (i) If we take representations of the form (4.1) with (S_1, ν_1) and (S_2, ν_2) for ϕ_1 and ϕ_2 and put $(S, \nu) = (S_1, \nu_1) \sqcup (S_2, \nu_2)$ for $\phi_1 + \phi_2$ with evident definition of $\alpha_1, \alpha_2, \dots$, then the equality (4.3) follows from Definition 4.1. Here $\sqcup$ denotes the disjoint sum of measure spaces.

(ii) If

$$\phi_j(\lambda_1, \lambda_2) = \int_{S_1} \alpha_j(\lambda_1, \sigma_1) \beta_j(\lambda_2, \sigma_1) \, d\nu_j(\sigma_1), \;\; j = 1, 2,$$

set

$$\phi(\lambda_1, \lambda_2) = \int_S \alpha(\lambda_1, \sigma) \beta(\lambda_2, \sigma) \, d\nu(\sigma),$$

where

$$(S, \nu) = (S_1, \nu_1) \times (S_2, \nu_2)$$

and

$$\alpha(\lambda, \sigma) = \alpha_1(\lambda, \sigma_1) \alpha_2(\lambda, \sigma_2), \quad \beta(\lambda, \sigma) = \beta_1(\lambda, \sigma_1) \beta_2(\lambda, \sigma_2).$$

Consequently,

$$T_{\phi_1}^{B_1,B_2}\left(T_{\phi_2}^{B_1,B_2}(V)\right)=\int_{S_1}\alpha_1(B_1,\sigma_1)T_{\phi_2}^{B_1,B_2}(V)\beta_1(B_2,\sigma_1)\,d\nu_1(\sigma_1)$$
$$=\int_{S_1}\alpha_1(B_1,\sigma_1)\left(\int_{S_2}\alpha_2(B_1,\sigma_2)V\beta_2(B_2,\sigma_2)\,d\nu_2(\sigma_2)\right)\beta_1(B_2,\sigma_1)\,d\nu_1(\sigma_1).$$

Now, Lemma 3.7 and Fubini's theorem (Lemma 3.8) imply

$$T_{\phi_1}^{B_1,B_2}\left(T_{\phi_2}^{B_1,B_2}(V)\right)=\int_{S_1\times S_2}\alpha_1(B_1,\sigma_1)\alpha_2(B_1,\sigma_2)V$$
$$\times\,\beta_2(B_2,\sigma_2)\beta_1(B_2,\sigma_1)\,d(\nu_1\times\nu_2)(\sigma_1,\sigma_2)=T_{\phi_1\phi_2}^{B_1,B_2}(V).\quad\blacksquare$$

5 Higher Order Fréchet Differentiability

We note that, by Stone's theorem [36, Theorem VIII.7] and joint continuity of multiplication of operators (from the unit ball) in the *so*-topology [6, Proposition 2.4.1], all operator-valued integrals occurring in this and subsequent sections are defined as in Section 3.

Lemma 5.1 *If A is a self-adjoint (possibly unbounded) operator on a Hilbert space $\mathcal{H}$ and if f is a function on $\mathbb{R}$ such that $f\in W_1(\mathbb{R})$, then*

$$f(A)=(2\pi)^{-1/2}\int_{\mathbb{R}}e^{isA}\,m_f(ds).$$

The proof is a simple application of Fubini's theorem. See [6, Theorem 3.2.32].

Lemma 5.2 *(Duhamel's formula). If B is an unbounded self-adjoint operator on a Hilbert space $\mathcal{H}$, if V is a bounded self-adjoint operator on $\mathcal{H}$ and if $A=B+V$, then*

$$e^{isA}-e^{isB}=\int_0^s e^{i(s-t)A}i(A-B)e^{itB}\,dt. \tag{5.1}$$

Proof Let $F(t)=e^{itA}e^{-itB}$. Taking the derivative of $F(t)$ in the *so*-topology gives

$$F'(t)=iAe^{itA}e^{-itB}+e^{itA}(-iB)e^{-itB}=e^{itA}i(A-B)e^{-itB}.$$

So,

$$\int_0^s e^{itA}i(A-B)e^{-itB}\,dt=F(s)-F(0)=e^{isA}e^{-isB}-1.$$

Multiplying the last equality by e^{isB} from the right gives (5.1). ■

https://doi.org/10.4153/CJM-2009-012-0 Published online by Cambridge University Press

Theorem 5.3 *Let $\mathcal{N}$ be a von Neumann algebra. Suppose that $B = B^*$ is affiliated with $\mathcal{N}$, that $V \in \mathcal{N}$ is self-adjoint and set $A = B + V$. If $f \in W_1(\mathbb{R})$, then*

$$f(A) - f(B) = T^{A,B}_{f^{[1]}}(V).$$

Proof It follows from Lemma 5.1 that

$$f(A) - f(B) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (e^{isA} - e^{isB})\, m_f(ds).$$

Hence, by Lemma 5.2,

$$f(A) - f(B) = \frac{i}{\sqrt{2\pi}} \int_{\mathbb{R}} m_f(ds) \int_0^s e^{i(s-t)A} V e^{itB}\, dt.$$

Since $f \in W_1(\mathbb{R})$, by Lemma 2.1 and Fubini's theorem (Lemma 3.8), the repeated integral can be replaced by a double integral, so that

$$(5.2)\quad f(A) - f(B) = \frac{i}{\sqrt{2\pi}} \iint_{\Pi} e^{i(s-t)A} V e^{itB}\, m_f(ds)\, dt = \iint_{\Pi} e^{i(s-t)A} V e^{itB}\, d\nu_f(\sigma).$$

It now follows from Lemma 4.5 that $f(A) - f(B) = T^{A,B}_{f^{[1]}}(V)$. ■

Remark 5.4. The preceding formula is due to Birman and Solomyak [5]. It is similar to [30, Corollary 7.2], which applies to a wider class of functions but is restricted to bounded operators in a semifinite von Neumann algebra $\mathcal{N}$.

Let X be a topological vector space, $\mathcal{E}$ be a normed space embedded in X. Let $x \in X$ and $f\colon x + \mathcal{E} \to f(x) + \mathcal{E}$. The function f is called affinely Fréchet differentiable at x along $\mathcal{E}$ if there exists a (necessarily unique) bounded operator $L\colon \mathcal{E} \to \mathcal{E}$ such that

$$f(x+h) - f(x) = L(h) + r(x,h),$$

where $\|r(x,h)\|_{\mathcal{E}} = o(\|h\|_{\mathcal{E}})$. We write $L = D_{\mathcal{E}} f(x)$.

Theorem 5.5 *Let $\mathcal{N}$ be a von Neumann algebra, acting in a Hilbert space $\mathcal{H}$. Let $B = B^*$ be affiliated with $\mathcal{N}$ and let $V \in \mathcal{E}_{sa}$, where $\mathcal{E}$ is an invariant operator ideal over $\mathcal{N}$ with property (F). If $f \in W_2(\mathbb{R})$, then the function $f\colon B' \in B + \mathcal{E}_{sa} \mapsto f(B') \in f(B) + \mathcal{E}_{sa}$ is affinely Fréchet differentiable along $\mathcal{E}_{sa}$ and $D_{\mathcal{E}} f(B) = T^{B,B}_{f^{[1]}}$. The function $X \mapsto D_{\mathcal{E}} f(B+X)$ is continuous in the norm of $\mathcal{E}$ and satisfies the estimate*

$$(5.3)\qquad \|D_{\mathcal{E}} f(B+X)(V) - D_{\mathcal{E}} f(B)(V)\|_{\mathcal{E}} \leqslant \|m_{f''}\|\, \|V\|_{\mathcal{E}}\, \|X\|_{\mathcal{E}}\,,\ X, V \in \mathcal{E}.$$

Proof By (5.2) we have, following [42],

$$\begin{aligned} f(B+V) - f(B) &= \iint_{\Pi} e^{i(s-t)(B+V)} V e^{itB}\, d\nu_f(s,t) \\ &= \iint_{\Pi} e^{i(s-t)B} V e^{itB}\, d\nu_f(s,t) + \iint_{\Pi} \left(e^{i(s-t)(B+V)} - e^{i(s-t)B}\right) V e^{itB}\, d\nu_f(s,t) \\ &= (\mathrm{I}) + (\mathrm{II}). \end{aligned}$$

https://doi.org/10.4153/CJM-2009-012-0 Published online by Cambridge University Press

(I) is equal to $T^{B,B}_{f^{[1]}}(V)$ and represents a continuous linear operator on $\mathcal{E}$ (see Lemmas 4.5 and 4.6), so that it will be a Fréchet derivative of $f\colon B+\mathcal{E} \to f(B)+\mathcal{E}$ provided it is shown that the second term is $o(\|V\|_{\mathcal{E}})$. Applying Duhamel's formula (5.1) yields

$$\text{(II)} = \iint_{\Pi} \left(\int_0^{s-t} e^{i(s-t-u)(B+V)} iVe^{iuB}\, du \right) Ve^{itB}\, d\nu_f(s,t). \tag{5.4}$$

Since $f \in W_2(\mathbb{R})$, Lemmas 2.4, 4.3, and 4.5 enable us to rewrite (5.4) as

$$\text{(II)} = \iiint_{\Pi^{(2)}} e^{i(s-t)(B+V)} Ve^{i(t-u)B} Ve^{iuB}\, d\nu_f^{(2)}(s,t,u),$$

where $(\Pi^{(2)}, \nu_f^{(2)})$ is the finite measure space defined in Lemma 2.1. The $\mathcal{E}$-norm of the last expression is estimated by $|\nu_f^{(2)}|\,\|V\|\,\|V\|_{\mathcal{E}} \leqslant |\nu_f^{(2)}|\,\|V\|^2_{\mathcal{E}}$. So, the function $f\colon B+\mathcal{E} \to f(B)+\mathcal{E}$ is Fréchet differentiable and $D_{\mathcal{E}} f(B) = T^{B,B}_{f^{[1]}}$.

The norm continuity of this derivative and the estimate (5.3) follow by a similar argument using Duhamel's formula (5.1). ■

Remark 5.6. It follows, in particular, from the preceding theorem via Lemma 4.6 that the operator $T^{B,B}_{f^{[1]}}\big|_{\mathcal{E}}$ is a bounded linear operator on $\mathcal{E}$.

Theorem 5.7 *Let $\mathcal{N}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$, let $B = B^*$ be affiliated with $\mathcal{N}$ and let $V_1, \dots, V_n \in \mathcal{E}_{sa}$. If $f \in W_{n+1}(\mathbb{R})$, then the function $f\colon B' \in B+\mathcal{E}_{sa} \mapsto f(B') \in f(B)+\mathcal{E}_{sa}$ is n-times affinely Fréchet differentiable along $\mathcal{E}_{sa}$ and*

$$D^n_{\mathcal{E}} f(B)(V_1,\dots,V_n) = \sum_{\sigma\in P_n} T^{B,\dots,B}_{f^{[n]}}(V_{\sigma(1)},\dots,V_{\sigma(n)}) \in \mathcal{E}, \tag{5.5}$$

where P_n is the standard permutation group.

Proof If $n = 1$ then this theorem is exactly Theorem 5.5. Set $\widetilde{B} = B + V_{n+1}$. By induction we have

$$\begin{aligned} &D^n f(\widetilde{B}; V_1,\dots,V_n) - D^n f(B; V_1,\dots,V_n) \\ &\qquad = \sum_{\sigma\in P_n} \Big(T^{\widetilde{B},\widetilde{B},\dots,\widetilde{B}}_{f^{[n]}}(V_{\sigma(1)},\dots,V_{\sigma(n)}) - T^{B,B,\dots,B}_{f^{[n]}}(V_{\sigma(1)},\dots,V_{\sigma(n)}) \Big). \end{aligned}$$

A single term of this sum is

$$\begin{aligned} &T^{\widetilde{B},\widetilde{B},\dots,\widetilde{B}}_{f^{[n]}}(V_{\sigma(1)},\dots,V_{\sigma(n)}) - T^{B,B,\dots,B}_{f^{[n]}}(V_{\sigma(1)},\dots,V_{\sigma(n)}) \\ &\qquad = \sum_{j=0}^{n} \Big(T^{\widetilde{B},\dots,\overset{(j)}{\widetilde{B}},B,\dots,B}_{f^{[n]}}(V_{\sigma(1)},\dots,V_{\sigma(n)}) - T^{\widetilde{B},\dots,\widetilde{B},\overset{(j)}{B},\dots,B}_{f^{[n]}}(V_{\sigma(1)},\dots,V_{\sigma(n)}) \Big). \end{aligned}$$

Now, the j-th summand is (Lemma 4.5)

$$
\begin{aligned}
&T^{\widetilde B,\dots,\overset{(j)}{\widetilde B},B,\dots,B}_{f^{[n]}}(V_{\sigma(1)},\dots,V_{\sigma(n)}) - T^{\widetilde B,\dots,\widetilde B,\overset{(j)}{B},\dots,B}_{f^{[n]}}(V_{\sigma(1)},\dots,V_{\sigma(n)}) \\
&\quad = \int_{\Pi^{(n)}} e^{i(s_0-s_1)\widetilde B} V_{\sigma(1)} \cdots V_{\sigma(j)} e^{i(s_j-s_{j+1})\widetilde B} V_{\sigma(j+1)} e^{i(s_{j+1}-s_{j+2})B} V_{\sigma(j+2)} \\
&\qquad\qquad \cdots V_{\sigma(n)} e^{is_nB}\, d\nu^{(n)}_f(s_0,\dots,s_n) \\
&\qquad - \int_{\Pi^{(n)}} e^{i(s_0-s_1)\widetilde B} V_{\sigma(1)} \cdots V_{\sigma(j-1)} e^{i(s_{j-1}-s_j)\widetilde B} V_{\sigma(j)} e^{i(s_j-s_{j+1})B} V_{\sigma(j+1)} \\
&\qquad\qquad \cdots V_{\sigma(n)} e^{is_nB}\, d\nu^{(n)}_f(s_0,\dots,s_n) \\
&\quad = \int_{\Pi^{(n)}} e^{i(s_0-s_1)\widetilde B} V_{\sigma(1)} \cdots V_{\sigma(j)} \left(e^{i(s_j-s_{j+1})\widetilde B} - e^{i(s_j-s_{j+1})B}\right) V_{\sigma(j+1)} \\
&\qquad\qquad \cdots V_{\sigma(n)} e^{is_nB}\, d\nu^{(n)}_f(s_0,\dots,s_n).
\end{aligned}
$$

By Duhamel's formula (Lemma 5.2), we have

$$
\begin{aligned}
&T^{\widetilde B,\dots,\overset{(j)}{\widetilde B},B,\dots,B}_{f^{[n]}}(V_{\sigma(1)},\dots,V_{\sigma(n)}) - T^{\widetilde B,\dots,\widetilde B,\overset{(j)}{B},\dots,B}_{f^{[n]}}(V_{\sigma(1)},\dots,V_{\sigma(n)}) = \\
&\qquad \int_{\Pi^{(n)}} e^{i(s_0-s_1)\widetilde B} V_{\sigma(1)} \cdots V_{\sigma(j)} \left(\int_0^{s_j-s_{j+1}} e^{iu\widetilde B} i V_{n+1} e^{i(s_j-s_{j+1}-u)B}\, du\right) \\
&\qquad\qquad V_{\sigma(j+1)} \cdots V_{\sigma(n)} e^{is_nB}\, d\nu^{(n)}_f(s_0,\dots,s_n).
\end{aligned}
$$

Applying Fubini's theorem (Lemma 3.8) we get

$$
\begin{aligned}
(5.6)\quad & T^{\widetilde B,\dots,\overset{(j)}{\widetilde B},B,\dots,B}_{f^{[n]}} - T^{\widetilde B,\dots,\widetilde B,\overset{(j)}{B},\dots,B}_{f^{[n]}} = i \int_{\Pi^{(n)}} \int_0^{s_j-s_{j+1}} e^{i(s_0-s_1)\widetilde B} V_{\sigma(1)} \\
&\qquad \cdots V_{\sigma(j)} e^{iu\widetilde B} V_{n+1} e^{i(s_j-s_{j+1}-u)B} V_{\sigma(j+1)} \cdots V_{\sigma(n)} e^{is_nB}\, du\, d\nu^{(n)}_f(s_0,\dots,s_n).
\end{aligned}
$$

Hence, it follows from formula (5.6), Lemma 2.4, and the fact that multiple operator integral is well-defined (Lemma 4.3) that

$$
\begin{aligned}
&T^{\widetilde B,\dots,\overset{(j)}{\widetilde B},B,\dots,\overset{(n)}{B}}_{f^{[n]}}(V_{\sigma(1)},\dots,V_{\sigma(n)}) - T^{\widetilde B,\dots,\widetilde B,\overset{(j)}{B},\dots,\overset{(n)}{B}}_{f^{[n]}}(V_{\sigma(1)},\dots,V_{\sigma(n)}) \\
&\qquad = T^{\widetilde B,\dots,\overset{(j)}{\widetilde B},B,\dots,\overset{(n+1)}{B}}_{f^{[n+1]}}(V_{\sigma(1)},\dots,V_{\sigma(j)},V_{n+1},V_{\sigma(j+1)},\dots,V_{\sigma(n)}).
\end{aligned}
$$

Since the multiple operator integral on the right hand side minus the same multiple operator integral with the last $\widetilde B$ replaced by B has the order of $o((\max \|V_j\|)^{n+2})$ by Duhamel's formula, we see that the theorem is proved.

That the value of the derivative (5.5) belongs to $\mathcal{E}$ follows from Lemma 4.6. ■

https://doi.org/10.4153/CJM-2009-012-0 Published online by Cambridge University Press

The argument of the last proof and Lemma 4.5 implies the following.

Corollary 5.8 *Let $\mathcal{N}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$. If $B = B^*$ is affiliated with $\mathcal{N}$, if $V \in \mathcal{E}_{sa}$ and if $f \in W_{n+1}(\mathbb{R})$, then*

$$f(B+V) - f(B) = T^{B,B}_{f^{[1]}}(V) + T^{B,B,B}_{f^{[2]}}(V,V) + \cdots + T^{B,\dots,B}_{f^{[n]}}(V,\dots,V) + O(\|V\|^{n+1}_{\mathcal{E}}).$$

Proof This corollary is a consequence of Theorem 5.7 and Taylor's formula [38, Theorem 1.43]. ∎

6 Spectral Shift and Spectral Averaging

The aim of this section is to prove a semifinite extension of a formula for spectral averaging due to Birman and Solomyak [4].

We first recall the following extension of the spectral shift formula of M. G. Krein from [2, Theorem 3.1].

Theorem 6.1 *If $B = B^*$ is affiliated with $\mathcal{N}$ and $V = V^* \in \mathcal{L}^1(\mathcal{N}, \tau)$, then there exists a unique function $\xi = \xi_{B+V,B}(\cdot) \in L^1(\mathbb{R})$ such that*

$$\|\xi\|_1 \leqslant \|V\|_1, \quad \int_{-\infty}^{\infty} \xi(\lambda)\, d\lambda = \tau(V),$$

$$-\tau(\operatorname{supp}(V_-)) \leqslant \xi(\lambda) \leqslant \tau(\operatorname{supp}(V_+)) \quad \textit{for a.e. } \lambda \in \mathbb{R},$$

and for any function $f \in C^1(\mathbb{R})$ whose derivative f' admits the representation

$$f'(\lambda) = \int_{-\infty}^{\infty} e^{-i\lambda t}\, dm(t), \ \lambda \in \mathbb{R}$$

for some finite (complex) Borel measure on $\mathbb{R}$, the operator $f(B+V) - f(B)$ is τ-trace class and

$$\tau\big(f(B+V) - f(B)\big) = \int_{-\infty}^{\infty} f'(\lambda)\xi(\lambda)\, d\lambda.$$

The function $\xi_{B+V,B}(\cdot)$ is called the *Krein spectral shift function* for the pair $(B+V, B)$.

Lemma 6.2 *If $(\mathcal{N}, \tau)$ is a semifinite von Neumann algebra, $B = B^*$ is affiliated with $\mathcal{N}$, and $V \in \mathcal{L}^1(\mathcal{N}, \tau)$, then the function $\gamma(\lambda, r) = \tau\left(VE^{B_r}_\lambda\right)$ is measurable, where $B_r := B + rV, \ r \in [0,1]$.*

Proof Let $\phi_{\lambda,\varepsilon}$ be a smooth approximation of $\chi_{(-\infty,\lambda]}$. We note that $\phi_{\lambda,\varepsilon}(B) = \phi_{0,\varepsilon}(B - \lambda)$, and that the unbounded-operator valued function $(\lambda, r) \in \mathbb{R}^2 \mapsto B_r - \lambda$ is resolvent uniformly continuous [36, VIII.7]. It follows from [36, Theorem VIII.20(b)] that the function $(\lambda, r) \mapsto \phi_{\lambda,\varepsilon}(B_r)$ is *so*-continuous, so that Lemma 2.5 implies that the function $(\lambda, r) \mapsto \tau\left(V\phi_{\lambda,\varepsilon}(B_r)\right)$ is continuous. Now, since $\phi_{\lambda,\varepsilon} \to \chi_{(-\infty,\lambda]}$ pointwise as $\varepsilon \to 0$, the operator $\phi_{\lambda,\varepsilon}(B_r) \to \chi_{(-\infty,\lambda]}(B_r)$ in *so*-topology. Hence, again by Lemma 2.5, the function $\tau\left(V\chi_{(-\infty,\lambda]}(B_r)\right)$ is measurable. ∎

Theorem 6.3 *Let $(\mathcal{N},\tau)$ be a semifinite von Neumann algebra on a Hilbert space $\mathcal{H}$ with a faithful normal semifinite trace τ. Let $B = B^*$ be affiliated with $\mathcal{N}$ and let $V = V^* \in \mathcal{L}^1(\mathcal{N},\tau)$. If $f \in W_2(\mathbb{R})$, then $f(B+V) - f(B) \in \mathcal{L}^1(\mathcal{N},\tau)$ and*

$$\tau(f(B+V) - f(B)) = \int_{\mathbb{R}} f'(\lambda)\, d\Xi(\lambda),$$

where the measure Ξ is given by

$$\Xi(a,b) = \int_0^1 \tau(VE^{B_r}_{(a,b)})\, dr, \quad a,b \in \mathbb{R}.$$

Here $B_r := B + rV$, $r \in [0,1]$ and $dE^{B_r}_\lambda$ is the spectral measure of B_r.

Due to Lemma 6.2, the measure Ξ is well defined.

Proof If $\phi(\lambda,\mu) = \alpha(\lambda)\beta(\mu)$, where α,β are continuous bounded functions on $\mathbb{R}$, then by the definition of the multiple operator integral $T^{B,B}_\phi(V) = \alpha(B)V\beta(B)$. Hence,

$$\tau\big(T^{B,B}_\phi(V)\big) = \tau(\alpha(B)V\beta(B)) = \tau(\alpha(B)\beta(B)V).$$

Since the function $\alpha(\cdot)\beta(\cdot)$ is bounded, the simple spectral approximations to the bounded operator $\alpha(B)\beta(B)$ converge uniformly and so, after multiplying by V, converge in norm of $\mathcal{L}^1(\mathcal{N},\tau)$. This implies that

$$\tau(\alpha(B)\beta(B)V) = \tau\left(\int_{\mathbb{R}} \alpha(\lambda)\beta(\lambda)\, dE^B_\lambda V\right) = \int_{\mathbb{R}} \alpha(\lambda)\beta(\lambda)\tau\left(dE^B_\lambda V\right).$$

Hence, for functions of the form $\phi(\lambda,\mu) = \alpha(\lambda)\beta(\mu)$, it follows that

$$\tau\left(T^{B,B}_\phi(V)\right) = \int_{\mathbb{R}} \phi(\lambda,\lambda)\tau\left(dE^B_\lambda V\right). \tag{6.1}$$

Let (S,Σ,ν) be a finite (complex) measure space, let $\alpha(\,\cdot\,,\,\cdot\,)$, $\beta(\,\cdot\,,\,\cdot\,)$ be bounded continuous functions on $\mathbb{R}\times S$ and suppose that

$$\phi(\lambda,\mu) = \int_S \alpha(\lambda,\sigma)\beta(\lambda,\sigma)\, d\nu(\sigma) \quad \text{for all } (\lambda,\mu) \in \mathbb{R}^2$$

is a representation of ϕ given by (4.1). Let $\phi_\sigma(\lambda,\mu) := \alpha(\lambda,\sigma)\beta(\mu,\sigma)$. It then follows from the definition of the multiple operator integral that

$$T^{B,B}_\phi(V) = \int_S T^{B,B}_{\phi_\sigma}(V)\, d\nu(\sigma),$$

and hence by Corollary 4.8

$$\tau\left(T^{B,B}_\phi(V)\right) = \int_S \tau\left(T^{B,B}_{\phi_\sigma}(V)\right)\, d\nu(\sigma).$$

https://doi.org/10.4153/CJM-2009-012-0 Published online by Cambridge University Press

It follows from (6.1) that

$$\tau\left(T_{\phi}^{B,B}(V)\right) = \int_S \int_{\mathbb{R}} \phi_\sigma(\lambda,\lambda)\tau\left(dE_\lambda^B V\right)\, d\nu(\sigma) \tag{6.2}$$
$$= \int_{\mathbb{R}} \int_S \phi_\sigma(\lambda,\lambda)\, d\nu(\sigma)\tau\left(dE_\lambda^B V\right)$$
$$= \int_{\mathbb{R}} \phi(\lambda,\lambda)\tau\left(dE_\lambda^B V\right).$$

The interchange of integrals in the second equality is justified by Lemma 2.6 and Fubini's theorem. Further, since $f \in W_2(\mathbb{R})$, it follows from Theorem 5.5 applied to $\mathcal{E} = \mathcal{L}^1(\mathcal{N},\tau)$ that the Fréchet derivative $D_{\mathcal{L}^1} f(B_r) = T_{f^{[1]}}^{B_r,B_r}$ exists for all $r \in [0,1]$. By the continuity of the Fréchet derivative given by the estimate (5.3) and the Newton–Leibnitz formula for the Fréchet derivative (see *e.g.*, [38, Theorem 1.43]) it follows that

$$\int_0^1 T_{f^{[1]}}^{B_r,B_r}(V)\, dr = \int_0^1 D_{\mathcal{L}^1} f(B_r)(V)\, dr = f(B+V) - f(B).$$

By Corollary 4.8 and taking traces it follows that

$$\int_0^1 \tau\left(T_{f^{[1]}}^{B_r,B_r}(V)\right)\, dr = \tau(f(B+V) - f(B)). \tag{6.3}$$

Since $f^{[1]}$ is continuous, $f^{[1]}(\lambda,\lambda) = f'(\lambda)$, so that (6.3) and (6.2) imply

$$\tau(f(B+V) - f(B)) = \int_0^1 \int_{\mathbb{R}} f^{[1]}(\lambda,\lambda)\tau\left(dE_\lambda^{B_r} V\right)\, dr$$
$$= \int_0^1 \int_{\mathbb{R}} f'(\lambda)\tau\left(dE_\lambda^{B_r} V\right)\, dr$$
$$= \int_{\mathbb{R}} f'(\lambda) \int_0^1 \tau\left(dE_\lambda^{B_r} V\right)\, dr,$$

the interchange of the integrals in the last equality being justified by Fubini's theorem [24, VI.2] via Lemma 2.6 and the fact that f' is a bounded function. ■

The next corollary in the case that $\mathcal{N} = \mathcal{B}(\mathcal{H})$ and $\tau = \mathrm{Tr}$ was established in [4].

Corollary 6.4 *The measure Ξ is absolutely continuous, and the equality $d\Xi(\lambda) = \xi(\lambda)\, d\lambda$ holds, where $\xi(t)$ is the spectral shift function for the pair $(B+V, B)$.*

Proof From Theorems 6.1 and 6.3, it follows that

$$\int_{\mathbb{R}} f'(\lambda)\, d\Xi(\lambda) = \int_{\mathbb{R}} f'(\lambda)\xi(\lambda)\, d\lambda$$

for all $f \in C_c^\infty(\mathbb{R})$. Consequently, the measures $d\Xi(\lambda)$ and $\xi(\lambda)\, d\lambda$ have the same derivative in the sense of generalized functions. By [22, Ch. I.2.6] there exists a constant c such that $d\Xi(\lambda) - \xi(\lambda)\, d\lambda = c \cdot d\lambda$. Since the measures $d\Xi(\lambda)$ and $\xi(\lambda)\, d\lambda$ are finite, it follows immediately that $c = 0$. ■

7 Spectral Shift and Spectral Flow

The second author and J. Phillips have established various analytic formulae for spectral flow along a path of self-adjoint unbounded Breuer–Fredholm operators affiliated with a semifinite von Neumann algebra. For special choices of path suggested by the theory of the Krein spectral shift function, one may study a spectral flow function on the real line $\mu \mapsto \mathrm{sf}(\mu, D_0, D_1)$, $\mu \in \mathbb{R}$, where D_1 and D_0 differ by a τ-trace class operator and the function measures spectral flow from $D_0 - \mu$ to $D_1 - \mu$. We now show that under these circumstances the spectral flow function actually coincides with the Krein spectral shift function.

Let us first recall preliminary material about spectral flow. For more details see [9, 33, 34]. In these papers the notion of type II spectral flow is introduced and an analytic approach is developed starting from ideas of Getzler [23]. The new approach of these papers allows the study of spectral flow between certain unbounded Breuer–Fredholm operators affiliated with a general semifinite von Neumann algebra [9]. We summarize the main features.

Let $\mathcal{N}$ be a semifinite von Neumann algebra with a faithful normal semifinite trace τ and $P, Q \in \mathcal{N}$ be two infinite projections. Let $\ker_Q T := \ker T \cap Q(\mathcal{H})$. An operator $T \in P\mathcal{N}Q$ is said to be (P, Q) τ-Fredholm if the subspaces $\ker_Q T$ and $\ker_P T^*$ are τ-finite and there exists a projection $P_1 \in \mathcal{N}$ such that $P_1 \leq P$, $\tau(P - P_1) < \infty$ and $P_1(\mathcal{H}) \subset T(\mathcal{H})$. In this case (P, Q)-index of the operator T is defined to be a number

$$\tau\text{-ind}_{P\text{-}Q}(T) := \tau[\ker_Q T] - \tau[\ker_P T^*].$$

Here $[\mathcal{K}]$ denotes the projection onto the subspace $\mathcal{K} \subseteq \mathcal{H}$. If $P = Q = 1$ we call T just τ-Fredholm. For details see [7, 8, 35].

Now, let $F\colon t \in [a, b] \mapsto F_t \in \mathcal{N}$ be a norm continuous path of τ-Fredholm operators and $P_t = \frac{1}{2}(1 + \mathrm{sign}(F_t))$. If a partition $t_0 = a < t_1 < \cdots < t_n = b$ of the segment $[a, b]$ is sufficiently small, then the operators $P_{j-1}P_j\colon P_j\mathcal{H} \to P_{j-1}\mathcal{H}$, $P_{j-1}P_j \in P_{j-1}\mathcal{N}P_j$, are (P_{j-1}, P_j) τ-Fredholm for $j = 1, \dots, n$ $(P_j := P_{t_j})$, so that the number

$$\mathrm{sf}(\{F_t\}) := \sum_{j=1}^{n} \tau\text{-ind}_{P_{j-1}\text{-}P_j}(P_{j-1}P_j)$$

is well defined and does not depend on the partition $\{t_j, j = 1, \dots, n\}$. Further, if two paths $\{F_t\}$ and $\{G_t\}$ with the same ends points are norm homotopic, then $\mathrm{sf}(\{F_t\}) = \mathrm{sf}(\{G_t\})$, so that the spectral flow $\mathrm{sf}(F_0, F_1)$ depends only on the endpoints.

We recall the definition of a semifinite spectral triple (see *e.g.*, [10]).

Definition 7.1 A semifinite spectral triple $(\mathcal{A}, \mathcal{N}, D)$ is given by a Hilbert space $\mathcal{H}$, a $*$-algebra $\mathcal{A} \subset \mathcal{N}$ where $\mathcal{N}$ is a semifinite von Neumann algebra acting on $\mathcal{H}$, and a densely defined unbounded self-adjoint operator D affiliated to $\mathcal{N}$ such that

(1) $[D, a]$ is densely defined and extends to a bounded operator for all $a \in \mathcal{A}$;
(2) $(\lambda - D)^{-1} \in \mathcal{K}(\mathcal{N}, \tau)$ for all $\lambda \notin \mathbb{R}$, where $\mathcal{K}(\mathcal{N}, \tau)$ is the set of all τ-compact operators.

A spectral triple $(\mathcal{A}, \mathcal{N}, D)$ is said to be θ-summable if for all $t > 0$ the operator e^{-tD^2} is τ-trace class.

Let $(\mathcal{A}, \mathcal{N}, D)$ be a θ-summable semifinite spectral triple, let $V \in \mathcal{L}^1(\mathcal{N}, \tau)$ and let

$$D_r = D + rV, \quad r \in [0, 1].$$

The Carey–Phillips formula [9] for spectral flow between $D_0 = D$ and $D_1 = D + V$ is given by

$$\mathrm{sf}(D_0, D_1) = \sqrt{\frac{\varepsilon}{\pi}} \int_0^1 \tau\left(Ve^{-\varepsilon D_r{}^2}\right) dr + \frac{1}{2}\left(\eta_\varepsilon(D_1) - \eta_\varepsilon(D_0)\right) + \frac{1}{2}\tau\left([\ker(D_1)] - [\ker(D_0)]\right),$$

where the η_ε-invariant of an unbounded self-adjoint operator D, such that e^{-tD^2} is τ-trace class for all $t > 0$, is defined as [9, Definition 8.1]

$$\eta_\varepsilon(D) := \frac{1}{\sqrt{\pi}} \int_\varepsilon^\infty \tau\left(De^{-tD^2}\right) t^{-1/2}\, dt.$$

Spectral flow may be interpreted as the "net amount" of spectrum crossing zero while moving from D_0 to D_1. So, it is natural to define the function $\mathrm{sf}(\lambda; D_0, D_1) := \mathrm{sf}(D_0 - \lambda, D_1 - \lambda)$ as spectral flow at a point λ.

It follows from the Carey–Phillips formula that

(7.1)
$$\mathrm{sf}(\mu; D_0, D_1) = \sqrt{\frac{\varepsilon}{\pi}} \int_0^1 \tau\left(Ve^{-\varepsilon(D_r - \mu)^2}\right) dr + \frac{1}{2}\left(\eta_\varepsilon(D_1 - \mu) - \eta_\varepsilon(D_0 - \mu)\right) + \frac{1}{2}\tau\left([\ker(D_1 - \mu)] - [\ker(D_0 - \mu)]\right), \quad \mu \in \mathbb{R}.$$

The following theorem establishes a connection between the spectral shift function for the pair $(D_0 + V, D_0)$ and the spectral flow function $\mathrm{sf}(\,\cdot\,, D_0, D_0 + V)$.

Theorem 7.2 *If $V \in \mathcal{N}$ belongs to the τ-trace class, then*

$$\mathrm{sf}(\mu; D_0, D_1) = \xi_{D_1, D_0}(\mu) + \frac{1}{2}\tau\left([\ker(D_1 - \mu)] - [\ker(D_0 - \mu)]\right)$$

for almost all $\mu \in \mathbb{R}$.

Proof The spectral theorem and Lemma 2.5 imply

$$\sqrt{\frac{\varepsilon}{\pi}}\tau\left(Ve^{-\varepsilon(D_r - \mu)^2}\right) = \sqrt{\frac{\varepsilon}{\pi}}\tau\left(V \int_{\mathbb{R}} e^{-\varepsilon(\lambda - \mu)^2}\, dE_\lambda^{D_r}\right) = \int_{\mathbb{R}} j_\varepsilon(\lambda - \mu)\, \tau\left(V dE_\lambda^{D_r}\right),$$

https://doi.org/10.4153/CJM-2009-012-0 Published online by Cambridge University Press

where $j_\varepsilon(x) = \sqrt{\frac{\varepsilon}{\pi}} e^{-\varepsilon x^2}$, $x \in \mathbb{R}$. By Corollary 6.4 and using the fact that the system $\{j_\varepsilon\}$ is an approximate identity, we obtain that

$$\int_0^1 \int_{\mathbb{R}} j_\varepsilon(\lambda - \mu)\, \tau\left(V dE_\lambda^{D_r}\right)\, dr = \int_{\mathbb{R}} j_\varepsilon(\lambda - \mu) \frac{d}{d\lambda} \int_0^1 \tau\left(V E_\lambda^{D_r}\right)\, dr\, d\lambda$$
$$= j_\varepsilon * \xi_{D_1,D_0}(\mu) \xrightarrow{L^1} \xi_{D_1,D_0}(\mu) \quad \text{when} \quad \varepsilon \to \infty.$$

The convergence in the last line can be justified by [37, Chapter 3, Section 5.6].

Since $\eta_\varepsilon(D_j - \mu) \to 0$, $j = 0, 1$, when $\varepsilon \to \infty$, it follows from the Carey–Phillips formula (7.1) that

$$\operatorname{sf}(\mu; D_0, D_1) = \xi_{D_1,D_0}(\mu) + \frac{1}{2}\tau\left([\ker(D_1 - \mu)] - [\ker(D_0 - \mu)]\right), \quad \mu \in \mathbb{R}. \quad \blacksquare$$

References

[1] M. F. Atiyah, V. K. Patodi, and I. M. Singer, *Spectral asymmetry and Riemannian geometry. III.* Math. Proc. Cambridge Philos. Soc. **79**(1976), no. 1, 71–99.
[2] N. A. Azamov, P. G. Dodds, and F. A. Sukochev, *The Krein spectral shift function in semifinite von Neumann algebras.* Integral Equations Operator Theory **55**(2006), no. 3, 347–362.
[3] M. Sh. Birman and A. B. Pushnitski, *Spectral shift function, amazing and multifaceted. Dedicated to the memory of Mark Grigorievich Krein (1907–1989).* Integral Equations Operator Theory **30**(1998), no. 2, 191–199.
[4] M. Sh. Birman and M. Z. Solomyak, *Remarks on the spectral shift function.* J. Soviet Math.**3**(1975), 408–419.
[5] ———, *Double Stieltjes operator integrals. I.* Izdat. Leningrad Univ., Leningrad, 1973, 27–53.
[6] O. Bratteli and D. Robinson, *Operator algebras and quantum statistical mechanics. I.* Springer-Verlag, New York-Berlin, 1979.
[7] M. Breuer, *Fredholm theories in von Neumann algebras. I.* Math. Ann. **178**(1968), 243–254.
[8] ———, *Fredholm theories in von Neumann algebras. II.* Math. Ann. **180**(1969), 313–325.
[9] A. L. Carey and J. Phillips, *Spectral flow in Fredholm modules, eta invariants and the JLO cocycle.* K-Theory **31**(2004), no. 2, 135–194.
[10] A. L. Carey, J. Phillips, A. Rennie, and F. A. Sukochev, *The Hochschild class of the Chern character of semifinite spectral triples.* J. Funct. Anal. **213**(2004), no. 1, 111–153.
[11] ———, *The local index formula in semifinite von Neumann algebras I. Spectral flow.* Adv. Math. **202**(2006), no. 2, 451–516.
[12] ———, *The local index formula in semifinite von Neumann algebras I. The even case.* Adv. Math. **202**(2006), no. 2, 517–554.
[13] A. L. Carey, J. Phillips, and F. A. Sukochev, *Spectral flow and Dixmier traces.* Adv. Math. **173**(2003), no. 1, 68–113.
[14] A. Connes, *Noncommutative Geometry.* Academic Press, San Diego, CA, 1994.
[15] A. Connes and H. Moscovici, *The local index formula in noncommutative geometry.* Geom. Funct. Anal. **5**(1995), no. 2, 174–243.
[16] Y. L. Daletskii and S. G. Krein, *Integration and differentiation of functions of Hermitian operators and applications to the theory of perturbations.* Voronež. Gos. Univ. Trudy Sem. Funkcional. Anal. **1956**(1956), no. 1, 81–105.
[17] J. Dixmier, *von Neumann Algebras.* North-Holland Mathematical Library 27, North-Holland Publishing Co., Amsterdam-New York, 1981.
[18] P. G. Dodds, T. K. Dodds, and B. de Pagter, *Noncommutative Köthe duality.* Trans. Amer. Math. Soc. **339**(1993), no. 2, 717–750.
[19] P. G. Dodds, T. K. Dodds, B. de Pagter, and F. A. Sukochev, *Lipschitz continuity of the absolute value and Riesz projections in symmetric operator spaces.* J. Funct. Anal. **148**(1997), no. 1, 28–69.
[20] N. Dunford and J. T. Schwartz, *Linear Operators, Part I: General theory.* John Wiley and Sons, Inc., New York, 1988.
[21] T. Fack and H. Kosaki, *Generalised s-numbers of τ-measurable operators.* Pacific J. Math. **123**(1986), no. 2, 269–300.

[22] I. M. Gelfand and G. E. Shilov, *Generalized functions. Vol. 1. Properties and operations.* Academic Press, New York-London, 1964.
[23] E. Getzler, *The odd Chern character in cyclic homology and spectral flow.* Topology **32**(1993), no. 3, 489–507.
[24] K. Jacobs, *Measure and integral. Probability and mathematical statistics.* Academic Press, New York-London, 1978.
[25] M. G. Krein, *On the trace formula in perturbation theory.* Mat. Sbornik N.S. **33(75)**(1953), 597–626.
[26] ———, *Some new studies in the theory of perturbations of self-adjoint operators.* First Math. Summer School, Part I, Izdat Naukova Dumka, Kiev, 1964, pp. 103–187.
[27] S. G. Krein, J. I. Petunin, and E. M. Semenov, *Interpolation of linear operators.* Translations of Mathematical Monographs 54, American Mathematical Society, Providence, RI, 1982.
[28] I. M. Lifshits, *On a problem in the theory of perturbations connected with quantum statistics.* Uspekhi Mat. Nauk **7**(1952), 171–180.
[29] B. de Pagter and F. A. Sukochev, *Differentiation of operator functions in non-commutative L_p-spaces.* J. Funct. Anal. **212**(2004), no. 1, 28–75.
[30] B. de Pagter, H. Witvliet, and F. A. Sukochev, *Double operator integrals.* J. Funct. Anal. **192**(2002), no. 1, 52–111.
[31] B. S. Pavlov, *Multidimensional operator integrals.* Problems of Math. Anal. 2: Linear operators and operator equations, Izdat. Leningrad. Univ., Leninegrad, 1969, pp. 99–122.
[32] V. V. Peller, *Multiple operator integrals and higher operator derivatives.* J. Funct. Anal. **233**(2006), no. 2, 515–544.
[33] J. Phillips, *Self-adjoint Fredholm operators and spectral flow.* Canad. Math. Bull. **39**(1996), no. 4, 460–467.
[34] ———, *Spectral flow in type I and type II factors—a new approach.* In: Cyclic cohomology and noncommutative geometry, Fields. Inst. Commun. 17, American Mathematical Society, Providence, RI, 1997, pp. 137–153.
[35] J. Phillips and I. Raeburn, *An index theorem for Toeplitz operators with noncommutative symbol space.* J. Funct. Anal. **120**(1994), no. 2, 239–263.
[36] M. Reed and B. Simon, *Methods of modern mathematical physics: 1. Functional analysis.* Academic Press, New York, 1972.
[37] H. Reiter, *Classical harmonic analysis and locally compact groups.* Clarendon Press, Oxford, 1968.
[38] J. T. Schwartz, *Nonlinear Functional Analysis.* Notes on Mathematics and its Applications, Gordon and Breach Science Publishers, New York-London-Paris, 1969.
[39] M. Z. Solomyak and V. V. Stenkin, *On a class of multiple operator Stieltjes integrals.* Problems of mathematical analysis, Leningrad University Publ. 2, 1969, 122–134.
[40] V. V. Stenkin, *Multiple operator integrals.* Izv. Vyss. Ucebn. Zaved. Matematika **4**(1977), 102–115.
[41] N. N. Vakhaniya, V. I. Tarieladze, and S. A. Chobanyan, *Probability distributions in Banach spaces.* Nauka. Moscow, 1985.
[42] H. Widom, *When are differentiable functions differentiable?* In: Linear and Complex Analysis Problem Book 3, Part 1, Lecture Notes in Mathematics 1573, Springer-Verlag, Berlin-Heidelberg-New York, 1994, pp. 266–271.
[43] K. Yosida, *Functional analysis.* Fundamental Principles of Mathematical Sciences 123, Springer-Verlag, Berlin-New York, 1980.

(Azamov, Dodds) *School of Computer Science, Engineering and Mathematics, Flinders University of South Australia, Bedford Park, 5042, SA, Australia*
e-mail: azam0001@csem.flinders.edu.au
peter@csem.flinders.edu.au

(Carey) *Mathematical Sciences Institute, Australian National University, Canberra, ACT 0200, Australia*
e-mail: acarey@maths.anu.edu.au

(Sukochev) *School of Mathematics and Statistics, University of New South Wales, Kensington, NSW 2052, Australia*
e-mail: f.sukochev@unsw.edu.au