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# AN EXTENSION OF THE PRINCIPLE OF SPATIAL AVERAGING FOR INERTIAL MANIFOLDS

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**Abstract**

In this paper we extend a theorem of Mallet-Paret and Sell for the existence of an inertial manifold for a scalar-valued reaction diffusion equation to new physical domains $\Omega_n \subset R^n$, $n = 2, 3$. For their result the Principle of Spatial Averaging (PSA), which certain domains may possess, plays a key role for the existence of an inertial manifold. Instead of the PSA, we define a weaker PSA and prove that the domains $\Omega_n$ with appropriate boundary conditions for the Laplace operator, $\Delta$, satisfy a weaker PSA. This weaker PSA is enough to ensure the existence of an inertial manifold for a specific class of scalar-valued reaction diffusion equations on each domain $\Omega_n$ under suitable conditions.

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## 1. Introduction

During the last twenty years there have been a number of major developments in the study of the long time behavior of solutions of a large class of nonlinear evolutionary equations. One of these advances was the discovery that a dissipative partial differential equation has a compact global attractor with finite Hausdorff dimension (Mallet-Paret [8], Mañé [10]). Because of this basic structure of the global attractor, it was widely believed that the long time behavior of the solutions should strongly resemble the behavior of the solutions of a finite system of ordinary differential equations. Recently, it was shown that, under suitable conditions, a dissipative nonlinear evolutionary equation possesses a finite dimensional inertial manifold. By an inertial manifold for the flow on a Hilbert space $H$, we mean a subset $\mathscr{M}$ of $H$ satisfying the





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following properties: $\mathscr{M}$ is a finite dimensional Lipschitz manifold, it is positively invariant, and it attracts all the solutions exponentially. Furthermore, the dynamics on this manifold can be determined completely by a finite dimensional system of ordinary differential equations, which we call an inertial form. By these properties, an inertial manifold can be a useful tool in the study of long time behaviors of solutions and has been studied by many authors. See, for example, Fabes, Luskin and Sell [1], Foias, Sell and Temam [2], Foias and Temam [3], Jolly [4], Kwak [5, 6], Kwean [7], Mallet-Paret and Sell [9] and Temam [13].

Of particular interest from the point of view taken in this paper is the problem of finding sufficient conditions for the existence of an inertial manifold for the differential equations which can be transformed to an abstract form of the nonlinear evolutionary equation

$$\frac{du}{dt} + \nu A u = R(u) \tag{1.1}$$

on a Hilbert space $H$, where $\nu > 0$ is a viscosity parameter. One of the typical results on this problem was made by Mallet-Paret and Sell [9]. Under suitable conditions, they proved the existence of inertial manifolds of a class of scalar-valued reaction diffusion equations of the form

$$u_t = \nu \Delta u + f(x, u), \quad x \in \Omega_n \subset R^n, \quad u \in R, \tag{1.2}$$

for any 2-dimensional rectangular domains and some cubic domains. For their results, they introduced a new concept: the Principle of Spatial Averaging (PSA). The PSA is a property which the Laplacian over a bounded Lipschitz region $\Omega \subset R^n, n \leq 3$, may or may not have. It is not clear at all for which domains and boundary conditions PSA holds.

The purpose of this paper is to extend the result of Mallet-Paret and Sell [9] into new physical domains $\Omega_n \subset R^n$, $n = 2, 3$, where $\Omega_n$ is a bounded domain of the following form:

$$\begin{cases} \Omega_2 = (\text{equilateral triangle of side } \pi) \\ \Omega_3 = (\text{equilateral triangle of side } \pi) \times [0, L\pi], \end{cases} \tag{1.3}$$

where $L^2$ is rational. For these domains, we do not know whether PSA holds or not. However, we formulate a weaker form of PSA and we prove that the weaker PSA is enough to guarantee the existence of inertial manifolds for (1.2) and (1.3).

## 2. An abstract invariant manifold theory

For convenience we present an abstract theory developed in Mallet-Paret and Sell [9].

Let $H$ be a Hilbert space with an inner product $\langle\cdot,\cdot\rangle$ and a norm $\|\cdot\|$, and let $\mathscr{P}$ be a finite dimensional subspace of $H$ with orthogonal projection $P$, and let $\mathscr{Q}$ be the orthogonal complement of $\mathscr{P}$ with complementary projection $Q = I - P$.

Writing $u \in H$ as $u = (p, q)$ where

$$p = Pu \in PH \equiv \mathscr{P}, \quad q = Qu \in QH \equiv \mathscr{Q},$$

we consider an abstract differential equation

$$\begin{aligned} p' &= F(p,q), \\ q' &= -Aq + G(p,q). \end{aligned} \tag{2.1}$$

We assume $A$ is a closed self-adjoint linear operator on $\mathscr{Q}$ with dense domain $D(A) \subset \mathscr{Q}$. We assume further that $-A$ generates a $C^0$-semigroup $e^{-At}$ in $\mathscr{Q}$ for $t > 0$, and that $A$ has a compact resolvent on $\mathscr{Q}$. The (nonlinear) functions

$$F : H \to \mathscr{P}, \qquad G : H \to \mathscr{Q}$$

are assumed to be locally Lipschitz continuous in $H$. These assumptions on $A$, $F$, and $G$ are standing assumptions throughout this section. Under these assumptions, the system (2.1) with initial condition $u_0 = (p_0, q_0) \in H$ has a unique, maximally defined solution $u(t) = u(t, u_0) = (p(t, p_0, q_0), q(t, p_0, q_0))$ on some interval $[0, \omega)$ where $\omega = \omega(p_0, q_0) \in (0, \infty]$.

Furthermore, the existence of an invariant manifold $\mathscr{M}$ for the system (2.1) can be proved under the following five hypotheses.

(I) (Regularity Condition) There exist constants $R_1$ and $R_2$ such that both $F$ and $G$ are $C^1$ in the convex set $\mathscr{A} \times \mathscr{C}$, where

$$\mathscr{A} = \{p \in \mathscr{P} : \|p\| \le R_1\}, \quad \mathscr{C} = \{q \in D(A) \subset \mathscr{Q} : \|Aq\| \le R_2\}.$$

(II) (Dissipative Condition) If $p \in \mathrm{cl}(\mathscr{P}\backslash\mathscr{A})$, then

$$\langle p, F(p,0)\rangle < 0 \quad \text{and} \quad G(p,0) = 0.$$

(III) (Sobolev Condition) If $p_0 \in \mathscr{A}$ and $t_0 > 0$ are such that $p(t, p_0, 0) \in \mathscr{A}$ holds in $[0, t_0]$, then $q(t, p_0, 0) \in \mathscr{C}$ in $[0, t_0]$.

(IV) (Linear Stability Condition) $\langle q, Aq\rangle \ge \Lambda\|q\|^2$ for all $q \in \mathscr{D}$, for some $\Lambda > 2\gamma$, where $\gamma = \sup\{\|DG(p,q)\|_{\mathscr{L}} : (p,q) \in \mathscr{A} \times \mathscr{C}\}$, and $\mathscr{L} = \mathscr{L}(H, \mathscr{Q})$.

(V) (Uniform Cone Condition) With $V \equiv \frac{1}{2}\|\sigma\|^2 - \frac{1}{2}\|\rho\|^2$ and $V' \equiv \langle\sigma,\sigma'\rangle - \langle\rho,\rho'\rangle$, where $\rho \in \mathscr{P}$, $\sigma \in \mathscr{D} \subset \mathscr{Q}$, and $\rho'$ and $\sigma'$ are given by the linear variational equation form of (2.1), that is,

$$\begin{aligned} \rho' &= DF(p,q)(\rho,\sigma), \\ \sigma' &= -A\sigma + DG(p,q)(\rho,\sigma), \end{aligned} \tag{2.2}$$

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there exists $\eta > 0$ such that $V' < -\eta$ for all $u \in \mathscr{A} \times \mathscr{C}$, $\rho \in \mathscr{P}$ and $\sigma \in \mathscr{D} \subset \mathscr{Q}$ with $\|\rho\| = \|\sigma\| \neq 0$.

Before we state the theorem we introduce the following notation. Let $\Phi : \mathscr{P} \to \mathscr{Q}$ be a function. The graph and the support of $\Phi$ are

$$\text{graph}\,(\Phi) = \{(p, \Phi(p)) : p \in \mathscr{P}\}, \quad \text{supp}\,(\Phi) = \text{cl}\,\{p \in \mathscr{P} : \Phi(p) \neq 0\}.$$

Let $\mathscr{E}$ denote the following subset of $\mathscr{A} \times \mathscr{C}$:

$$\mathscr{E} = \{(p, q) \in \mathscr{A} \times \mathscr{C} : \|q\| \leq \text{dist}\,(p, \partial\mathscr{A})\}.$$

Finally we let

$$\mathscr{G} = \mathscr{E} \cup (\mathscr{P} \times \{0\}). \tag{2.3}$$

THEOREM 2.1. *Assume that the differential equation* (2.1) *satisfies conditions* (I)–(V) *in addition to the standing assumptions on $A$, $F$ and $G$ given above. Then there exists a $C^1$-function $\Phi : \mathscr{P} \to \mathscr{Q}$ with $\|D\Phi\|_\infty \leq 1$ satisfying*

$$\text{supp}\,\Phi \subset \mathscr{A}, \qquad \Phi(p) \in \mathscr{C} \qquad \textit{for} \quad p \in \mathscr{P},$$

*and such that the graph, $\mathscr{M} = \text{graph}\,(\Phi)$, is an invariant manifold for* (2.1) *with $\mathscr{M} \subset \mathscr{G}$, where $\mathscr{G}$ is given by* (2.3)*. Furthermore $\mathscr{M}$ is locally attracting in the following sense: there exists $\alpha > 0$ such that if $u(t) = (p(t), q(t))$ is a solution of* (2.1) *satisfying $u(t) \in \mathscr{E}$ for all $t > 0$, then*

$$\text{dist}\,(u(t), \mathscr{M}) \leq 2e^{-\alpha t}\text{diam}\,\mathscr{C}, \quad t > 0.$$

*That is, $u(t)$ approaches $\mathscr{M}$ at a uniform exponential rate.*

See Mallet-Paret and Sell [9] for the proof.

## 3. Eigenvalues and eigenfunctions of the Laplacian and geometric properties of lattices

**3.1. Eigenvalues and eigenfunctions of the Laplacian.** Since the weaker PSA depends on the eigenvalues and the eigenfunctions of the Laplacian, we need to find those of the Laplace operator for the domains $\Omega_n \subset R^n$ given in (1.3) with suitable boundary conditions. Let $Z_+$ denote the positive integers and $Z_\oplus = Z_+ \cup \{0\}$. Then we obtain the following results.

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LEMMA 3.1. *Let $\Omega_n \subset R^n$ be given in* (1.3) *for* $n = 2, 3$. *Then the eigenvalues and the eigenfunctions of $-\Delta$ for Dirichlet boundary conditions are of the form: for $\Omega_2 \subset R^2$,*

$$\text{(3.1)} \qquad \begin{cases} \lambda_k = \dfrac{16}{27}(k_1^2 + k_2^2 - k_1 k_2), \\ f_k(x_1, x_2) = \displaystyle\sum_{(k_1,k_2)} \pm \exp\left(\frac{2i}{3}\right)\left(k_2 x_1 + \frac{2k_1 - k_2}{\sqrt{3}} x_2\right), \end{cases}$$

*and for $\Omega_3 \subset R^3$,*

$$\text{(3.2)} \qquad \begin{cases} \lambda_k = \dfrac{16}{27}(k_1^2 + k_2^2 - k_1 k_2) + \dfrac{k_3^2}{L^2}, \\ f_k(x_1, x_2, x_3) = \sin\left(\dfrac{k_3}{L} x_3\right) \displaystyle\sum_{(k_1,k_2)} \pm \exp\left(\frac{2i}{3}\right)\left(k_2 x_1 + \frac{2k_1 - k_2}{\sqrt{3}} x_2\right), \end{cases}$$

*where $k = (k_1, k_2) \in Z^2$ ($k = (k_1, k_2, k_3) \in Z^2 \times Z_+$ for $n = 3$) satisfies* (i) *$k_1 + k_2$ is multiple of* 3, (ii) *$k_1 \neq 2k_2$*, (iii) *$k_2 \neq 2k_1$, and $(k_1, k_2)$ in the summation ranges over $\mathscr{S} \subset Z^2$, $|\mathscr{S}| = 6$, and $\pm$ is determined by the following rules:*

$$\begin{array}{ccccc} & & (k_1, k_2) & & \\ & \nearrow & & \searrow & \\ (k_2 - k_1, k_2) & & & & (k_1, k_1 - k_2) \\ \uparrow & & & & \downarrow \\ (k_2 - k_1, -k_1) & & & & (-k_2, k_1 - k_2) \\ & \nwarrow & & \swarrow & \\ & & (-k_2, -k_1) & & \end{array}$$

*Each leg of the cycle induces a change of the sign in the $(k_1, k_2)$ entry of* (3.1) *and* (3.2). *For example, if $(k_1, k_2)$, $(k_2 - k_1, -k_1)$, $(-k_2, k_1 - k_2)$ have positive signs then the others have negative signs.*

LEMMA 3.2. *Let $\Omega_n \subset R^n$ be given in* (1.3) *for* $n = 2, 3$. *Then the eigenvalues and the eigenfunctions of $-\Delta$ for Neumann boundary conditions are of the form: for $\Omega_2 \subset R^2$,*

$$\text{(3.3)} \qquad \begin{cases} \lambda_k = \dfrac{16}{27}(k_1^2 + k_2^2 - k_1 k_2), \\ g_k(x_1, x_2) = \displaystyle\sum_{(k_1,k_2)} \exp\left(\frac{2i}{3}\right)\left(k_2 x_1 + \frac{2k_1 - k_2}{\sqrt{3}} x_2\right), \end{cases}$$

*and for $\Omega_3 \subset R^3$,*

$$\text{(3.4)} \qquad \begin{cases} \lambda_k = \dfrac{16}{27}(k_1^2 + k_2^2 - k_1 k_2) + \dfrac{k_3^2}{L^2}, \\ g_k(x_1, x_2, x_3) = \cos\left(\dfrac{k_3}{L} x_3\right) \displaystyle\sum_{(k_1,k_2)} \exp\left(\frac{2i}{3}\right)\left(k_2 x_1 + \frac{2k_1 - k_2}{\sqrt{3}} x_2\right), \end{cases}$$

*where* $k = (k_1, k_2) \in R^2$ ($k \in Z^2 \times Z_{\oplus}$ *for* $n = 3$) *is such that* $k_1 + k_2$ *is multiple of* 3, *and the summation has the same restrictions as in Lemma* 3.1 *except the sign.*

REMARK. The eigenvalues and the eigenfunctions in (3.1) and (3.3) are the direct consequences of Pinsky [11]. Then for the 3-dimensional case, we obtain (3.2) and (3.4) by applying the separation of variable method.

**3.2. Geometric properties of lattice points.** Here we introduce two geometric properties of a lattice in $R^3$. In particular, the second property is crucial to the proof of a weaker PSA (see Section 4) for each domain $\Omega_n \subset R^n$.

The first property is a Gap Theorem of Mallet-Paret and Sell [9]; see also Richards [12].

THEOREM 3.1. *Let* $\mathcal{T}$ *be a finite collection of functions* $T$ *of the form*

$$T(k_1, k_2) = ak_1^2 + bk_1k_2 + ck_2^2 + sk_1 + tk_2 + r,$$

*with rational coefficients and negative discriminant, that is,* $b^2 - 4ac < 0$. *Then given any* $h > 0$ *there exists arbitrarily large* $m$ *such that*

$$T(k_1, k_2) \notin [m, m + h], \tag{3.5}$$

*for all* $T \in \mathcal{T}$ *and* $k_1,\ k_2 \in Z$.

For the next theorem, we consider the three linearly independent vectors in $R^3$ :

$$e_1 = \left(\frac{4}{3\sqrt{3}}, 0, 0\right), \quad e_2 = \left(-\frac{2}{3\sqrt{3}}, \frac{2}{3}, 0\right), \quad e_3 = \left(0, 0, \frac{1}{L}\right)$$

and we define a new inner product and a norm induced by

$$\langle x, y\rangle = \left(\sum_{s=1}^{3} x_s e_s, \sum_{t=1}^{3} y_t e_t\right), \quad |[x]|^2 = \langle x, x\rangle \tag{3.6}$$

where $x, y \in R^3$ and $(\cdot, \cdot)$ is the usual inner product in $R^3$.

Now we prove the following theorem.

THEOREM 3.2. *Assume that* $L^2$ *is a rational number. Let* $k = (k_1, k_2, k_3) \in Z^3$ *and consider*

$$|[k]|^2 \equiv \frac{16}{27}(k_1^2 + k_2^2 - k_1k_2) + \frac{k_3^2}{L^2}. \tag{3.7}$$

*Then there exists* $\xi > 0$ *such that for any* $\kappa > 1$ *and* $d > 0$, *there exists an arbitrarily large* $\lambda$ *satisfying two conditions*:

(i) *whenever* $|[k]|^2, |[l]|^2 \in (\lambda - \kappa, \lambda + \kappa]$ *with* $k,\ l \in Z^3$, *one has either* $k = l$ *or* $|[k-l]| \geq d$,

(ii) $|[k]|^2 \notin (\lambda - \xi/2, \lambda + \xi/2)$ *for each* $k \in Z^3$.

PROOF. We follow Mallet-Paret and Sell's approach [9]. Let $L^2 = q/p$ where $p$ and $q$ are relative prime integers. Let $\alpha = \text{LCM}\{27, q\}$ be fixed where LCM means least common multiple. Then for any $k \in Z^3$, there exist integers $n$ and $r$ such that

$$|[k]|^2 = n + \frac{r}{\alpha}, \qquad 0 \leq r < \alpha.$$

Therefore, with $\xi = 1/(2\alpha)$ we see that there exists arbitrarily large $\lambda$ such that $|[k]|^2 \notin (\lambda - \xi/2, \lambda + \xi/2)$. For the rest of the proof we consider only such $\lambda$. Let $\lambda$ be fixed and let $N_0^\lambda$ be the annular region

$$N_0^\lambda \equiv \{x \in R^3 : \lambda - \kappa < |[x]|^2 \leq \lambda + \kappa\}.$$

Suppose that $k, l \in N_0^\lambda \cap Z^3$ and $0 \leq |[k-l]| < d$. Then for $j = l - k$,

$$|[l]|^2 = |[j+k]|^2 = |[j]|^2 + 2\langle k, j\rangle + |[k]|^2,$$

where $\langle \cdot, \cdot \rangle$ is defined in (3.6). As a result,

$$\begin{aligned} |\langle k, j\rangle| &\leq \frac{1}{2}\left||[l]|^2 - |[k]|^2 - |[j]|^2\right| \\ &\leq \frac{1}{2}\left||[l]|^2 - |[k]|^2| + \frac{1}{2}|[j]\right|^2 \\ &< \kappa + \frac{d^2}{2}. \end{aligned}$$

For each $j$ with $0 < |[j]| < d$, let $S_j = \{x \in R^3 : |\langle x, j\rangle| < \kappa + d^2/2\}$ and let $S = \bigcup_{0<|[j]|<d} S_j$. If the property (i) fails for some $\lambda$, then $S \cap N_0^\lambda \cap Z^3 \neq \emptyset$. If $k \in S \cap N_0^\lambda \cap Z^3$, then

$$|\langle k, j\rangle| < \kappa + \frac{d^2}{2}$$

and

$$\gamma \equiv \langle k, j\rangle = \frac{8k_1}{27}(2j_1 - j_2) + \frac{8k_2}{27}(2j_2 - j_1) + \frac{k_3 j_3}{L^2}$$

for some $j$ and some $\gamma = n/\alpha$ where $0 < |[j]| < d$ and $n$ is an integer such that $|n/\alpha| < \kappa + d^2/2$. Since $\gamma = n/\alpha$ for some integer n, there is only a finite number of

$\gamma$ satisfying $|\gamma| = |n/\alpha| < \kappa + d^2/2$. On the other hand since $j \neq 0$, we may assume that $j_3 \neq 0$. Then by solving $\langle k, j \rangle = \gamma$ for $k_3$, it is found that

$$k_3 = -\frac{L^2}{j_3}\left(\frac{8k_1}{27}(2j_1 - j_2) + \frac{8k_2}{27}(2j_2 - j_1) - \gamma\right)$$

and hence by substituting $k_3$,

$$\begin{aligned}
|[k]|^2 &= \frac{16}{27}(k_1^2 + k_2^2 - k_1k_2) + \frac{k_3^2}{L^2} \\
&= \left(\frac{16}{27} + \frac{L^2}{j_3^2}\left(\left(\frac{16}{27}\right)^2 j_1^2 + \left(\frac{16}{27}\right)^2 j_2^2 - \left(\frac{16}{27}\right)^2 j_1j_2\right)\right)k_1^2 \\
&\quad + \left(-\frac{16}{27} + \frac{L^2}{j_3^2}\left(2\left(\frac{8}{27}\right)^2 j_1j_2 - \left(\frac{16}{27}\right)^2 j_1^2\right.\right. \\
&\quad \left.\left. +2\left(\frac{16}{27}\right)^2 j_1j_2 - \left(\frac{8}{27}\right)^2 j_2^2\right)\right)k_1k_2 \\
&\quad + \left(\frac{16}{27} + \frac{L^2}{j_3^2}\left(\left(\frac{8}{27}\right)^2 j_1^2 + \left(\frac{16}{27}\right)^2 j_2^2 - \left(\frac{16}{27}\right)^2 j_1j_2\right)\right)k_2^2 \\
&\quad + s_{j,\gamma}k_1 + t_{j,\gamma}k_2 + r_{j,\gamma}
\end{aligned} \tag{3.8}$$

where $s_{j,\gamma}$, $t_{j,\gamma}$ and $r_{j,\gamma}$ are rationals depending only on $j$ and $\gamma$. Now by taking coefficients in (3.8), we define a quadratic function $T_{j,\gamma}$ on $Z^2$ with rational coefficients of the form

$$T_{j,\gamma}(l_1, l_2) = a_j l_1^2 + b_j l_1 l_2 + c_j l_2^2 + s_{j,\gamma} l_1 + t_{j,\gamma} l_2 + r_{j,\gamma}. \tag{3.9}$$

Then the discriminant of $T_{j,\gamma}$ in (3.9) is negative. Also since $k \in N_0^\lambda$,

$$T_{j,\gamma}(k_1, k_2) \in (\lambda - \kappa, \lambda + \kappa]. \tag{3.10}$$

Now let $\mathcal{T}$ be the set of all quadratic functions $T_{j,\gamma}$ of the form in (3.9) for $j \in Z^3$ and $\gamma = n/\alpha$ with $0 < |[j]| < d$, $|\gamma| = |n/\alpha| < \kappa + d^2/2$. Then since the indices $j$ and $\gamma$ range over finite sets, $\mathcal{T}$ is a finite collection of functions $T_{j,\gamma}$ satisfying all hypotheses of Theorem 3.1. With $h = 2 + 2\kappa$, there exists $m$ in the statement of Theorem 3.1 such that for any $T_{j,\gamma} \in \mathcal{T}$ and $l \in Z^2$

$$T_{j,\gamma}(l_1, l_2) \notin [m, m+h], \quad (\lambda - \kappa, \lambda + \kappa] \subset [m, m+h]$$

for some $\lambda$ satisfying the second assertion (ii). Therefore (3.10) is impossible for this $\lambda$. As $m$ can be chosen arbitrarily large, the proof is now complete. □

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## 4. The existence of inertial manifolds

We turn our attention now to the specific class of scalar partial differential equations of the form

$$(4.1)\qquad \frac{\partial u}{\partial t} = \nu \Delta u + f(x, u), \quad x \in \Omega_n \subset R^n,\ u \in R$$

where the domain $\Omega_n$ is given in (1.3). The main goal is to show the existence of an inertial manifold for (4.1) on each domain $\Omega_n$ given in (1.3). In particular, for $\Omega_3$ in (1.3), we assume that $L^2$ is rational number.

The nonlinearity

$$f : \overline{\Omega}_n \times R \to R$$

is assumed to satisfy the following conditions for some positive constants $K_1$ and $K_2$:

$$(4.2)\qquad \begin{cases} f \text{ is } C^1 \text{ in } \overline{\Omega}_n \times R, \\ |f(x, u)|, |D_x f(x, u)| \le K_1 |u| + K_2 \text{ in } \overline{\Omega}_n \times R, \text{ and} \\ |D_u f(x, u)| \le K_1 \text{ in } \overline{\Omega}_n \times R. \end{cases}$$

We consider one of the following boundary conditions for the equation (4.1):

$$(4.3)\qquad \begin{cases} \text{Dirichlet}: & u = 0 \quad \text{on } \partial\Omega_n, \\ \text{Neumann}: & \dfrac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega_n. \end{cases}$$

Then equation (4.1) can be written as an abstract differential equation

$$(4.4)\qquad \frac{du}{dt} = \nu \Delta u + \tilde{f}(u)$$

in the phase space $H = L^2(\Omega_n)$ and $\tilde{f}$ is a Lipschitz continuous mapping on $H$ such that

$$(4.5)\qquad \begin{aligned} \|\tilde{f}(u_1) - \tilde{f}(u_2)\| &\le K_1 \|u_1 - u_2\|, && \text{for all } u_1, u_2 \in H \\ \|\tilde{f}(u)\| &\le K_1 \|u\| + K_3, && \text{for all } u \in H \end{aligned}$$

where $K_3 = (\operatorname{vol} \Omega_n)^{1/2} K_2$. In this setting $\mathscr{D}$ denotes the Laplace operator with the domain

$$(4.6)\qquad \mathscr{D} = \{u \in H^2 : \text{ the boundary conditions (4.3) hold}\}.$$

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For simplicity we assume $\nu = 1$ and for any $\lambda > 0$, let $P_\lambda$ denote the canonical orthogonal projection onto the finite dimensional subspace

$$\mathscr{P}_\lambda = \operatorname{Span}\{e_m : \lambda_m \leq \lambda\}$$

of $H$ where $\{e_j : j = 1, 2, \dots\}$ is a complete orthonormal set of eigenfunctions $e_j$ corresponding to eigenvalues $\lambda_j$ of $-\Delta$ and let $Q_\lambda = I - P_\lambda$. Then by applying $P_\lambda$ and $Q_\lambda$ to equation (4.4), we obtain the system

$$\begin{cases} p' = \Delta p + P_\lambda \tilde{f}(p, q) \\ q' = \Delta q + Q_\lambda \tilde{f}(p, q) \end{cases} \tag{4.7}$$

where $p = P_\lambda u$ and $q = Q_\lambda u$.

The modified equations, to which Theorem 2.1 will be applied, are

$$\begin{aligned} p' &= -\phi(\|Ap\|^2)Ap + \psi(\|p\|^2)[p + P_\lambda \tilde{f}(p, q)], \\ q' &= -Aq + q + \psi(\|p\|^2) Q_\lambda \tilde{f}(p, q) \end{aligned} \tag{4.8}$$

where $\lambda > 0$ is appropriately chosen and $A$ is the positive self-adjoint operator

$$A = I - \Delta,$$

$\phi, \psi : [0, \infty) \to [0, 1]$ are $C^1$ functions such that with a sufficiently large fixed $R > 0, \phi, \psi$ satisfy

$$\begin{cases} \phi'(\tau) \leq 0 & \text{in} \quad [0, \infty), \\ \tau\phi'(\tau) + \phi(\tau) \geq 0 & \text{in} \quad [0, \infty), \\ \phi(\tau) = 1 & \text{in} \quad [0, R^2], \\ \phi(\tau) = \dfrac{1}{2} & \text{in} \quad [K_4 R^2, \infty) \quad \text{for some} \quad K_4 > 1, \\ \psi(\tau) = 1 & \text{in} \quad [0, K_4 R^2], \\ \psi(\tau) = 0 & \text{in} \quad [K_5 R^2, \infty) \quad \text{for some} \quad K_5 > K_4. \end{cases} \tag{4.9}$$

First, we prove that the system (4.8) satisfies main hypotheses (I)–(V) of Theorem 2.1. To do this, we introduce a weaker PSA as follows: for any $v \in L^\infty$ we let $B_v$ denote the operator on $L^2$ defined by

$$(B_v u)(x) = v(x)u(x), \quad u \in L^2,$$

and let $\tilde{v}$ denote the mean value

$$\tilde{v} = (\operatorname{vol}\Omega)^{-1} \int_\Omega v(x)dx.$$

DEFINITION. For a given (bounded Lipschitz) domain $\Omega \subset R^n$, $n \leq 3$, and choice of boundary conditions for the Laplacian, we say the *weaker* principle of spatial averaging holds if there exists a quantity $\xi > 0$ such that for every $\epsilon > 0$, $\kappa > 0$ and any bounded subset $\mathscr{B} \subset H^2$, there exists arbitrarily large $\lambda = \lambda(\mathscr{B}) > \kappa$, such that

$$\|(P_{\lambda+\kappa} - P_{\lambda-\kappa})(B_v - \tilde{v}I)(P_{\lambda+\kappa} - P_{\lambda-\kappa})\|_{\text{op}} \leq \epsilon \tag{4.10}$$

holds for any $v \in \mathscr{B}$; and such that

$$\lambda_{m+1} - \lambda_m \geq \xi \tag{4.11}$$

where $m$ satisfies $\lambda_m \leq \lambda < \lambda_{m+1}$.

The main difference between the weaker PSA and PSA is the choice of $\lambda$ and the upper bound of the estimate (4.10). In the weaker PSA, the quantity $\lambda$ is allowed to depend on the bounded subset $\mathscr{B}$ while the original PSA requires the existence of $\lambda > \kappa$ such that

$$\|(P_{\lambda+\kappa} - P_{\lambda-\kappa})(B_v - \tilde{v}I)(P_{\lambda+\kappa} - P_{\lambda-\kappa})\|_{\text{op}} \leq \epsilon \|v\|_{H^2}$$

holds whenever $v \in H^2$. In the point of our concern, what we really need is to show that the operator norm of (4.10) can be arbitrarily close to 0 on any bounded subset of $H^2$. However in the original PSA, the dependence of the estimate on $\|v\|_{H^2}$ and the requirement of the inequality for all $v \in H^2$ cause some difficulties for proving PSA. Actually Mallet-Paret and Sell used some technical lemmas (see [9]) which are difficult to prove for general domains. The advantage of the weaker PSA is that it can not only replace PSA for the same result but also enables us to drop all their technical lemmas.

The importance of the weaker PSA is that it implies the Uniform Cone Condition and hence the existence of invariant manifold for the system (4.8).

THEOREM 4.1. *Assume that the domain $\Omega$ and the boundary conditions for $-\Delta$ satisfy the weaker* PSA. *Assume $f$ satisfies the regularity and growth conditions* (4.2); *assume also that the function $D_u f$ is $C^2$ on $\overline{\Omega} \times R$. Fix functions $\phi$ and $\psi$ satisfying* (4.9). *Then there exist arbitrary large $\lambda$ such that the system* (4.8) *satisfies all the hypotheses of the invariant manifold Theorem* 2.1.

Note that the conclusion of Theorem 4.1 holds for certain large $\lambda$, not necessarily for all large $\lambda$. Since the proofs of the main conditions (I)–(IV) are exactly the same as in Mallet-Paret and Sell [9], we only prove the Uniform Cone Condition. Moreover, if $\|Ap\| \geq K_4^{1/2}R$, we can easily show the Uniform Cone Condition. On the other hand, if $\|Ap\| \leq K_4^{1/2}R$, we have

$$V' = \langle \sigma, \sigma' \rangle - \langle \rho, \rho' \rangle \leq -\langle \sigma, A\sigma \rangle + \|A\rho\| + \langle (-\rho, \sigma), D\tilde{f}(u)(\rho, \sigma) \rangle \tag{4.12}$$

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where $D\tilde{f}(u)(x) = D_u f(x, u(x))$, and $\rho'$ and $\sigma'$ are given by the variational form of (4.8). Therefore, to complete the proof of the Uniform Cone Condition, it suffices to show that the right hand side of (4.12) is negative and bounded away from zero.

LEMMA 4.1. *Assume that all assumptions of Theorem* 4.1 *hold. Let* $\mathscr{B}$ *be any bounded subset of* $\mathscr{D}$. *Then there exists* $\eta > 0$ *such that for all* $u \in \mathscr{B}$ *and* $(\rho, \sigma) \in \mathscr{P}_\lambda \times \mathscr{Q}_\lambda$, *with* $\|\rho\| = \|\sigma\| = 1$, *there exists arbitrarily large* $\lambda$ *such that*

$$-\langle \sigma, A\sigma \rangle + \|A\rho\| + \langle (-\rho, \sigma), D\tilde{f}(u)(\rho, \sigma) \rangle < -\eta.$$

PROOF. By the smoothness of $f$ and the boundedness of $\mathscr{B} \subset \mathscr{D} \subset H^2$, for each $u \in \mathscr{B}$

$$D\tilde{f}(u)(x) = D_u f(x, u(x)) \in H^2$$

and $\mathscr{B}_1 = \{v : v(x) = D\tilde{f}(u)(x),\ u \in \mathscr{B}\}$ is also bounded subset of $H^2$. Moreover, as a multiplication operator $B_v$ for each $v \in \mathscr{B}_1$, it is bounded on $\mathscr{B}_1$, that is, there exists a number $K_6 > 0$ such that

$$\|B_v\|_{\text{op}} \le K_6, \qquad \text{for all} \quad v \in \mathscr{B}_1, \tag{4.13}$$

where $\|\cdot\|_{\text{op}}$ is the operator norm on $L^2$. Choose quantities $\kappa > 0$ and $r_0 > 0$ so that

$$\kappa - 2K_6 > r_0. \tag{4.14}$$

From now on, we fix $\kappa$, $K_6$, $r_0$ and a bounded subset $\mathscr{B}_1$ throughout the rest of proof. Then by the weaker PSA, there exists a quantity $\xi > 0$ such that for $\epsilon > 0$, $\kappa > 0$ and $\mathscr{B}_1 \subset H^2$, there exists arbitrarily large $\lambda > 0$ satisfying (4.10) and (4.11). We only consider such $\lambda$. Let $u \in \mathscr{B}$ and $(\rho, \sigma) \in \mathscr{P}_\lambda \times \mathscr{Q}_\lambda$ with $\|\rho\| = \|\sigma\| = 1$. We consider two cases:

(i) $(P_{\lambda+\kappa} - P_{\lambda-\kappa})\rho = 0$ or $(P_{\lambda+\kappa} - P_{\lambda-\kappa})\sigma = 0$,
(ii) $(P_{\lambda+\kappa} - P_{\lambda-\kappa})\rho = \rho$ and $(P_{\lambda+\kappa} - P_{\lambda-\kappa})\sigma = \sigma$.

For the case (i), without loss of generality, we assume that $(P_{\lambda+\kappa} - P_{\lambda-\kappa})\rho = 0$. Let $v \equiv D\tilde{f}(u)$. Then by (4.13) and the choice of $\kappa$, $r_0$,

$$\begin{aligned} -\langle \sigma, A\sigma \rangle + \|A\rho\| + \langle (-\rho, \sigma), v(\rho, \sigma) \rangle &\le -\kappa + \left| \int_{\Omega_3} v(x)(\sigma^2 - \rho^2)\,dx \right| \\ &\le -\kappa + 2K_6 \le -r_0. \end{aligned}$$

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For the case (ii), with the choice of $\lambda$ and the property of the weaker PSA, we have

$$\begin{aligned} &- \langle \sigma, A\sigma \rangle + \|A\rho\| + \langle (-\rho, \sigma), v(\rho, \sigma) \rangle \\ &\quad \le -(1 + \lambda_{m+1}) + (1 + \lambda_m) + \left| \int_\Omega v(x)(\sigma^2 - \rho^2)dx \right| \\ &\quad \le -\xi + \left| \int_\Omega (v(x) - \tilde{v})(\sigma^2 - \rho^2)dx \right| + \left| \int_\Omega \tilde{v}(\sigma^2 - \rho^2)dx \right| \\ &\quad \le -\xi + 2\epsilon. \end{aligned}$$

Since the existence of $\xi > 0$ is independent of the choice of $\epsilon > 0$, there exist arbitrarily large $\lambda > 0$ satisfying (4.10) and (4.11) for $\epsilon$ which is less than $\xi/4$ and hence

$$-\xi + 2\epsilon < -\frac{\xi}{2}.$$

Therefore, by choosing $\eta = \min\{r_0, \xi/2\}$ the proof is complete. □

Next we prove a main objective in this paper.

THEOREM 4.2. *The weaker* PSA *holds for each domain* $\Omega_n$ *given in* (1.3) *with either Dirichlet or Neumann boundary conditions.*

For $n = 2$, by Theorem 3.1 and Lemmas 3.1 and 3.2, we have arbitrarily large gaps in the spectrum of $A = I - \Delta$ and hence we can choose $\lambda > 0$ so that the interval $(\lambda - \kappa, \lambda + \kappa]$ contains no $\lambda_m$ and hence $P_{\lambda+\kappa} - P_{\lambda-\kappa} = 0$. If $n = 3$, we do not have this property. However, the next lemma yields the weaker PSA for $n = 3$.

LEMMA 4.2. *Let* $\Omega \subset R^3$ *be given in* (1.3). *Fix Neumann boundary conditions for Laplace and let* $\mathscr{B}$ *be a bounded subset of* $H^2$. *Then for any* $\epsilon > 0$ *and* $\kappa > 1$, *there exist arbitrarily large* $\lambda = \lambda(\mathscr{B}) > \kappa$ *such that*

$$\left| \int_\Omega (v - \tilde{v})\rho^2 dx \right| \le \epsilon \tag{4.15}$$

*for any* $v \in \mathscr{B}$ *and* $\rho \in \text{Range}\,(P_{\lambda+\kappa} - P_{\lambda-\kappa}) \subset L^2$ *with* $\|\rho\| = 1$.

PROOF. Let $\{e_k : k \in Z^2 \times Z_\oplus\}$ be a complete orthonormal set of eigenfunctions of $-\Delta$ for the domain $\Omega$. Since $\mathscr{B}$ is a bounded subset in $H^2$, the compact imbedding $H^2 \hookrightarrow L^\infty$ implies that $\mathscr{B}$ is a compact set in $L^\infty$. Hence, for a given $\epsilon > 0$, there exist $v_1, v_2, \dots, v_n \in \mathscr{B}$ such that for any $v \in \mathscr{B}$,

$$\|v - v_j\|_{L^\infty} \le \frac{\epsilon}{3}, \quad \text{for some} \quad j = 1, 2, \dots, n. \tag{4.16}$$

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On the other hand, since $H^2 \subset L^2$ and $\{e_k : k \in Z^2 \times Z_\oplus\}$ is a complete orthonormal basis, there exists a $d > 0$ such that for any $j$, $1 \le j \le n$,

$$\left(\sum_{|[k]| \ge d} |v_{j,k}|^2\right)^{1/2} \le \frac{\epsilon}{18}(\operatorname{vol}\Omega) \tag{4.17}$$

where $v_{j,k} = \langle v_j, e_k\rangle$ is Fourier coefficient with respect to $e_k$ for each $k \in Z^2 \times Z_\oplus$. Fix this $d > 0$ throughout this proof. Then for this $d > 0$ and $\kappa > 1$, we can choose and fix arbitrarily large $\lambda > \kappa$ satisfying (i) of Theorem 3.2. Now consider a function

$$\rho \in \text{ Range } (P_{\lambda+\kappa} - P_{\lambda-\kappa}) : \quad \|\rho\| = 1.$$

Its Fourier expansion

$$\rho(x) = \sum \rho_{(k)} e_k$$

involves only terms for which

$$\lambda - \kappa < |[k]|^2 \le \lambda + \kappa.$$

Then, for any $v_j$, $1 \le j \le n$,

$$\begin{aligned} \left|\int_\Omega (v_j - \tilde{v}_j)\rho^2\right| &= \left|\int_\Omega (v_j - \tilde{v}_j)\left(\sum_k \rho_{(k)} e_k\right)\left(\sum_l \bar{\rho}_{(l)} \bar{e}_l\right)\right| \\ &= \left|\sum_k \sum_l \rho_{(k)} \bar{\rho}_{(l)} \left(\sum_{s=1}^{6} \int_\Omega (v_j - \tilde{v}_j) e_{k-\delta_s}\right)\right| \end{aligned} \tag{4.18}$$

where $\delta_s$ are given by:

$$\begin{aligned} &\delta_1 = (l_1, l_2, l_3), && \delta_2 = (l_1, l_1 - l_2, l_3), && \delta_3 = (-l_2, l_1 - l_2, l_3), \\ &\delta_4 = (-l_2, -l_1, l_3), && \delta_5 = (-l_1 + l_2, -l_1, l_3), && \delta_6 = (l_2 - l_1, l_2, l_3). \end{aligned} \tag{4.19}$$

However, since $|[l]|^2 \in (\lambda - \kappa, \lambda + \kappa]$ and $|[\delta_s]|^2 = |[l]|^2$ for all $s = 1, 2, \dots, 6$, by (i) of Theorem 3.2, we obtain, for each $s$, either

$$|[k - \delta_s]| \ge d, \quad \text{or} \quad k = \delta_s. \tag{4.20}$$

Also, since

$$\int_\Omega (v_j - \tilde{v}_j) e_{k-\delta_s} \equiv w_{j,k-\delta_s} = \begin{cases} 0 & \text{if} \quad k = \delta_s, \\ v_{j,k-\delta_s} & \text{if} \quad k \ne \delta_s, \end{cases} \tag{4.21}$$

(4.18) and (4.19) imply

$$\left|\int_\Omega (v_j - \tilde{v}_j)\rho^2\right| \le \sum_{s=1}^{6}\sum_{k,l} |\rho_{(k)}||\rho_{(l)}||w_{j,k-h_s(l)}|$$

where $h_s$ is a function on $Z^3$ given by $h_s(l) = \delta_s$, $1 \le s \le 6$, where $\delta_s$ is given in (4.19). By the Schwarz inequality, (4.20), (4.21) and (4.17), we obtain

$$\begin{aligned}\left|\int_\Omega (v_j - \tilde{v}_j)\rho^2\right| &\le \sum_{s=1}^{6}\left(\left(\sum_{k,l}|\rho_{(k)}|^2|\rho_{(l)}|^2\right)^{1/2}\left(\sum_{k,l}|w_{j,k-h_s(l)}|^2\right)^{1/2}\right)\\ &\le \sum_{s=1}^{6}\left(\sum_{k}|\rho_{(k)}|^2\right)\left(\sum_{k,l}|w_{j,k-h_s(l)}|^2\right)^{1/2}\\ &\le (\operatorname{vol}\Omega)^{-1}\sum_{s=1}^{6}\left(\sum_{\|[r]\|\ge d}|v_{j,r}|^2\right)^{1/2} \le \frac{\epsilon}{3}\end{aligned}$$

by the choice of $d$, $\lambda$. Then for any $v \in \mathscr{B}$, there exist some $j$, $1 \le j \le n$, such that

$$\|v - v_j\|_{L^\infty} \le \frac{\epsilon}{3}$$

and hence

$$\left|\int_\Omega (v - \tilde{v})\rho^2\right| \le \left|\int_\Omega (v - v_j)\rho^2\right| + \left|\int_\Omega (\tilde{v} - \tilde{v}_j)\rho^2\right| + \left|\int_\Omega (v_j - \tilde{v}_j)\rho^2\right| \le \epsilon.$$

□

REMARK. The proof for Dirichlet boundary conditions is the same as for the Neumann case except for minor modifications.

PROOF OF THEOREM 4.2. Let $\epsilon > 0$ and $\kappa > 1$ be given. Fix a bounded set $\mathscr{B}$ in $H^2$. Then by Theorem 3.2 and Lemma 4.2, there exists a quantity $\xi > 0$ satisfying (4.10) and (4.11). □

From all these results, one can prove the existence of an inertial manifold for (4.1).

THEOREM 4.3. *Assume that* (4.1) *is dissipative and that* $f : \overline{\Omega}_n \times R \to R$ *is* $C^3$ *on the domain* $\Omega_n \subset R^n$ *given in* (1.3). *Then for every* $\nu > 0$ *and for suitable choice of boundary conditions, there exists an inertial manifold* $\mathscr{M}$ *for* (4.1).

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PROOF. Let $S(t)$ be a nonlinear semiflow on $H = L^2(\Omega_n)$ generated by the system (4.7). Since (4.7) is dissipative there exists a global attractor $\mathscr{U}$ which is a bounded set in $L^\infty$ and a compact set in $L^2$. Now we choose $R > 0$ large enough so that

$$\mathscr{U} \subset \mathscr{B} \equiv \{u \in \mathscr{D}(A) : \|Au\| < R\}.$$

The existence of an attractor and its regularity property are fairly standard matters, so we will not address these issues here. Now we fix this $R > 0$.

On the other hand, the system (4.8) also induces a semiflow, which we denote here by $\tilde{S}(t)$. Then for a given $R > 0$, we can construct and fix $C^1$ functions $\phi$ and $\psi$ satisfying (4.9) and then systems (4.7) and (4.8) coincide on $\mathscr{B}$. Let $\lambda$ be chosen so that the conclusion of Theorem 4.1 is true. Then by the Abstract Invariant Manifold Theorem, there exists an invariant manifold $\mathscr{M}$ for (4.8), which is given by the graph of a Lipschitz function

$$\Phi : \mathscr{P} \to \mathscr{Q}.$$

In order to show that $\mathscr{M}$ is an inertial manifold, it suffices to prove that it attracts all the solutions of (4.1) and (4.3) exponentially.

Let $r > 0$ be any fixed constant and let

$$B_r = \{u \in H : \|u\| \leq r\}.$$

Then we know that there is a $T = T(B_r)$ such that the solution of (4.7), $u(t) = (p(t), q(t))$, starting $u_0 = (p_0, q_0) \in B_r$, satisfies that

$$u(t) = (p(t), q(t)) \in \mathscr{B}$$

for any $t \geq T$. This implies that

$$S(t)u_0 = \tilde{S}(t - T)(p(T), q(T)), \quad t \geq T.$$

We can choose $R_1$ and $R_2$ satisfying the regularity condition such that

$$R_1 > r, \quad R_2 > r.$$

Then this gives us

$$S(t)u_0 = \tilde{S}(t - T)(p(T), q(T)) \in \mathscr{E},$$

where $\mathscr{E}$ is defined in Section 2. Therefore, by Theorem 2.1, we have

$$\text{dist}\left(\tilde{S}(t - T)(p(T), q(T))\right) \leq 2(\text{diam}\,\mathscr{C})e^{-\alpha(t-T)}$$

which implies that $\mathscr{M}$ attracts all the solution with uniform exponential rate. Therefore $\mathscr{M}$ is an inertial manifold for (4.7) and (4.8). □

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REMARK. Although we only consider the type of domains in (1.3), for appropriate boundary conditions, we also have the same conclusion for domains of the form:

$$\begin{cases} \Omega_2 = \left\{(x_1, x_2) : 0 \le x_2 \le \sqrt{3}x_1, 0 \le x_1 \le \dfrac{\pi}{2}\right\} \\ \Omega_3 = \left\{(x_1, x_2) : 0 \le x_2 \le \sqrt{3}x_1, 0 \le x_1 \le \dfrac{\pi}{2}\right\} \times [0, L\pi] \end{cases}$$

$$\begin{cases} \Omega_2 = \left\{x \in R^2 : x = \displaystyle\sum_{k=1}^{2} t_k e_k, 0 \le t_k \le 1\right\} \\ \Omega_3 = \left\{x \in R^3 : x = \displaystyle\sum_{k=1}^{3} t_k e_k, 0 \le t_k \le 1\right\} \end{cases}$$

where the $e_k$'s are linearly independent vectors.

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