BICONTINUOUS ISOMORPHISMS BETWEEN TWO CLOSED LEFT IDEALS OF A COMPACT DUAL RING

LING-ERL E. T. WU

A quasi-Frobenius ring is a ring with minimum condition satisfying the conditions $r(l(H)) = H$ and $l(r(L)) = L$ for right ideals H and left ideals L where $r(S)$ $(l(S))$ denotes the right (left) annihilator of a subset S of the ring. Nakayama first defined and studied such rings **(8; 9)** and they have been studied by a number of authors **(2; 3; 4; 6)**. A dual ring is a topological ring satisfying the conditions $r(l(H)) = H$ and $l(r(H)) = L$ for closed right ideals H and closed left ideals L. Baer **(1)** and Kaplansky **(7)** introduced the notion of such rings, which is a natural generalization of that of quaso-Frobenius rings. Numakura studied the analogy between dual rings and quasi-Frobenius rings in **(10)**.

Ikeda **(3)** proved that every isomorphism between two left ideals in a quasi-Frobenius ring can be extended to an isomorphism of R. The purpose of this note is to prove the following theorem, which is analogous to Ikeda's theorem: *In a compact dual ring R every bicontinuous isomorphism between two closed left ideals can be extended to a bicontinuous isomorphism of R.*

The proof is not similar to Ikeda's proof because of the topological structure. We shall begin with the following result due to Kaplansky.

THEOREM. *If R is a compact dual ring, then R has an identity and R has a complete system of ideal neighbourhoods of* 0 **(7**, Corollary to Theorem 4; **10**, Lemma 3.2**)**.

COROLLARY. *Every principal left (right) ideal in a compact dual ring is closed.*

LEMMA 1. *Let L be a proper closed left (right) ideal in a compact dual ring R. Then L is contained in a maximal open left (right) ideal M in R.*

Proof. Let $\{V_\alpha | \alpha \in \mathfrak{A}\}$ be a complete system of ideal neighbourhoods of 0. Then $\bar{L} = L = \bigcap_{\alpha\in\mathfrak{A}}(L + V_\alpha)$. Since $L \neq R$, there exists an index $\alpha \in \mathfrak{A}$ such that $L + V_\alpha \neq R$. $L + V_\alpha$ is an open ideal containing L. By the compactness of R, $R/(L + V_\alpha)$ is finite. Hence there exists a maximal open left ideal M such that $L \subseteq L + V_\alpha \subseteq M$, completing the proof.

LEMMA 2. *In a compact dual ring R every closed left (right) ideal L with $0 \neq L \neq R$ contains a closed simple left (right) subideal.*

Proof. By Lemma 1, there exists a maximal open right ideal M of R containing $r(L)$. Then $l(M) \subseteq l(r(L)) = L$.

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Let $a \in l(M)$ and $a \neq 0$. Ra is closed and hence

$$Ra = l(r(Ra)) \subseteq l(M).$$

Since M is maximal, this implies that $r(Ra) = M$ and $Ra = l(M)$. Thus we have shown that $l(M)$ is a simple ideal contained in L.

LEMMA 3. *If S is a simple left ideal of a compact dual ring R, then S and SR are closed and finite.*

Proof. First we observe that S is closed, since $S = Ra$ where a is a non-zero element of S. Now $r(S)$ is an open maximal right ideal of R. Hence there exists an open two-sided ideal $U \subseteq r(S)$. This implies that $S \subseteq l(U)$ and also $SR \subseteq l(U)$ since $l(U)$ is a two-sided ideal. By (**10**, Theorem 1), $l(U)$ is finite and thus S and SR are finite. It follows that SR is closed.

LEMMA 4. *Let L be a proper closed left ideal in a compact dual ring R. Suppose θ is a bicontinuous isomorphism between L and a left ideal L'. Then θ can be extended to a bicontinuous isomorphism on H where $L \subset H$, $L \neq H$.*

Proof. By a theorem of Numakura (**10**, Theorem 4), there exists an element a in R which defines θ. If $l(a) = 0$, then the right multiplication by a is a monomorphism of R. Suppose $l(a) \neq 0$. Let $N = l(a) + L$. Since

$$x \xrightarrow{\theta} xa$$

is an isomorphism between L and L', $0 = \operatorname{Ker} \theta = l(a) \cap L$. Therefore $N = l(a) \oplus L$. Let S be a closed simple left subideal of $l(a)$ and let $H = S \oplus L$. We shall show that there exists a closed simple left ideal S' of R such that $S \cong S'$ (bicontinuously) and $S' \cap L' = 0$.

Suppose the contrary: that is, for all closed simple left ideals S'' either $S \cong S''$ (bicontinuously) is false or we have $S'' \subseteq L'$. If x_0 is an element in R such that $Sx_0 \neq 0$, then $S \cong Sx_0$ and by our assumption $Sx_0 \subseteq L'$. Hence, we have $SR \subseteq L'$. By Lemma 3 SR is finite and closed. Now $\theta^{-1}|_{SR}$ is a continuous homomorphism of SR into L; by (**10**, Theorem 4), there exists an element b in R such that $\theta^{-1}(x) = xb$ for all $x \in SR$. $(SR)b \subseteq SR$, and since θ^{-1} is 1–1 and SR is finite, we have $(SR)b = SR$. This shows that $SR \subseteq L$; in particular, $S \subseteq L$. This contradicts the fact that $S \cap L = 0$. Thus we have shown that there exists a closed simple left ideal S' of R and a bicontinuous isomorphism $\alpha: S \to S'$ such that $S' \cap L' = 0$.

Now define $\gamma : S \oplus L \to S' \oplus L'$ by

$$\gamma(s, x) = (\alpha(s), \theta(x)), \qquad s \in S \quad \text{and} \quad x \in L,$$

where $\oplus$ is a topological direct sum. Then γ is a bicontinuous isomorphism of $H = S \oplus L \supset L$ which extends θ. This completes the proof.

THEOREM 4. *Let R be a compact dual ring. Then for every bicontinuous isomorphism θ between two closed left ideals we can choose a suitable unit which* *defines θ; that is, every bicontinuous isomorphism between two closed left ideals can be extended to a bicontinuous isomorphism of R.*

Proof. Let θ be a bicontinuous isomorphism between closed left ideals L and L'. Let

$$\mathfrak{p} = \{(L_\alpha, a_\alpha, b_\alpha, L'_\alpha) \mid \alpha \in \mathfrak{A}\}$$

be the collection of all $(L_\alpha, a_\alpha, b_\alpha, L'_\alpha)$ where L_α and L'_α are closed left ideals containing L and L' respectively, and there is a bicontinuous isomorphism θ_α between L_α and L'_α which extends θ:

$$\theta_\alpha(x) = xa_\alpha \text{ for all } x \in L, \qquad \theta_\alpha^{-1}(x') = x'b_\alpha \text{ for all } x' \in L'.$$

We partial order $\mathfrak{p}$ in the following way: $(L_\alpha, a_\alpha, b_\alpha, L'_\alpha) \leqslant (L_\beta, a_\beta, b_\beta, L'_\beta)$ if and only if

(i) $L_\alpha \subseteq L_\beta, L'_\alpha \subseteq L'_\beta)$;

(ii) $x_\alpha a_\alpha = x_\alpha a_\beta$ for all $x_\alpha \in L_\alpha$, $x'_\alpha b_\alpha = x'_\alpha b_\beta$ for all $x'_\alpha \in L'_\alpha$.

We also partial order the index set $\mathfrak{A} : \alpha \leqslant \beta$ if and only if

$$(L_\alpha, a_\alpha, b_\alpha, L'_\alpha) \leqslant (L_\beta, a_\beta, b_\beta, L'_\beta).$$

Let

$$\{(L_\alpha, a_\alpha, b_\alpha, L'_\alpha) \mid \alpha \in \mathfrak{C} \subseteq \mathfrak{A}\}$$

be a chain in $\mathfrak{p}$. Let

$$L_0 = \overline{\bigcup_{\alpha\in\mathfrak{C}} L_\alpha}, \qquad L'_0 = \overline{\bigcup_{\alpha\in\mathfrak{C}} L'_\alpha}.$$

For each $\alpha \in \mathfrak{C}$, define

$$A_\gamma = \{\alpha_\mu \mid \mu \geqslant \gamma\}, \qquad B_\gamma = \{b_\mu \mid \mu \geqslant \gamma\}.$$

We observe that $A_\gamma = a_\gamma + r(L_\gamma)$ and $B_\gamma = b_\gamma + r(L_\gamma)$ and hence each A_γ, B_γ is closed. Then $\{A_\alpha \mid \alpha \in \mathfrak{C}\}$ and $\{B_\alpha \mid \alpha \in \mathfrak{C}\}$ both have the finite intersection property; for if $A_{\gamma_1}, \ldots, A_{\gamma_n}$ are in $\{A_\gamma \mid \gamma \in \mathfrak{C}\}$, there exists a largest index, say γ_n; then $a_{\gamma_n} \in A_{\gamma_i}$ for each $i = 1, 2, \ldots, n$. Similarly, $\{B_\gamma | \gamma \in \mathfrak{C}\}$ has the finite intersection property. Therefore, by the compactness of R, there exist $a_0 \in \bigcap_{\gamma\in\mathfrak{C}} A_\gamma$ and $b_0 \in \bigcap_{\gamma\in\mathfrak{C}} B_\gamma$. Let $\theta_0 : L_0 \to L'_0$ be defined by $\theta_0(x) = xa_0$ for all $x \in L_0$ and let $\theta'_0 : L'_0 \to L_0$ be defined by $\theta'_0(x') = x'b_0$, $x' \in L'_0$. Then we see that θ_0 (θ'_0) restricted to L_γ (L'_γ) coincides with the right multiplication by a_γ (b_γ) for each $\gamma \in \mathfrak{C}$. Therefore θ_0 restricted to $\bigcap_{\gamma\in\mathfrak{C}} L_\gamma$ and θ'_0 restricted to $\bigcap_{\gamma\in\mathfrak{C}} L'_\gamma$ are isomorphisms, inverse to each other. Hence θ_0 is a bicontinuous isomorphism between L_0 and L'_0. Thus we have shown that the chain $\{(L_\alpha, a_\alpha, b_\alpha, L'_\alpha) \mid \alpha \in \mathfrak{C}\}$ has an upper bound (L_0, a_0, b_0, L'_0) in $\mathfrak{p}$.

By Zorn's lemma, there exists a maximal element (L_m, a_m, b_m, L'_m) in $\mathfrak{p}$. We shall show that $L_m = R$.

Suppose $L_m \neq R$. It follows from Lemma 4 that there exists a closed left ideal L_t properly containing L_m such that the right multiplication by a_m can

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be extended to a bicontinuous isomorphism λ on L_t, say λ is given by the right multiplication by a_t. Let $L'_t = L_t a_t$. By (**10**, Theorem 4), there exists an element b_t which defines λ^{-1}. Then $(L_t, a_t, b_t, L'_t) \in \mathfrak{p}$ and (L_m, a_m, b_m, L'_m) is less than (L_t, a_t, b_t, L'_t), contradicting the maximality of (L_m, a_m, b_m, L'_m). Thus we have $L_m = R$ and this completes the proof of the theorem.

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Cowell College,
University of California at Santa Cruz