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ASYMPTOTIC PROPERTIES OF SEMILINEAR EQUATIONS

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ABSTRACT. We study the asymptotic properties of positive solutions to the semilinear equation $-\Delta u = f(x, u)$. Existence and asymptotic estimates are obtained for solutions in exterior domains, as well as entire solutions, for $n \geqq 2$. The study uses integral operator equations in R^n, and convergence theorems for solutions of Poisson's equation in bounded domains. A consequence of the method is that more precise estimates can be obtained for the growth of solutions at infinity, than have been obtained by other methods. As a special case the results are applied to the generalized Emden-Fowler equation $-\Delta u = p(x)u^\gamma$, for $\gamma > 0$.

1. **Introduction.** The equation under study is the semilinear elliptic equation

$$-\Delta u = f(x, u),\ x \in \Omega. \tag{1.1}$$

Ω can be either R^n or an exterior domain in R^n, $n \geqq 2$. Δ is the n-dimensional Laplacian, and $f \in C^\alpha_{\text{loc}}(\Omega \times R)$, $0 < \alpha < 1$. We assume $f(x, u) > 0$ for $|x| > 0$ and $u \neq 0$. An important example, and one which has been widely studied, is the Emden-Fowler prototype

$$-\Delta u = p(x)u^\gamma,\ \gamma > 0. \tag{1.2}$$

Linear equations of the form (1.1) have been studied by Meyers and Serrin [6], with respect to the Dirichlet problem. We consider two problems:

(I) Ω an exterior domain, and f either sublinear or superlinear.

(II) $\Omega = R^n$, and f sublinear.

Problems (I) and (II) have been studied by Kusano and Swanson [4], and by Noussair and Swanson [9], using the barrier method. This study will be based on classical integral operator equations in R^n. A solution of the integral equation will be shown to solve (1.1) by means of convergence theorems for solutions of Poisson's equation in bounded domains.

Our basic existence theorems for the exterior domain problem are analogous to those of [4]. The method is of interest, however, because it allows us to obtain more precise estimates for the rate of decay at infinity. As in the case of

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bounded harmonic functions in exterior domains, we derive conditions sufficient for solutions to have limits at infinity. Furthermore, we do not require the assumption of radial symmetry. For the linear problem, this limiting value uniquely specifies the solution to the Dirichlet problem.

DEFINITIONS AND NOTATION

(i) $\Omega_M = \{y \in R^n: M \leqq |y|\}$
(ii) $\Omega_{M,N} = \{y \in R^n: M \leqq |x| \leqq N\}$
(iii) $B_r(x) = \{y \in R^n: |x - y| \leqq r\}$, $B_r = B_r(0)$
(iv) $S_r(x) = \partial B_r(x) = \{y \in R^n: |x - y| = r\}$, $S_r = S_r(0)$
(v) $Q_c(\Omega) = \{q \in C^\alpha_{\mathrm{loc}}(\Omega; I): q(x) < c, x \in \Omega\}$, $I = (0, \infty)$
(vi) $Q_{c,c'}(\Omega) = \{q \in C^\alpha_{\mathrm{loc}}(\Omega; I): 0 < c \leqq u(x) \leqq c'\}$
(vii) $Q^o_c(\Omega) = \{q \in C^\alpha_{\mathrm{loc}}(\Omega; I): \lim_{|x|\to\infty} q(x) = c\}$
(viii) $Q_{f(x),g(x)}(\Omega) = \{q \in C^\alpha_{\mathrm{loc}}(\Omega; I): o < f(x) \leqq q(x) \leqq g(x)\}$
(ix) ω_n = volume of unit ball $= 2\pi^{n/2}/n\Gamma(n/2)$

$$\Gamma(x - y) = \frac{1}{n(2 - n)\omega_n}|x - y|^{2-n}, \; n > 2 \tag{1.3}$$
$$= (1/2\pi)\log|x - y|, \; n = 2.$$

We assume throughout that $f(x, u)$ satisfies the following condition:
(H) There exist functions P, p, $\in C^\alpha_{\mathrm{loc}}(I \times I; I)$ with $P(r, t)$ and $p(r, t)$ either non-increasing or non-decreasing in t, satisfying

$$p(|x|, u) \leqq f(x, u) \leqq P(|x|, u).$$

In Section 2 we obtain conditions sufficient for the existence of solutions to (1.1), positive in an exterior domain, and satisfying the asymptotic conditions

$$o < c < u(x) < c', \; n \geqq 3 \tag{1.4a}$$

$$o < c \log|x| < u(x) < c' \log|x|, \; n = 2. \tag{1.4b}$$

Section 3 concerns the existence of solutions satisfying

$$o < c|x|^{2-n} < u(x) < c'|x|^{2-n}, \; n \geqq 3 \tag{1.5a}$$

$$o < c < u(x) < c', \; n = 2. \tag{1.5b}$$

For linear elliptic equations, solutions to the exterior Dirichlet problem are uniquely determined by their limiting values as $|x| \to \infty$. We will show in section 4 the solutions given here also satisfy the asymptotic conditions $\lim_{|x|\to\infty} u(x) = c$, and $\lim_{|x|\to\infty} |x|^{n-2}u(x) = c$.

In Section 5 we consider the problem of global solutions to (1.1), with $f(x, u)$ sublinear in u. These results generalize those of [4], which required radial symmetry of f.

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2. **Existence of (maximal) positive solutions.** Consider first the case of positive solutions to (1.1), which for $|x|$ sufficiently large satisfy (1.4). Proofs will be given for the case P, p non-decreasing. The non-increasing case is analogous. Define integral operators

$$T[u](x) = c - \int_{\Omega_M} \Gamma(x-y) f(y, u(y))dy, \; n \geqq 3 \tag{2.1a}$$

$$T[u](x) = c \log|x| - \int_{\Omega_M} \log \frac{|x-y|}{|y|} f(y, u(y))dy, \; n = 2. \tag{2.1b}$$

LEMMA 2.1. *The operator T of* (2.1) *maps* $Q_{c,c'}(\Omega_M)$ ($n \geqq 3$) [*resp.* $Q_{c\log|x|, c'\log|x|}(\Omega_M)$ ($n = 2$)] *into itself, for sufficiently large M, provided*

$$\int^{\infty} tP(t, c')dt < \infty, \; (n \geqq 3) \tag{2.2a}$$

$$\int^{\infty} tP(t, c' \log t)dt < \infty, \; (n = 2) \tag{2.2b}$$

PROOF. For $u \in Q_{c,c'}(\Omega_M)$, we have

$$\begin{aligned} T[u](x) &= c - \int_{\Omega_M} \Gamma(x-y) f(y, u(y))dy \\ &\leqq c + \frac{1}{n(n-2)\omega_n} \int_M^{|x|} P(\rho, c') \int_{S_\rho} |x-y|^{2-n} ds d\rho \\ &+ \frac{1}{n(n-2)\omega_n} \int_{|x|}^{\infty} P(\rho, c') \int_{S_\rho} |x-y|^{2-n} ds d\rho \\ &\leqq c + \frac{1}{n(n-2)\omega_n} \left(\int_M^{|x|} \rho^{n-1} |x|^{2-n} P(\rho, c')d\rho + \int_{|x|}^{\infty} \rho P(\rho, c')d\rho \right) \\ &\leqq c + \frac{1}{n(n-2)\omega_n} \int_M^{\infty} \rho P(\rho, c')d\rho < c' \end{aligned}$$

provided M is sufficiently large. Note that if $P(r, t)$ and $p(r, t)$ are non-increasing in t, the proof is completely analogous.

The case $n = 2$ is significantly more complicated, due to the non-positivity of the fundamental solution. We define

$$\Omega^+ [\text{resp. } \Omega^-] = \left\{ y \colon \log \frac{|x-y|}{|y|} > 0 \; [\text{resp. } < 0] \right\}. \tag{2.5}$$

Then Ω^+ is a half plane on which the inequality

$$\frac{|x-y|}{|y|} \leqq \frac{|x| + |y|}{|y|} \leqq \frac{|x| + M}{M}, \; y \in \Omega^+$$

is satisfied. Therefore for $u \in Q_{c\log|x|, c'\log|x|}(\Omega_M)$,

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$$\int_{\Omega_M} \log \frac{|x-y|}{|y|} f(y, u(y))dy < \int_{\Omega^+} \log \frac{|x-y|}{|y|} f(y, u(y))dy$$

$$< \log \frac{|x|+M}{M} \int_{\Omega+} P(|y|, u(y))dy$$

$$\leqq 2\pi \log \frac{|x|+M}{M} \int_M^\infty \rho P(\rho, c' \log \rho)d\rho$$

and this is less than $(c'-c)\log|x|$ if M is sufficiently large.

To obtain a lower bound we define

$$(2.6) \qquad \Omega_+[\text{resp. } \Omega_-] = \{y \in \Omega_M: \log|x-y| > 0 \; [\text{resp.} < 0]\}.$$

In the region $M \leqq |y| \leqq 2|x|$ we can write

$$\int \log \frac{|x-y|}{|y|} f(y, u(y))dy = \int \log|x-y|\, f(y, u(y))dy - \int \log|y|\, f(y, u(y))dy,$$

and therefore

$$\int_{\Omega_M} \log \frac{|x-y|}{|y|} f(y, u(y))dy = \int_{\substack{y\in\Omega_+ \\ |y|<2|x|}} \log|x-y|\, f(y, u(y))dy$$

$$- \int_{\substack{y\in\Omega_+ \\ |y|<2|x|}} \log|y|\, f(y, u(y))dy + \int_{2|x|<|y|} \log \frac{|x-y|}{|y|} f(y, u(y))dy$$

$$+ \int_{\Omega_-} \log \frac{|x-y|}{|y|} f(y, u(y)) = I_1 - I_2 + I_3 + I_4.$$

We can estimate these integrals as follows:

(i) $$I_1 > 2\pi \log|x| \int_M^{|x|-1} \rho p(\rho, c \log \rho)d\rho$$

(ii) $$I_2 < 2\pi \log|x| \int_M^{|x|} \rho P(\rho, c' \log \rho)d\rho + 2\pi \log 2 \int_{|x|}^{2|x|} \rho P(\rho, c' \log \rho)d\rho$$

(iii) For $2|x| < |y| < \infty$ we have

$$\frac{1}{2} \leqq \frac{|y|-|x|}{|y|} \leqq \frac{|x-y|}{|y|} \leqq \frac{|y|+|x|}{|y|} \leqq \frac{3}{2}.$$

Therefore

$$|I_3| \leqq 2\pi \log 2 \int_{2|x|}^{\infty} \rho P(\rho, c' \log \rho) d\rho$$

(iv)
$$I_4 = \int_{\Omega_-} \log|x - y| \, f(y, u(y))dy$$
$$- \int_{\Omega_-} \log|y| \, f(y, u(y))dy.$$

The first integral in (iv) is $o(|x|)$ and $-\int \log|y| \, f(y, u(y))dy > 0$. It follows that for M sufficiently large, $T[u](x) > c \log|x|$, and this establishes the inequalities $c \log|x| < u(x) < c' \log|x|$.

LEMMA 2.2. *Under the hypotheses of Lemma* 2.1, *T is continuous and compact, with respect to the topology of uniform convergence on compact sets.*

PROOF. Let $\{u_m\}$ be a sequence in $Q_{c,c'}(\Omega_M)$ with $\lim_{m\to\infty} u_m = u$. Then

$$|T[u_m](x) - T[u](x)|$$
$$\leqq \int_{\Omega_M} \Gamma(x - y)|f(y, u_m(y)) - f(y, u(y))|dy$$
$$= \int_{\Omega_{M,N}} \Gamma(x - y)|f(y, u_m(y)) - f(y, u(y))|dy$$
$$+ \int_{\Omega_N} \Gamma(x - y)|f(y, u_m(y)) - f(y, u(y))|dy, \ 0 < M < N.$$

The second integral is

$$\leqq 2 \int_{\Omega_N} \Gamma(x - y)P(|y|, c')dy < \frac{\epsilon}{2}$$

provided N is sufficiently large. Since $f \in C^{\alpha}_{\text{loc}}$ there is a constant K, depending on N, such that $|f(y, u_m(y)) - f(y, u(y))| \leqq K|u_m(y) - u(y)|^{\alpha}$, uniformly in $\Omega_{M,N}$. Let m_o be so large that for

$$m > m_o, \ |u_m - u|^{\alpha} < \frac{\epsilon}{2}\left[K \int_{\Omega_{M,N}} \Gamma(x - y)dy\right]^{-1}.$$

Then the first integral is less than $\epsilon/2$, and this shows that T is continuous.

We will show that T is compact. Clearly, the image of T is uniformly bounded. The Ascoli-Arzèla theorem can be applied if the image is equicontinuous in Ω_M. For $x, x' \in \Omega_M$, let $S = |x - x'|$, $\xi = (1/2)(x + x')$. Then for

$$u \in Q_{c,c'}(\Omega_M), \ T[u](x) - T[u](x')$$
$$= \int_{\Omega_M} (\Gamma(x - y) - \Gamma(x' - y))f(y, u(y))dy = I_1 + I_2,$$

where

$$I_1 = \int_{B_\delta(\xi)} (\Gamma(x - y) - \Gamma(x' - y))f(y, u(y))dy$$

$$I_2 = \int_{\Omega_M \setminus B_\delta(\xi)} (\Gamma(x - y) - \Gamma(x' - y))f(y, u(y))dy.$$

Estimating I_2 we obtain

$$|I_2| \leqq \frac{2}{n(n-2)\omega_n} \int_{\Omega_M \setminus B_\delta(x)} |x - y|^{2-n} f(y, u(y))dy$$

$$\leqq \frac{2}{n(n-2)\omega_n} \frac{\delta^{n-2}}{2} \int_M^\infty \int_{S_\rho} P(\rho, c')d\sigma d\rho.$$

To estimate I_1 we write $B_\delta(\xi) = B_{\delta/4}(x)B_{\delta/4}(x')U\Omega$, where $\Omega = B_\delta(\xi) \setminus (B_{\delta/4}(x) \cup B_{\delta/4}(x'))$. So

$$I_1 = \left(\int_{B_{\delta/4}(x)} + \int_{B_{\delta/4}(x')} + \int_\Omega\right)(\Gamma(x - y) - \Gamma(x' - y))f(y, u(y))dy$$

$$= I_3 + I_4 + I_5.$$

$$|I_3| = \int_{B_{\delta/4}(x)} |\Gamma(x - y) - \Gamma(x' - y)| f(y, u(y))dy$$

$$\leqq \frac{1}{n(n-2)\omega_n} \int_{B_{\delta/4}(x)} |x - y|^{2-n} P(|y|, c')dy$$

$$+ \frac{1}{n(n-2)\omega_n} \int_{B_{\delta/4}(x)} |x' - y|^{2-n} P(|y|, c')dy.$$

The classical arguments show that these integrals tend to zero as δ tends to zero, uniformly with respect to x, x' in compact sets.

For $n = 2$, the proof of Lemma 2.2 is similar, and will be omitted.

THEOREM 2.1. *Equation* (1.1) *has a solution u in* $Q_{c,c'}(\Omega_M) \cap C^{2+\alpha}(\Omega_M)$ $(n \geqq 3)$ [*resp.* $Q_{c\log|x|,c'\log|x|}(\Omega_M) \cap C^{2+\alpha}(\Omega_M)$ $(n = 2)$] *provided* (2.2) *is satisfied and M is sufficiently large.*

PROOF. By Lemmas 2.1 and 2.2 T is a continuous, compact mapping of $Q_{c,c'}(\Omega_M)$ $(n \geqq 3)$ [resp. $Q_{c\log|x|,c'\log|x|}(\Omega_M)$ $(n = 2)$] into itself. These are closed, convex sets, and by Schauder fixed point theorem there exists a solution of the integral equation $u(x) = T[u](x)$, $x \in \Omega_M$. We will show that $u \in C^{2+\alpha}_{\mathrm{loc}}(\Omega_M)$ and that u is a solution of (1.1).

Because u is a continuous solution of the integral equation, it is certainly $C^\alpha_{\mathrm{loc}}(\Omega_M)$. We will express u as the limit of a sequence of solutions of the Poisson equation

$$\Delta v + f(x, u(x)) = 0. \tag{2.7}$$

Fix $N_o > M$, and for $N > N_o$ let u_N be the solution of (2.7) defined by

$$u_N(x) = \int_{\Omega_{M,N}} \Gamma(x-y)f(y, u(y))dy.$$

Then $u_N \in C^{2+\alpha}(\Omega_{M,N_o})$ and by Corollary 4.7 of [6], some subsequence of $\{u_N\}$ converges uniformly on Ω_{M,N_o} to a solution of (1.1). But, the entire sequence $\{u_N\}$ converges uniformly to u on Ω_{M,N_o}. To verify this,

$$|u(x) - u_N(x)| = \int_{\Omega_N} \Gamma(x-y)f(y, u(y))dy \leqq \int_{\Omega_N} \Gamma(x-y)P(|y|, c')dy$$

and this $\to 0$ as $N \to \infty$. It follows that u satisfies (1.1) in Ω_{M,N_o}, and hence in all of Ω_M.

3. **Existence of (minimal) positive solutions.** In this section we discuss briefly the question of the existence of positive solutions to (1.1) satisfying the asymptotic conditions

(3.1a) $$o < c|x|^{2-n} < u(x) < c'|x|^{2-n},\ n \geqq 3$$

(3.1b) $$o < c < u(x) < c',\ n = 2.$$

The methods are similar to those used in Section 2, so many details will be omitted. We again define integral operators

(3.2a) $$S[u](x) = c|x|^{2-n} - \int_{\Omega_M} \Gamma(x-y)f(y, u(y))dy,\ n \geqq 3$$

(3.2b) $$S[u](x) = c + \int_{\Omega_M} \log\frac{|x-y|}{|x|} f(y, u(y))dy,\ n = 2.$$

For $n \geqq 3$ we work in the space $Q_{c|x|^{2-n},c'|x|^{2-n}}$, topologized by $\|u\| = \sup_{|x|\geqq M} |u(x)|\,|x|^{n-2}$, and for $n = 2$ the space $Q_{c,c'}$.

LEMMA 3.1. *The operator S is a continuous, compact mapping of $Q_{c|x|^{2-n},c'|x|^{2-n}}(\Omega_M)$ $(n \geqq 3)$ [resp. $Q_{c,c'}(\Omega_M)$ $(n = 2)$] provided M is sufficiently large and*

(3.3a) $$\int^{\infty} t^{n-1}P(t, c't^{2-n})dt < \infty\ (n \geqq 3)$$

(3.3b) $$\int^{\infty} t \log tP(t, c')dt < \infty\ (n = 2).$$

THEOREM 3.1. *Equation* (1.1) *has a positive solution satisfying* (3.1) *in an exterior domain Ω_M, provided M is sufficiently large and* (3.3) *is satisfied.*

4. **Existence of limits at infinity.** For linear elliptic equations, solutions to the exterior Dirichlet problem are uniquely determined by their limiting values as $|x| \to \infty$. We will show that for $n \geqq 3$, solutions given by Theorem 2.1 satisfy

(4.1) $$\lim_{|x|\to\infty} u(x) = c,$$

and that those given by Theorem 3.1 satisfy

(4.2) $$\lim_{|x|\to\infty} |x|^{2-n}u(x) = c.$$

Considering first the case of maximal solutions, recall that the solution given by Theorem 2.1 satisfies the integral equation

$$u(x) = c - \int_{\Omega_M} \Gamma(x - y)f(y, u(y))dy.$$

Assuming the growth condition (2.2a), we have

(4.3) $$\lim_{|x|\to\infty} \int_{\Omega_M} \Gamma(x - y)f(y, u(y))dy = 0.$$

First observe that for $\epsilon_1 > 0$, there exists an $N_1(\epsilon_1) > M$ such that

(4.4) $$\int_{\Omega_{N_1}} \Gamma(x - y)f(y, u(y))dy < \frac{1}{3}\epsilon_1, \forall x \in \Omega_M.$$

In fact, following the proof of Lemma 2.1 it is clear that (4.4) can be satisfied for $x \in \Omega_{N_1}$. It is only necessary to estimate the integral for $x \in \Omega_{M,N_1}$. But

$$\int_{\Omega_{N_1}} \Gamma(x - y)f(y, u(y))dy = \frac{1}{n(2-n)\omega_n} \int_{N_1}^{\infty} P(\rho, c') \int_{S_\rho} |x - y|^{2-n} ds d\rho$$
$$= \frac{1}{n(2-n)\omega_n} \int_{N_1}^{\infty} \rho P(\rho, c') d\rho,$$

and clearly this is small for N_1 sufficiently large, for all $x \in \Omega_{M,N_1}$.

Next we define a function Γ_{ϵ_2} by

$$\Gamma_{\epsilon_2}(x - y) = \Gamma(x - y), |x - y| \geqq \epsilon_2$$
$$= \frac{1}{n(2-n)\omega_n}\epsilon_2^{2-n}, |x - y| < \epsilon_2.$$

By the dominated convergence theorem,

$$\lim_{|x|\to\infty} \int_{\Omega_{M,N_1}} \Gamma_{\epsilon_2}(x - y)f(y, u(y))dy = 0.$$

Therefore there exists an $N_2(N_1, \epsilon_1)$ such that for $|x| > N_2$,

(4.5) $$\int_{\Omega_{M,N_1}} \Gamma_{\epsilon_2}(x - y)f(y, u(y))dy < \frac{1}{3}\epsilon_1.$$

Finally

$$\lim_{|x|\to\infty} \int_{B_{\epsilon_2}(x)} \Gamma(x - y)f(y, u(y))dy = 0,$$

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therefore there exists an $N_3(\epsilon_1, \epsilon_2)$ such that

$$\int_{B_{\epsilon_2}(x)} \Gamma(x - y)f(y, u(y))dy < \frac{1}{3}\epsilon_1, \; |x| > N_3. \tag{4.6}$$

From (4.4)-(4.6) we have that for $|x| > N = \max_{i=1}^{3}\{N\}$,

$$\begin{aligned}
&\int_{\Omega_M} \Gamma(x - y)f(y, u(y))dy \\
&= \left(\int_{\Omega_{M,N_1}} + \int_{\Omega_{N_1}}\right)\Gamma(x - y)f(y, u(y))dy \\
&\leqq \int_{\Omega_{M,N_1}} \Gamma_{\epsilon_2}(x - y)f(y, u(y))dy + \int_{B_{\epsilon_2}(x)} \Gamma(x - y)f(y, u(y))dy \\
&+ \int_{\Omega_{N_1}} \Gamma(x - y)f(y, u(y))dy < \epsilon_1.
\end{aligned}$$

THEOREM 4.1. *For $n \geqq 3$ equation* (1.1) *has a solution u in $C^{2+\alpha}(\Omega_M)$ which satisfies* (4.1) *provided* (2.2a) *is satisfied.*

The analogous result holds for minimal solutions.

THEOREM 4.2. *For $n \geqq 3$ equation* (1.1) *has a solution u in $C^{2+\alpha}(\Omega_M)$ which satisfies* (4.2) *provided* (3.3a) *holds.*

PROOF. As in the previous case, for $\epsilon_1 > 0$ there exists an $N_1(\epsilon_1) > M$ such that for all $x \in \Omega_M$,

(4.6) (i) $$|x|^{n-2} \int_{\Omega_{N_1}} \Gamma(x - y)f(y, u(y))dy < \frac{1}{4}\epsilon_1,$$

(ii) $$\int_{\Omega_{N_1}} f(y, u(y))dy < \frac{1}{4}\epsilon_1.$$

With Γ_{ϵ_2} defined as before, the dominated convergence theorem implies

$$\begin{aligned}
&\lim_{|x|\to\infty} |x|^{n-2} \int_{\Omega_{M,N_1}} \Gamma_{\epsilon_2}(x - y)f(y, u(y))dy \\
&= \int_{\Omega_{M,N_1}} f(y, u(y))dy.
\end{aligned} \tag{4.7}$$

Then

$$\begin{aligned}
&\left| |x|^{n-2} \int_{\Omega_M} \Gamma(x - y)f(y, u(y))dy - \int_{\Omega_M} f(y, u(y))dy \right| \\
&= \left| |x|^{n-2} \int_{\Omega_{M,N_1}} \Gamma(x - y)f(y, u(y))dy - \int_{\Omega_{M,N_1}} f(y, u(y))dy \right| \\
&+ \left| |x|^{n-2} \int_{\Omega_{N_1}} \Gamma(x - y)f(y, u(y))dy - \int_{\Omega_{N_1}} f(y, u(y))dy \right|
\end{aligned}$$

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By (4.6) the second term above is $<(1/2)\epsilon_1$, for all $x \in \Omega_M$. Exactly as in the previous case, we can approximate $\Gamma(x - y)$ by $\Gamma_{\epsilon_2}(x - y)$, and show that for $|x| > N(\epsilon_1)$, the first term is $<(1/2)\epsilon_1$. This implies (4.7), and completes the proof of Theorem 4.2.

5. **Entire solutions to sublinear equations.** We will prove the existence of entire, positive solutions to the sublinear equation

$$(5.1) \qquad -\Delta u = f(x)u^\lambda, \quad 0 < \lambda < 1, \quad x \in R^n.$$

The solutions we obtain decay uniformly to zero as $|x| \to \infty$. They are in fact minimal positive solutions, in terms of the rate of growth of the spherical mean. We assume $f \in C^\alpha_{\text{loc}}(R^n)$, and define,

$$(5.2) \qquad p(t) = \inf_{|x|=t} f(x), \quad P(t) = \sup_{|x|=t} f(x).$$

THEOREM 5.1. *Equation* (5.1) *has an entire, positive solution which decays uniformly to zero as* $|x| \to \infty$ *provided*

$$(5.3) \qquad \int^\infty t^{n-1+\lambda(2-n)}P(t)dt < \infty.$$

This solution satisfies the condition

$$(5.4) \qquad \lim_{|x|\to\infty} |x|^{n-2}u(x) = c,$$

for some positive constant c.

It is known (see [4]) that if $f(x)$ satisfies the additional hypothesis $\limsup_{t\to\infty} [P(t)/p(t)] < \infty$, then (5.3) is in fact a necessary condition for the existence of minimal solutions, even in an *exterior* domain.

Theorem 5.1 will be proved by means of the integral operator

$$T[u](x) = \int_{R^n} \Gamma(x - y)f(y)u^\lambda(y)$$

operating in the space

$$Q = \{u \in C(R^n): k^{-1}\phi(|x|) < u(x) < k\phi(|x|)\}.$$

Here $k > 1$ is a constant and $\phi(t) = \min\{1, t^{2-n}\}$. The major effort in proving Theorem 5.1 is contained in the following result.

LEMMA 5.2. *For k sufficiently large, T maps Q into Q.*

PROOF. For $u \in Q$, we estimate $T[u](x)$.

$$0 \leqq |x| \leqq 1.$$

$$T[u](x) = \left(\int_{B_{|x|}(0)} + \int_{\Omega_{|x|,1}} + \int_{\Omega_1}\right)\Gamma(x - y)f(y)u^\lambda(y)dy$$

$$= I_1 + I_2 + I_3.$$

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Estimating each of these integrals;

$$I_1 \leqq k^\lambda \int_{\rho=0}^{|x|} P(\rho) \int_{S_\rho} \Gamma(x-y)ds d\rho = k^\lambda \omega_{n-1}|x|^{2-n} \int_0^{|x|} \rho^{n-1}P(\rho)d\rho$$

$$\leqq k^\lambda \omega_{n-1} \int_0^{|x|} \rho P(\rho)d\rho.$$

$$I_2 \leqq k^\lambda \int_{\rho=|x|}^{1} P(\rho) \int_{S_\rho} \Gamma(x-y)ds d\rho \leqq k^\lambda \omega_{n-1} \int_{|x|}^{1} \rho P(\rho)d\rho.$$

$$I_3 \leqq k^\lambda \int_{\rho=1}^{\infty} \rho^{2-n^\lambda} P(\rho) \int_{S_\rho} \Gamma(x-y)ds d\rho$$

$$\leqq k^\lambda \omega_{n-1} \int_1^\infty \rho^{(\lambda+1)(2-n)}P(\rho)d\rho.$$

Therefore

$$T[u](x) \leqq k^\lambda \omega_{n-1}\left\{\int_0^1 \rho P(\rho)d\rho + \int_1^\infty \rho^{(\lambda+1)(2-n)}P(\rho)d\rho\right\}$$

and this is $\leqq k\phi(|x|)$ provided

$$\int_0^1 \rho P(\rho)d\rho + \int_1^\infty \rho^{(\lambda+1)(2-n)}P(\rho)d\rho \leqq \omega_{n-1}^{-1}k^{1-\lambda}. \tag{5.3}$$

$$1 \leqq |x| \leqq \infty.$$

$$T[u](x) = \left(\int_{B_1(0)} + \int_{\Omega_{1,|x|}} + \int_{\Omega_{|x|}}\right)\Gamma(x-y)f(y)u^\lambda(y)dy$$

$$= I_1 + I_2 + I_3.$$

$$I_1 \leqq k^\lambda \int_{\rho=0}^{1} P(\rho) \int_{S_\rho} \Gamma(x-y)ds d\rho \leqq k^\lambda \omega_{n-1}|x|^{2-n} \int_0^1 \rho^{n-1}P(\rho)d\rho.$$

$$I_2 \leqq k^\lambda \int_{\rho=1}^{|x|} \rho^{\lambda(2-n)}P(\rho)d\rho \int_{S_\rho} \Gamma(x-y)ds d\rho$$

$$\leqq k^\lambda \omega_{n-1}|x|^{2-n} \int_1^{|x|} \rho^{n-1+\lambda(2-n)}P(\rho)d\rho.$$

$$I_3 \leqq k^\lambda \int_{\rho=|x|}^{\infty} \rho^{\lambda(2-n)}P(\rho) \int_{S_\rho} \Gamma(x-y)ds d\rho$$

$$\leqq k^\lambda \omega_{n-1}|x|^{2-n} \int_{|x|}^{\infty} \rho^{n-1+\lambda(2-n)}.$$

Therefore

$$T[u](x) \leqq k^\lambda \omega_{n-1}|x|^{2-n}\left\{\int_0^1 \rho^{n-1}P(\rho)d\rho + \int_1^\infty \rho^{n-1+\lambda(2-n)}P(\rho)d\rho\right\}$$

and this is $\leqq k\phi(|x|)$ provided

$$\int_0^1 \rho^{n-1}P(\rho)d\rho + \int_1^\infty \rho^{n-1+\lambda(2-n)}P(\rho)d\rho \leqq \omega_{n-1}^{-1}k^{1-\lambda}. \tag{5.4}$$

Having established the upper estimates (5.3) and (5.4), we consider lower estimates. Using the same notation as for the upper estimates we have the following:

$$0 \leqq |x| \leqq 1.$$

$$I_1 \geqq k^{-\lambda} \int_{\rho=0}^{|x|} p(\rho)\phi(\rho)^{\lambda} \int_{S_\rho} \Gamma(x-y)ds d\rho$$

$$\geqq k^{-\lambda}\omega_{n-1}|x|^{2-n} \int_0^{|x|} \rho^{n-1}p(\rho)d\rho \geqq k^{-\lambda}\omega_{n-1} \int_0^{|x|} \rho^{n-1}p(\rho)d\rho.$$

$$I_2 \geqq k^{-\lambda} \int_{\rho=|x|}^{1} p(\rho)\phi(\rho)^{\lambda} \int_{S_\rho} \Gamma(x-y)ds d\rho$$

$$\geqq k^{-\lambda}\omega_{n-1} \int_{|x|}^{1} \rho p(\rho)d\rho.$$

$$I_3 \geqq k^{-\lambda} \int_{\rho=1}^{\infty} \rho^{\lambda(2-n)}p(\rho) \int_{S_\rho} \Gamma(x-y)ds d\rho$$

$$\geqq k^{-\lambda}\omega_{n-1} \int_1^{\infty} \rho^{1+\lambda(2-n)}p(\rho)d\rho.$$

It follows that

$$T[u](x) \geqq k^{-\lambda}\omega_{n-1}\left\{\int_0^{|x|} \rho^{n-1}p(\rho)d\rho + \int_{|x|}^{1} \rho p(\rho)d\rho + \int_1^{\infty} \rho^{1+\lambda(2-n)}p(\rho)d\rho\right\}$$

$$\geqq k^{-\lambda}\omega_{n-1}\phi(|x|)\left\{\int_0^{1} \rho^{n-1}p(\rho)d\rho + \int_1^{\infty} \rho^{1+\lambda(2-n)}p(\rho)d\rho\right\}.$$

Therefore $T[u](x) \geqq k^{-1}\phi(|x|)$ provided

$$\int_0^{1} \rho^{n-1}p(\rho)d\rho + \int_1^{\infty} \rho^{1+\lambda(2-n)}p(\rho)d\rho \geqq \omega_{n-1}^{-1}k^{\lambda-1}. \tag{5.5}$$

$$1 \leqq |x| \leqq \infty.$$

$$I_1 \geqq k^{-\lambda} \int_{\rho=0}^{1} p(\rho) \int_{S_\rho} \Gamma(x-y)ds d\rho \geqq k^{-\lambda}\omega_{n-1}|x|^{2-n} \int_0^{1} \rho^{n-1}p(\rho)d\rho.$$

$$I_2 \geqq k^{-\lambda} \int_{\rho=1}^{|x|} \rho^{\lambda(2-n)}p(\rho) \int_{S_\rho} \Gamma(x-y)ds d\rho$$

$$\geqq k^{-\lambda}\omega_{n-1}|x|^{2-n} \int_1^{|x|} \rho^{n-1+\lambda(2-n)}p(\rho)d\rho.$$

$$I_3 \geqq k^{-\lambda} \int_{\rho=|x|}^{\infty} \rho^{\lambda(2-n)}p(\rho) \int_{S_\rho} \Gamma(x-y)ds d\rho$$

$$\geqq k^{-\lambda}\omega_{n-1}|x|^{2-n}\left\{\int_{|x|}^{\infty} \rho^{n-1+\lambda(2-n)}p(\rho)d\rho\right\}.$$

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So

$$T[u](x) \geqq k^{-\lambda}\omega_{n-1}\phi(|x|)\left\{\int_0^1 \rho^{n-1}p(\rho)d\rho + \int_1^\infty \rho^{n-1+\lambda(2-n)}p(\rho)d\rho\right\}.$$

Therefore $T[u](x) \geqq k^{-1}\phi(|x|)$ provided

$$\int_0^1 \rho^{n-1}p(\rho)d\rho + \int_1^\infty \rho^{n-1+\lambda(2-n)}p(\rho)d\rho \geqq \omega_{n-1}^{-1}k^{\lambda-1}. \tag{5.6}$$

Now observe that for $0 < \lambda < 1$, the inequalities 5.3-5.6 can all be satisfied if k is sufficiently large. This proves Lemma 5.2.

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