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Jacquet Modules of Parabolically Induced Representations and Weyl Groups

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Abstract. The representation parabolically induced from an irreducible supercuspidal representation is considered. Irreducible components of Jacquet modules with respect to induction in stages are given. The results are used for consideration of generalized Steinberg representations.

1 Introduction

Jacquet modules of parabolically induced representations can be applied to some problems in representation theory, for example, the question of reducibility of parabolically induced representations [T3], [J3]. They are also used for important work on description of discrete series for classical p-adic groups [MT], [J1], [J2]. These articles concern the classical groups $\mathrm{Sp}(n, F)$ and $\mathrm{SO}(2n+1, F)$. For calculating Jacquet modules of parabolically induced representations, they are using a structure on representations of the groups $\mathrm{Sp}(n, F)$ and $\mathrm{SO}(2n+1, F)$, described in [T1].

The purpose of this article is to find new techniques for calculating Jacquet modules for any connected p-adic group. Consequently, we also describe the structure of parabolically induced representations, their irreducible subrepresentations and irreducible subquotients.

Let G be a connected reductive p-adic group, $P = MU$ a standard parabolic subgroup of G and σ an irreducible supercuspidal representation of M. The geometric lemma ([BZ], [C], here Theorem 2.1) describes composition factors of

$$r_{M,G} \circ i_{G,M}(\sigma),$$

where $i_{G,M}$ denotes functor of parabolic induction and $r_{M,G}$ the Jacquet functor [BZ]. After eliminating all zero components in $r_{M,G} \circ i_{G,M}(\sigma)$ ([BZ], [C], here Theorem 2.2), we prove some interesting facts about the structure of $i_{G,M}(\sigma)$ and $r_{M,G} \circ i_{G,M}(\sigma)$ (Lemma 3.1 and Corollary 4.3).

In Section 5, we consider an intermediate standard Levi subgroup N such that $M < N < G$ and describe how information about $i_{N,M}(\sigma)$ and $r_{M,N} \circ i_{N,M}(\sigma)$ can be used in determining $i_{G,M}(\sigma)$ and $r_{M,G} \circ i_{G,M}(\sigma)$ (Theorem 5.1, Corollaries 5.2 and 5.3).

Using results of Sections 3, 4 and 5, choosing appropriate intermediate Levi subgroups, we can deduce information about parabolically induced representations and

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their Jacquet modules. In Section 7, we carry this out for some examples of representations of $\mathrm{Sp}(m, F)$, $\mathrm{SO}(2m+1, F)$ and $\mathrm{SO}(2m, F)$. More precisely, we consider the representation

$$\nu^{\alpha+n}\rho \times \cdots \times \nu^{\alpha}\rho \rtimes \sigma,$$

where ρ, σ are supercuspidal and $\nu^{\alpha}\rho \rtimes \sigma$ is reducible (see Section 7 for notation). For $\alpha > 0$, this representation is of length 2^{n+1} (Proposition 7.2) and it has the unique irreducible subrepresentation. This subrepresentation is square integrable and it is the unique square integrable subquotient of $\nu^{\alpha+n}\rho \times \cdots \times \nu^{\alpha}\rho \rtimes \sigma$ (Propositions 7.1 and 7.2). We shall call it a generalized Steinberg representation (see Remark 7.1).

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2 Preliminaries

In this section, we introduce notation and recall some results from [BZ] and [C] on Jacquet modules of parabolically induced representations.

Let F be a p-adic field and G the group of F-points of a reductive algebraic group defined over F. Fix a minimal parabolic subgroup P_0 and a maximal split torus $A_0 \subset P_0$. Let Δ be the corresponding set of simple roots. If $\Theta \subset \Delta$, then we write $P_\Theta = M_\Theta U_\Theta$ for the standard parabolic subgroup determined by Θ.

Let $P = MU$ be a standard parabolic subgroup of G. If σ is a smooth representation of M, then we denote by $i_{G,M}(\sigma)$ the representation parabolically induced by σ. For a smooth representation π of G, $r_{M,G}(\pi)$ is normalized Jacquet module of π with respect to M [BZ].

For a smooth finite length representation π we denote by $s.s.(\pi)$ the semi-simplified representation of π. It is the direct sum of the irreducible components of π. Let $\leq$ denote the natural partial order on the Grothendieck group of the category of all smooth finite length representations of G. For smooth finite length representations π_1 and π_2, we write $\pi_1 \leq \pi_2$ if $s.s.(\pi_1) \leq s.s.(\pi_2)$ in the Grothendieck group.

Let W be the Weyl group of G. For $\Theta \subset \Delta$, we denote by $W_\Theta \subset W$ the Weyl group of M_Θ. Let $\Omega \subset \Delta$. Set [C]

$$[W_\Theta \setminus W / W_\Omega] = \{w \in W \mid w\alpha > 0, \forall \alpha \in \Omega, w^{-1}\beta > 0, \forall \beta \in \Theta\}.$$

Theorem 2.1 **(Geometric Lemma, [BZ], [C])** *Let G be a connected reductive p-adic group, $P = P_\Theta = MU$, $Q = P_\Omega = NV$ parabolic subgroups. Let σ be an admissible representation of M. Then $r_{N,G} \circ i_{G,M}(\sigma)$ has a composition series with factors*

$$i_{N,N'} \circ w^{-1} \circ r_{M',M}(\sigma)$$

where $M' = M \cap w(N)$, $N' = w^{-1}(M) \cap N$ and $w \in [W_\Theta \setminus W / W_\Omega]$.

For $\Theta, \Theta' \subseteq \Delta$, we define

$$W(\Theta, \Theta') = \{w \in W \mid w\Theta' = \Theta\}.$$

If $\Theta = \Theta'$, then we set $W(\Theta) = W(\Theta, \Theta)$, and this is a subgroup of W.

Let $\Theta, \Theta' \subseteq \Omega \subseteq \Delta$. Define

$$W_\Omega(\Theta, \Theta') = \{w \in W_\Omega \mid w\Theta' = \Theta\},$$
$$W_\Omega(\Theta) = W_\Omega(\Theta, \Theta).$$

Theorem 2.2 (**[BZ]**, **[C]**) *Let σ be a supercuspidal representation of $M = M_\Theta$. Then*

$$s.s.\big(r_{M,G} \circ i_{G,M}(\sigma)\big) = \sum_{w \in W(\Theta)} w^{-1}\sigma = \sum_{w \in W(\Theta)} w\sigma.$$

Define

$$W(\sigma) = \{w \in W(\Theta) \mid w\sigma \cong \sigma\}.$$

We call σ *regular* if $W(\sigma) = \{1\}$. Otherwise, we say that σ is *non-regular*.

Theorem 2.3 (**[C, Proposition 6.4.1]**) *Let σ be a regular supercuspidal representation of $M = M_\Theta$. Then*

$$r_{M,G} \circ i_{G,M}(\sigma) \cong \bigoplus_{w \in W(\Theta)} w^{-1}\sigma = \bigoplus_{w \in W(\Theta)} w\sigma.$$

3 Regular Case

Lemma 3.1 *Let σ be a regular irreducible supercuspidal representation of M. Then:*

(1) $i_{G,M}(\sigma)$ has a unique irreducible subrepresentation.
(2) All irreducible subquotients of $i_{G,M}(\sigma)$ are mutually inequivalent.
(3) For any $w_1, w_2 \in W(\Theta)$,

$$\dim_{\mathbb{C}} \operatorname{Hom}_G\big(i_{G,M}(w_1\sigma), i_{G,M}(w_2\sigma)\big) = 1.$$

(4) Let p be the unique irreducible subrepresentation of $i_{G,M}(\sigma)$. Then

$$w\sigma \leq r_{M,G}(p) \Longleftrightarrow p \hookrightarrow i_{G,M}(w\sigma) \Longleftrightarrow i_{G,M}(w\sigma) \cong i_{G,M}(\sigma).$$

Remark 3.1 For σ a regular character, Lemma 3.1. is proved in [R].

Proof (1) is well-known [C], and (2) and (3) can be proved using Jacquet modules, Frobenius reciprocity and Theorem 2.3.

(4) The first equivalence follows from Frobenius reciprocity and Theorem 2.3. Now, suppose that $p \hookrightarrow i_{G,M}(w\sigma)$. By (3), we have a unique (up to a scalar)

non-trivial intertwining operator φ between $i_{G,M}(\sigma)$ and $i_{G,M}(w\sigma)$. Then Ker $\varphi \hookrightarrow i_{G,M}(\sigma)$, $\Im\varphi \hookrightarrow i_{G,M}(w\sigma)$. If Ker $\varphi \neq \{0\}$, then $p \hookrightarrow$ Ker φ, so, by (2), $\Im\varphi = \{0\}$. But this contradicts the assumption that φ is non-trivial. We conclude that Ker $\varphi = \{0\}$, so $i_{G,M}(w\sigma) \cong i_{G,M}(\sigma)$. ■

Let $P = MU$ be a standard parabolic subgroup of G. Denote by P^- the opposite parabolic subgroup of P, *i.e.*, the unique parabolic subgroup intersecting P in M. Let $\bar{P} = \bar{M}\bar{U}$ be the unique standard parabolic subgroup conjugate to P^- [C]; we can have either $\bar{P} = P$ or $\bar{P} \neq P$.

Proposition 3.2 *Let $P = MU$ be a standard parabolic subgroup of G, and let σ be an irreducible supercuspidal regular representation of M. Take $w \in W$ such that $w(P^-) = \bar{P}$. If q is an irreducible subrepresentation of $i_{G,M}(\sigma)$, then $\tilde{q}$ is a subrepresentation of $i_{G,\bar{M}}(w\tilde{\sigma})$.*

(Here $\tilde{\sigma}$ denotes the contragredient representation of σ.)

In the proof of Proposition 3.2, we shall use non-standard parabolic induction, in the notation of [BZ]: if $P = MU$ is a parabolic subgroup of G, and σ is a representation of M, we denote by $i_{U,1}(\sigma)$ the representation parabolically induced by σ from $P = MU$. If P is a standard parabolic subgroup, then $i_{U,1}(\sigma) = i_{G,M}(\sigma)$.

The following proposition can be proved directly:

Proposition 3.3 *Let $P = MU$ be a parabolic subgroup of G. Let $h\colon G \to G$ be an automorphism of the topological group G. For smooth representations σ of M and π of G, we have*

$$i_{h(U),1}\big(h(\sigma)\big) \cong h\big(i_{U,1}(\sigma)\big),$$
$$r_{h(U),1}\big(h(\pi)\big) \cong h\big(r_{U,1}(\pi)\big).$$

(The representation $h(\pi)$ is given by $h(\pi)(g) = \pi\big(h^{-1}(g)\big)$.

Corollary 3.4 *Let $P = MU$ be a parabolic subgroup of G, and $w \in W$. For smooth representations σ of M and π of G, we have*

$$i_{w(U),1}(w\sigma) \cong i_{U,1}(\sigma),$$
$$r_{w(U),1}(\pi) \cong w\big(r_{U,1}(\pi)\big).$$

Proof of Proposition 3.2 If π is an admissible representation of G, then we have from [C, Corollary 4.2.5].

$$(**) \qquad \widetilde{r_{U,1}(\pi)} \cong r_{U^-,1}(\tilde{\pi}).$$

Let $q \hookrightarrow i_{G,M}(\sigma)$, q irreducible. Then

$$\mathrm{Hom}_M\big(r_{M,G}(q), \sigma\big) \neq \{0\},$$

so $\mathrm{Hom}_M\big(\tilde{\sigma}, \widetilde{r_{M,G}(q)}\big) \neq \{0\}$ and $\mathrm{Hom}_M\big(\widetilde{r_{M,G}(q)}, \tilde{\sigma}\big) \neq \{0\}$, because $\widetilde{r_{M,G}(q)}$ is a direct sum of irreducible representations. Now, using (**), Frobenius reciprocity and Corollary 3.4, we get

$$\begin{aligned}\{0\} \neq \mathrm{Hom}_M\big(\widetilde{r_{M,G}(q)}, \tilde{\sigma}\big) &\cong \mathrm{Hom}_M\big(r_{U^-,1}(\tilde{q}), \tilde{\sigma}\big) \\ \cong \mathrm{Hom}_G\big(\tilde{q}, i_{U^-,1}(\tilde{\sigma})\big) &\cong \mathrm{Hom}_G\big(\tilde{q}, i_{G,\bar{M}}(w\tilde{\sigma})\big).\end{aligned}$$

■

4 Non-Regular Case

Recall some notation from [C]. Denote by Z the center of G. Let (π, V) be an admissible representation of G, and ω a character of Z. For each integer $n > 1$, we define

$$V_{\omega,n} = \left\{ v \in V \mid \big(\pi(z) - \omega(z)\big)^n v = 0, z \in Z \right\},$$

and also define

$$V_{\omega,\infty} = \bigcup_{n\in\mathbb{N}} V_{\omega,n},$$

$$V_\omega = V_{\omega,1}.$$

Each $V_{\omega,n}$ is G-stable. The representation (π, V) is called an *ω-representation* if $V = V_\omega$. We will call (π, V) an *(ω, n)-representation* if $V = V_{\omega,n}$.

Denote by $\mathrm{JH}(\pi)$ the set of equivalence classes of irreducible subquotients of π.

Proposition 4.1 *Let (π, V) be an admissible supercuspidal representation of G of finite length. Then there exists a direct sum decomposition*

$$V = \bigoplus_{\rho\in \mathrm{JH}(\pi)} V_\rho,$$

such that $\mathrm{JH}(V_\rho) = \{\rho\}$.

We will prove the proposition using a direct sum decomposition $V = \bigoplus V_{\omega,\infty}$ [C, Proposition 2.1.9], but Philip Kutzko hinted that it can also be proved using Bernstein decomposition [B].

Lemma 4.2 *Let π be an admissible supercuspidal finite length (ω, n)-representation. Then the following are equivalent:*

(1) $\rho \in \mathrm{JH}(\pi)$;
(2) $\rho \hookrightarrow V_\omega$;
(3) $\rho \hookrightarrow \pi$.

Proof Obviously, (2) $\Leftrightarrow$ (3) and (2), (3) $\Rightarrow$ (1). The implication (1) $\Rightarrow$ (2) is given in [BZ, Theorem 2.4.(b)]. ■

Proof of Proposition 4.1 According to [C, Proposition 2.1.9], we may assume that (π, V) is (ω, n)-representation for some central character ω and $n \in \mathbb{N}$. Take $\rho \in \mathrm{JH}(\pi)$. Then, by Lemma 4.2, there exists a subspace $V_1 \subset V$ such that (ρ, V_1) is a subrepresentation of V. If $\rho \in \mathrm{JH}(V/V_1)$, then $\rho \hookrightarrow V/V_1$. Hence, there exists a finite sequence

$$\{0\} = V_0 \subset V_1 \subset \cdots \subset V_k \subset V$$

such that $V_i/V_{i-1} \cong \rho$, $i = 1, \ldots, k$, and $\rho \notin \mathrm{JH}(V/V_k)$. Set $V_k = V_\rho$. We can do the same for any $\rho \in \mathrm{JH}(\pi)$. It follows $V = \bigoplus_{\rho \in \mathrm{JH}(\pi)} V_\rho$. ■

Corollary 4.3 *Let σ be an irreducible supersupercuspidal representation of M. Then*

(1) There exists a direct sum decomposition

$$r_{M,G} \circ i_{G,M}(\sigma) \cong \bigoplus_{w \in W(\Theta)/W(\sigma)} V_w$$

such that s.s.$(V_w) = kw\sigma$, where $k = \operatorname{card}\big(W(\sigma)\big)$.

(2) For all $w \in W(\Theta)$, $w\sigma$ is a subrepresentation and a quotient of $r_{M,G} \circ i_{G,M}(\sigma)$.

(3) For all $w_1, w_2 \in W(\Theta)$,

$$\dim_{\mathbb{C}} \operatorname{Hom}_G\big(i_{G,M}(w_1\sigma), i_{G,M}(w_2\sigma)\big) \geq 1.$$

(4) Let π be an irreducible subquotient of $i_{G,M}(\sigma)$. For any $w \in W(\Theta)$, we have

$$\pi \hookrightarrow i_{G,M}(w\sigma) \Longleftrightarrow w\sigma \leq r_{M,G}(\pi).$$

Proof (1), (2) follow from Proposition 4.1.

(3) follows from (2), since

$$\operatorname{Hom}_G\big(i_{G,M}(w_1\sigma), i_{G,M}(w_2\sigma)\big) \cong \operatorname{Hom}_M\big(r_{M,G} \circ i_{G,M}(w_1\sigma), w_2\sigma\big).$$

(4) The first implication follows from Frobenius reciprocity. The second follows from Frobenius reciprocity and 1. ■

Now, we consider the case when $P = P_\Theta = MU$ is a maximal standard parabolic subgroup and σ is an irreducible supercuspidal representation of M.

If $P \neq \bar{P}$, then $W(\Theta) = \{1\}$. This implies that σ is regular and $i_{G,M}(\sigma)$ irreducible (also, *cf.* [C, Theorem 7.1.4]).

If $P = \bar{P}$, then $W(\Theta) = \{1, w\}$ and *s.s.*$\big(r_{M,G} \circ i_{G,M}(\sigma)\big) = \sigma + w\sigma$. Suppose that $w\sigma \cong \sigma$ (hence, σ is non-regular). Then a simple application of Frobenius reciprocity gives the following:

(1) If $i_{G,M}(\sigma)$ is irreducible, then the sum $r_{M,G} \circ i_{G,M}(\sigma) = \sigma + \sigma$ is not direct.
(2) If $i_{G,M}(\sigma)$ is reducible, then it is of length 2 [C, Cor. 7.1.2] and $r_{M,G} \circ i_{G,M}(\sigma) = \sigma \oplus \sigma$.
   (a) If $i_{G,M}(\sigma)$ is completely reducible, then it is the direct sum of two inequivalent irreducible subrepresentations

$$i_{G,M}(\sigma) = p_1 \oplus p_2, \quad p_1 \not\cong p_2.$$

   (b) If $i_{G,M}(\sigma)$ is reducible, but not semi-simple, then $i_{G,M}(\sigma)$ has one irreducible subquotient and its multiplicity is two.

Proposition 4.4 *Let $P = P_\Theta = MU$ be a standard parabolic subgroup of G, and let σ be an irreducible supercuspidal representation of M. Suppose that $W(\sigma) = \{1, w\} = W_\Omega(\Theta)$ for some $\Omega, \Theta \subseteq \Omega \subseteq \Delta$. Let $Q = P_\Omega = NV$.*

(1) If $i_{N,M}(\sigma)$ is completely reducible, then $i_{N,M}(\sigma)$ is the direct sum of two inequivalent irreducible subrepresentations, $i_{N,M}(\sigma) = p_1 \oplus p_2$, and $i_{G,M}(\sigma) = i_{G,N}(p_1) \oplus i_{G,N}(p_2)$. Further, $i_{G,N}(p_i)$, $i = 1, 2$, has a unique irreducible subrepresentation q_i, $i = 1, 2$, and $q_1 \not\cong q_2$.
(2) If $i_{N,M}(\sigma)$ is irreducible, then $i_{G,M}(\sigma)$ has a unique irreducible subrepresentation p. Further, $\sigma + \sigma \hookrightarrow r_{M,G}(p)$, and this sum is not direct.

Proof For (1), let q_i, $i = 1, 2$ be a subrepresentation of $i_{G,N}(p_i)$. Then $q_i \hookrightarrow i_{G,M}(\sigma)$ and Frobenius reciprocity give $\sigma \le r_{M,G}(q_i)$. But the multiplicity of σ in $r_{M,G} \circ i_{G,M}(\sigma)$ is two, so q_1, q_2 are the only two irreducible subrepresentations of $i_{G,M}(\sigma)$. Since

$$\dim_{\mathbb{C}} \operatorname{Hom}_G\big(q_1, i_{G,M}(\sigma)\big) = \dim_{\mathbb{C}} \operatorname{Hom}_M\big(r_{M,G}(q_1), \sigma\big) = 1,$$

it follows that $q_1 \not\cong q_2$. This proves 1.

For (2), let p be an irreducible subrepresentation of $i_{G,M}(\sigma)$. We have

$$\operatorname{Hom}_G\big(p, i_{G,M}(\sigma)\big) \cong \operatorname{Hom}_N\big(r_{N,G}(p), i_{N,M}(\sigma)\big).$$

Since $i_{N,M}(\sigma)$ is irreducible, it is a quotient of $r_{N,G}(p)$ and we have

$$2\sigma \le r_{M,N} \circ i_{N,M}(\sigma) \le r_{M,G}(p).$$

It follows that p is the unique irreducible subrepresentation of $i_{G,M}(\sigma)$. ■

5 Decomposition of Weyl Group

Suppose that M and N are standard Levi subgroups of G, $M < N$, corresponding to $\Theta \subseteq \Omega \subseteq \Delta$.

Theorem 5.1 *Let σ be an irreducible supercuspidal representation of M. Then*

$$\begin{aligned} s.s.\big(r_{M,G}\circ i_{G,M}(\sigma)\big) &= \sum_{\substack{w\in[W_\Omega\backslash W/W_\Theta]\\ w(\Theta)\subseteq\Omega}} \sum_{v\in W_\Omega(\Theta,w\Theta)} w^{-1}v^{-1}\sigma \\ &= \sum_{\substack{w\in[W_\Theta\backslash W/W_\Omega]\\ w^{-1}(\Theta)\subseteq\Omega}} \sum_{v\in W_\Omega(w\Theta,\Theta)} wv\sigma. \end{aligned}$$

Proof

$$\begin{aligned} s.s.\big(r_{M,G}\circ i_{G,M}(\sigma)\big) &= s.s.\big(r_{M,G}\circ i_{G,N}\circ i_{N,M}(\sigma)\big) \\ &= \sum_{w\in[W_\Omega\backslash W/W_\Theta]} i_{M,M'}\circ w^{-1}\circ r_{N',N}\circ i_{N,M}(\sigma) \end{aligned}$$

where $M' = M\cap w^{-1}(N)$, $N' = w(M)\cap N$. From Theorem 2.2, we have

$$s.s.\big(r_{M,G}\circ i_{G,M}(\sigma)\big) = \sum_{w'\in W(\Theta)} w'\sigma,$$

and this is a sum of supercuspidal representations. Hence, if

$$i_{M,M'}\circ w^{-1}\circ r_{N',N}\circ i_{N,M}(\sigma)$$

is different from zero, it is a sum of supercuspidal representations,

$$i_{M,M'}\circ w^{-1}\circ r_{N',N}\circ i_{N,M}(\sigma) = \sum_{w'\in S\subseteq W(\Theta)} w'\sigma.$$

We conclude $M = M' = M\cap w^{-1}(N)$, so $M\subset w^{-1}(N)$, $w(M)\subset N$. It follows that $w(\Theta)\subset\Omega$, $N' = w(M)\cap N = w(M)$. Hence,

$$w^{-1}\circ r_{N',N}\circ i_{N,M}(\sigma) = \sum_{w'\in S\subseteq W(\Theta)} w'\sigma$$

$$r_{N',N}\circ i_{N,M}(\sigma) = \sum_{w'\in S\subseteq W(\Theta)} ww'\sigma$$

$$\sum_{v\in[W_{w(M)}\backslash W_\Omega/W_M]} i_{N',N''}\circ v^{-1}\circ r_{M'',M}(\sigma) = \sum_{w'\in S\subseteq W(\Theta)} ww'\sigma$$

where $M'' = M\cap v(N')$, $N'' = v^{-1}(M)\cap N'$. This is again a sum of supercuspidal representations, so

$$i_{N',N''}\circ v^{-1}\circ r_{M'',M}(\sigma)\neq 0$$

implies $N' = N''$, $M = M''$. Now it follows from $N'' = v^{-1}(M)\cap N'$ and $N' = w(M)$ that $N' = N'' = v^{-1}(M) = w(M)$. ■

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Corollary 5.2 *If $w\Theta \subset \Omega$ implies $w\Theta = \Theta$, then*

$$s.s.\big(r_{M,G} \circ i_{G,M}(\sigma)\big) = \sum_{w \in [W_\Omega \backslash W / W_\Theta] \cap W(\Theta)} \sum_{v \in W_\Omega(\Theta)} w^{-1} v^{-1} \sigma$$
$$= \sum_{w \in [W_\Theta \backslash W / W_\Omega] \cap W(\Theta)} \sum_{v \in W_\Omega(\Theta)} w v \sigma. \qquad \blacksquare$$

Corollary 5.3 *Suppose that $w\Theta \subset \Omega$ implies $w\Theta = \Theta$. Let σ be an irreducible supercuspidal representation of M. Let δ be a subquotient of $i_{N,M}(\sigma)$. Suppose that*

$$s.s.\big(r_{M,N}(\delta)\big) = \sum_{v \in S \subseteq W_\Omega(\Theta)} v\sigma.$$

Then

$$s.s.\big(r_{M,G} \circ i_{G,N}(\delta)\big) = \sum_{w \in [W_\Theta \backslash W / W_\Omega] \cap W(\Theta)} \sum_{v \in S} w v \sigma.$$

Proof We have

$$s.s.\big(r_{M,G} \circ i_{G,N}(\delta)\big) = \sum_{w \in [W_\Theta \backslash W / W_\Omega] \cap W(\Theta)} w \circ r_{N',N}(\delta),$$

where $N' = M$. $\blacksquare$

Remark 5.1 If Θ and Θ' are subsets of Δ, they are called associates if the set $W_\Omega(\Theta, \Theta') = \{w \in W_\Omega \mid w\Theta' = \Theta\}$ is not empty. For any $\Theta \subseteq \Delta$, denote by $\{\Theta\}_\Delta$ the set of its associates [C] and by $\{\Theta\}_\Omega$ the set of its associates in Ω.

Let σ be an irreducible supercuspidal representation of M. Let δ be a subquotient of $i_{N,M}(\sigma)$. Suppose that for every $\Theta' \in \{\Theta\}_\Omega$ we are given

$$s.s.\big(r_{M_{\Theta'},N}(\delta)\big) = \sum_{v \in S(\Theta')} v\sigma.$$

Then

$$s.s.\big(r_{M,G} \circ i_{G,N}(\delta)\big) = \sum_{\Theta' \in \{\Theta\}_\Omega} \sum_{\substack{w \in [W_\Theta \backslash W / W_\Omega] \\ w(\Theta') = \Theta}} \sum_{v \in S(\Theta')} w v \sigma.$$

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6 Description of Some Subsets of Weyl Groups for Classical Groups

If we want to apply results of Section 5 to Levi subgroups $M < N < G$ corresponding to the subsets of the simple roots $\Theta \subseteq \Omega \subseteq \Delta$, we need to understand precisely $W(\Theta)$ and $[W_\Theta \setminus W/W_\Omega] \cap W(\Theta)$. In this section, we will describe these sets for certain Levi subgroups of classical p-adic groups.

(a) We consider the group

$$M = M_\Theta = \underbrace{\mathrm{GL}(k,F) \times \cdots \times \mathrm{GL}(k,F)}_{n} \times S_m,$$

where

$$S_m = \begin{cases} \mathrm{Sp}(m,F), \\ \mathrm{SO}(2m+1,F), \\ \mathrm{SO}(2m,F), & m \geq 1 \text{ or } m = 0,\ k\text{-even}. \end{cases}$$

M is isomorphic to a standard Levi subgroup of $G = S_K$, $K = kn+m$ [T1], [Ba2]. The description of the Weyl group for G can be found in [T1], [Ba1]. A simple calculation gives

$$W(\Theta) \cong \mathrm{Sym}(n) \ltimes \{\pm 1\}^n.$$

Here $(\epsilon_1, \ldots, \epsilon_n) \in W(\Theta)$, $\epsilon_i = \pm 1$ for $i = 1, \ldots, n$, corresponds to

$$(\underbrace{\epsilon_1, \ldots \epsilon_1}_{k}, \ldots, \underbrace{\epsilon_n, \ldots \epsilon_n}_{k}, \underbrace{1, \ldots, 1}_{m}) \in W,$$

for $\mathrm{Sp}(m,F)$, $\mathrm{SO}(2m+1,F)$ for every k, and for $\mathrm{SO}(2m,F)$, k even, and

$$\Big(\underbrace{\epsilon_1, \ldots \epsilon_1}_{k}, \ldots, \underbrace{\epsilon_n, \ldots \epsilon_n}_{k}, \underbrace{1, \ldots, 1}_{(m-1)}, \prod_{i=1}^{n} \epsilon_i\Big) \in W,$$

for $\mathrm{SO}(2m,F)$, k odd.

For an ordered partition $(n_1, \ldots, n_q = n)$ of n, denote by

$$\mathrm{Sh}_{(n_1,\ldots,n_q)}$$

the set of all shuffles of sets

$$\{1, \ldots, n_1\}, \{n_1+1, \ldots, n_2\}, \ldots, \{n_{q-1}+1, \ldots, n_q\}.$$

(Suppose that $S_1, S_2, \ldots, S_q$ are disjoint ordered sets. A shuffle of the sets $S_1, S_2, \ldots, S_q$ is a permutation p of the set $S = S_1 \cup S_2 \cup \cdots \cup S_q$ which preserves the order on each of the sets S_k, $k = 1, 2, \ldots, q$, *i.e.*, if s_1, s_2 are contained in the same set S_k, then $s_1 < s_2$ implies $p(s_1) < p(s_2)$.)

For $k \leq l \leq n$, define a permutation $z_{(k,l)}$ with

$$z_{(k,l)}(j) = \begin{cases} j, & j < k; \\ k + l - j, & k \leq j \leq l; \\ j, & j > l. \end{cases}$$

If $k > l$, we define $z_{(k,l)} = \mathrm{id}$. For $l \leq n$, set

$$N_l = \mathrm{GL}(kl, F) \times S_{(n-l)k+m},$$
$$M < N_l < G.$$

We have the following:

Lemma 6.1

$$[W_M \setminus W/W_{N_l}] \cap W(\Theta) = \bigcup_{i=0}^{l} \mathrm{Sh}_{(l-i,l,n)}\, z_{(l-i+1,l)}(1_{l-i}, -1_i, 1_{n-l}).$$

Proof The simple roots are $\alpha_i = e_i - e_{i+1}$, when $i < kn + m$, and $\alpha_{kn+m} = e_{kn+m-1} + e_{kn+m}$ [T1], [Ba2]. The set $\Omega \subset \Delta$ corresponding to N is $\Omega = \Delta \setminus \{\alpha_{lk}\}$. Since

$$\begin{aligned}[W_M \setminus W/W_{N_l}] \cap W(\Theta) &= [W_\Theta \setminus W/W_\Omega] \cap W(\Theta) \\ &= \{w \in W(\Theta) \mid w\alpha > 0, \forall \alpha \in \Omega\},\end{aligned}$$

$w \in W(\Theta)$ is an element of $[W_\Theta \setminus W/W_\Omega]$ if and only if the following condition is satisfied:

$$w\alpha_{ik} > 0 \quad \text{for } i = 1, 2, \ldots, l-1, l+1, \ldots, n.$$

The positive roots are $e_j - e_l$, $j < l$ and $e_j + e_l$. This gives the lemma. ∎

(b) Suppose that k is odd. Let

$$M = M_\Theta = \underbrace{\mathrm{GL}(k, F) \times \cdots \times \mathrm{GL}(k, F)}_{n}$$

be a standard Levi subgroup of $G = \mathrm{SO}(2nk, F)$. Then

$$W(\Theta) \cong \{\pm 1\}^{n-1} \rtimes \mathrm{Sym}(n),$$

where $\{\pm 1\}^{n-1} = \{(\epsilon_1, \ldots, \epsilon_n) \mid \prod \epsilon_i = 1\}$.

For $l \leq n$, set

$$N_l = \mathrm{GL}(kl, F) \times \mathrm{SO}\big(2(n-l)k, F\big).$$

Let $\Omega \subseteq \Delta$ correspond to N.

If $l = n$, then $w(\Theta) \subseteq \Omega$ implies $w(\Theta) = \Theta$. In other words, the set $\{\Theta\}_\Omega$ of associates of Θ in Ω is equal to $\{\Theta\}$.

If $l < n$, then $\{\Theta\}_\Omega = \{\Theta, s(\Theta)\}$. Here s denotes the automorphism of the root system which interchanges α_{kn-1} and α_{kn} [Ba1], [Ba2]. We have

$$M_{s(\Theta)} = sM_\Theta s^{-1} = s(M),$$

where

$$s = \begin{bmatrix} I & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & I \end{bmatrix}.$$

Lemma 6.2

(1) The set $[W_M \setminus W/W_{N_l}] \cap W(\Theta)$ *is equal to*

$$\bigcup_{\substack{i=0 \\ i \text{ even}}}^{l} \mathrm{Sh}_{(l-i,l,n)}\, z_{(l-i+1,l)}(1_{l-i}, -1_i, 1_{n-l}).$$

(2) The set of all $w \in [W_M \setminus W/W_{N_l}]$ *such that* $w\big(s(\Theta)\big) = \Theta$ *is equal to*

$$\bigcup_{\substack{i=0 \\ i \text{ odd}}}^{l} \mathrm{Sh}_{(l-i,l,n)}\, z_{(l-i+1,l)}(1_{l-i}, -1_i, 1_{n-l})s.$$

7 Applications

We shall apply our results to study representations of classical p-adic groups which we refer to as generalized Steinbergs (see Remark 7.1).

Recall some notation from [Z] and [T1]. For admissible representations ρ_1, ρ_2 of $\mathrm{GL}(k_1, F)$, $\mathrm{GL}(k_2, F)$ respectively, we define

$$\rho_1 \times \rho_2 = i_{G,M}(\rho_1 \otimes \rho_2),$$

where $M \cong \mathrm{GL}(k_1, F) \times \mathrm{GL}(k_2, F)$ is a standard Levi subgroup of $G = \mathrm{GL}(k_1 + k_2, F)$. Also, set

$$S_m = \begin{cases} \mathrm{Sp}(m, F), \\ \mathrm{SO}(2m+1, F), \\ \mathrm{SO}(2m, F). \end{cases}$$

If ρ is an admissible representation of $\mathrm{GL}(k, F)$ and σ is an admissible representation of S_m, then we define

$$\rho \rtimes \sigma = i_{G,M}(\rho \otimes \sigma),$$

where $M \cong \mathrm{GL}(k, F) \times S_m$ is a standard Levi subgroup of $G = S_{k+m}$ [T1], [Ba2]. We have

$$(\rho_1 \times \rho_2) \rtimes \sigma = \rho_1 \rtimes (\rho_2 \rtimes \sigma).$$

Let ν denote $|\det|$. Let ρ be an irreducible supercuspidal representation of $\mathrm{GL}(k, F)$ and n a non-negative integer. The set $[\rho, \nu^n\rho] = \{\rho, \nu\rho, \dots, \nu^n\rho\}$ is called a segment. We know from [Z] that the representation $\nu^n\rho \times \nu^{n-1}\rho \times \cdots \times \rho$ has a unique irreducible subrepresentation $\delta([\rho, \nu^n\rho])$ and

$$r_{M,G}\big(\delta([\rho, \nu^n\rho])\big) = \nu^n\rho \otimes \nu^{n-1}\rho \otimes \cdots \otimes \rho.$$

The following proposition is similar to Proposition 3.1 in [T2], here extended to the case of $\mathrm{SO}(2n, F)$. Also, we allow $\alpha = 0$.

Proposition 7.1 *Let ρ be an irreducible unitarizable supercuspidal representation of $\mathrm{GL}(k, F)$ and let σ be an irreducible supercuspidal representation of S_m. Let $\alpha \geq 0$. Suppose that $\nu^\alpha\rho \rtimes \sigma$ is reducible. Let n be a non-negative integer. Then:*

(1) $\rho \cong \tilde{\rho}$.
(2) $\nu^{-\alpha}\rho \rtimes \sigma$ is reducible. $\nu^\beta\rho \rtimes \sigma$ is irreducible for any real number $\beta \neq \pm\alpha$.
(3) If $\alpha > 0$, then the representation $\nu^{\alpha+n}\rho \times \cdots \times \nu^\alpha\rho \rtimes \sigma$ contains a unique irreducible subrepresentation, denote it by $\delta([\nu^\alpha\rho, \nu^{\alpha+n}\rho], \sigma)$. This subrepresentation is square-integrable. We have

$$r_{M,G}\big(\delta([\nu^\alpha\rho, \nu^{\alpha+n}\rho], \sigma)\big) = \nu^{\alpha+n}\rho \otimes \cdots \otimes \nu^\alpha\rho \otimes \sigma,$$
$$\delta([\nu^\alpha\rho, \nu^{\alpha+n}\rho], \sigma)^\sim \cong \delta([\nu^\alpha\rho, \nu^{\alpha+n}\rho], \tilde{\sigma}).$$

(4) If $\alpha = 0$, then the representation $\nu^n\rho \times \cdots \times \rho \rtimes \sigma$ contains two inequivalent irreducible subrepresentations q_1 and q_2. For $n > 0$, q_1 and q_2 are square integrable, while for $n = 0$ they are tempered (but not square integrable). We have

$$r_{M,G}(q_1) = r_{M,G}(q_2) = \nu^n\rho \otimes \cdots \otimes \rho \otimes \sigma.$$

Note that we are not requiring α to be half-integral.

Proof For $m = 0$ and k odd, the representation $\nu^\alpha\rho \rtimes 1$ of $S_k = \mathrm{SO}(2k, F)$ is irreducible [Sh, Prop. 3.5]. Thus, the assumption on the reducibility of $\nu^\alpha\rho \rtimes \sigma$ excludes the case $m = 0$ and k odd for $\mathrm{SO}(2m, F)$, and we are in the situation described in the case (a) of Section 6.

(1) follows from [T2], [Ba2], and (2) follows from [S].

(3) Let

$$\tau = \nu^{\alpha+n}\rho \otimes \cdots \otimes \nu^\alpha\rho \otimes \sigma.$$

Since τ is regular, it follows from Lemma 3.1 that $i_{G,M}(\tau)$ has a unique irreducible subrepresentation. Set

$$N_{n+1} = \mathrm{GL}\big((n+1)k, F\big) \times S_m,$$
$$N_n = \mathrm{GL}(nk, F) \times S_{k+m}.$$

Let $q \hookrightarrow \nu^\alpha\rho \rtimes \sigma$. Then $r_{\mathrm{GL}(k,F)\times S_m}(q) = \nu^\alpha\rho \otimes \sigma$. Let

$$q_0 = \delta([\nu^{\alpha+1}\rho, \nu^{\alpha+n}\rho]) \otimes q.$$

This is a representation of N_n, and $r_{M,N_n}(q_0) = \tau$, $i_{G,N_n}(q_0) \hookrightarrow i_{G,M}(\tau)$. On the other hand,

$$q_1 = \delta([\nu^\alpha\rho, \nu^{\alpha+n}\rho]) \otimes \sigma$$

is a representation of N_{n+1}, and $r_{M,N_{n+1}}(q_1) = \tau$, $i_{G,N_{n+1}}(q_1) \hookrightarrow i_{G,M}(\tau)$. Let $\delta = \delta([\nu^\alpha\rho, \nu^{\alpha+n}\rho], \sigma)$ be the unique irreducible subrepresentation of $i_{G,M}(\tau)$. Then

$$\delta \hookrightarrow i_{G,N_n}(q_0),$$
$$\delta \hookrightarrow i_{G,N_{n+1}}(q_1),$$
$$r_{M,G}(\delta) \le r_{M,G} \circ i_{G,N_n}(q_0),$$
$$r_{M,G}(\delta) \le r_{M,G} \circ i_{G,N_{n+1}}(q_1).$$

From Corollary 5.2 and Lemma 6.1, we have the following

$$r_{M,G} \circ i_{G,N_n}(q_0) = \sum_{i=0}^{n} \mathrm{Sh}_{(n-i,n,n+1)}\, z_{(n-i+1,n)}(1_{n-i}, -1_i, 1)\tau,$$

$$r_{M,G} \circ i_{G,N_{n+1}}(q_1) = \sum_{i=0}^{n+1} \mathrm{Sh}_{(n-i+1,n+1)}\, z_{(n-i+2,n+1)}(1_{n+1-i}, -1_i)\tau.$$

In the first sum, all terms contain the factor $\nu^\alpha\rho$, but in the second sum, we have $\nu^\alpha\rho$ only for $i = 0$. Since $\mathrm{Sh}_{(n+1,n+1)} = \{1\}$, we conclude that the only common factor for $r_{M,G} \circ i_{G,N_n}(q_0)$ and $r_{M,G} \circ i_{G,N_{n+1}}(q_1)$ is τ. Hence, $r_{M,G}(\delta) = \tau$. The Casselman square integrability criterion [C], [Ta2], [Ba2] tells us that δ is square integrable.

Let w_l be the longest element in W and $w_{l,\Theta}$ the longest element in W_Θ [C]. Take $w = w_l w_{l,\Theta}$. Then $w \in W(\Theta)$, $w(P) = P^-$, and $w(\rho_1 \otimes \cdots \otimes \rho_k \otimes \sigma) = \tilde\rho_1 \otimes \cdots \otimes \tilde\rho_k \otimes \sigma$. Now it follows from Proposition 3.2. that $\delta([\nu^\alpha\rho, \nu^{\alpha+n}\rho], \sigma)^\sim \cong \delta([\nu^\alpha\rho, \nu^{\alpha+n}\rho], \tilde\sigma)$.

(4) According to Proposition 4.4, $\nu^n\rho \times \cdots \times \rho \rtimes \sigma = i_{G,N}(p_1) \oplus i_{G,N}(p_2)$, where p_1 and p_2 are two inequivalent subrepresentations of $\nu^n\rho \otimes \cdots \nu\rho \otimes (\rho \rtimes \sigma)$. Further, $i_{G,N}(p_i)$, $i = 1, 2$ contains a unique irreducible subrepresentation q_i, $i = 1, 2$, and $q_1 \not\cong q_2$. The proof that

$$r_{M,G}(q_1) = r_{M,G}(q_2) = \nu^n\rho \otimes \cdots \otimes \rho \otimes \sigma$$

is by induction on n. The proof is similar to that of (3), applied to the groups N_n and $N_1 = \mathrm{GL}(k, F) \times S_{nk+m}$. We are using the fact that $\nu^{n-1}\rho \times \cdots \times \rho \rtimes \sigma$ has two inequivalent subrepresentations, with Jacquet modules

$$\nu^{n-1}\rho \otimes \cdots \otimes \rho \otimes \sigma,$$

which is assured by inductive assumption. ■

Additional properties, similar to those for the Steinberg representation [C], [BoW], are given in the following proposition.

Proposition 7.2 *Let ρ be an irreducible unitarizable supercuspidal representation of $\mathrm{GL}(k, F)$, σ an irreducible supercuspidal representation of S_m. Let $\alpha > 0$. Suppose that $\nu^\alpha\rho \rtimes \sigma$ is reducible. Then:*

(1) $\nu^{\alpha+n}\rho \times \cdots \times \nu^\alpha\rho \rtimes \sigma$ is a multiplicity one representation.
(2) The length of $\nu^{\alpha+n}\rho \times \cdots \times \nu^\alpha\rho \rtimes \sigma$ is 2^{n+1}.
(3) $\delta([\nu^\alpha\rho, \nu^{\alpha+n}\rho], \sigma)$ is the unique square integrable subquotient of $\nu^{\alpha+n}\rho \times \cdots \times \nu^\alpha\rho \rtimes \sigma$.

Proof (1) follows from Lemma 3.1, because $\nu^{\alpha+n}\rho \otimes \cdots \otimes \nu^\alpha\rho \otimes \sigma$ is regular.

(2) By induction on n. Let $n \geq 1$. Set

$$N = S_{nk+m},$$
$$\tau = \nu^{\alpha+n-1}\rho \otimes \cdots \otimes \nu^\alpha\rho \otimes \sigma.$$

Let q be an irreducible subquotient of $\nu^{\alpha+n-1}\rho \times \cdots \times \nu^\alpha\rho \rtimes \sigma$. Write

$$r_{M',N}(q) = \sum_{w \in S} w\tau.$$

Fix $w_0 \in S$. We know from Lemma 3.1 that

$$w \in S \iff i_{N,M'}(w\tau) \cong i_{N,M'}(w_0\tau).$$

Now, we consider the representation $\nu^{\alpha+n}\rho \rtimes q$. According to Corollary 5.3 and Lemma 6.1, we have

$$r_{M,G}(\nu^{\alpha+n}\rho \rtimes q) = \left(\mathrm{Sh}_{(1,n+1)} \cup \mathrm{Sh}_{(1,n+1)}(-1, 1_n)\right)\left(\nu^{\alpha+n}\rho \otimes r_{M',N}(q)\right).$$

We consider the action of $\mathrm{Sh}_{(1,n+1)} \cup \mathrm{Sh}_{(1,n+1)}(-1, 1_n)$ on $\nu^{\alpha+n}\rho \otimes w_0\tau$. $w_0\tau$ is the tensor product of a permutation of elements $\nu^{\epsilon_1(\alpha+n-1)}\rho, \ldots, \nu^{\epsilon_n\alpha}\rho$, where ϵ_i is 1 or -1, all tensored with σ. We will assume that $\epsilon_1 = 1$. (The same basic argument works when $\epsilon_1 = -1$.) Hence,

$$w_0\tau \cong \nu^{\beta_1}\rho \otimes \cdots \nu^{\beta_k}\rho \otimes \nu^{\alpha+n-1}\rho \otimes \nu^{\beta_{k+1}}\rho \otimes \cdots \nu^{\beta_{n-1}}\rho \otimes \sigma.$$

Now, we have

$$\begin{aligned}
&\nu^{\alpha+n}\rho \rtimes i_{N,M'}(w_0\tau)\\
&\quad = \nu^{\alpha+n}\rho \times \nu^{\beta_1}\rho \times \cdots \times \nu^{\beta_k}\rho \times \nu^{\alpha+n-1}\rho \times \nu^{\beta_{k+1}}\rho \times \cdots \times \nu^{\beta_{n-1}}\rho \rtimes \sigma\\
&\quad \cong \nu^{\beta_1}\rho \times \nu^{\alpha+n}\rho \times \cdots \times \nu^{\beta_k}\rho \times \nu^{\alpha+n-1}\rho \times \nu^{\beta_{k+1}}\rho \times \cdots \times \nu^{\beta_{n-1}}\rho \rtimes \sigma\\
&\quad \vdots\\
&\quad \cong \nu^{\beta_1}\rho \times \cdots \times \nu^{\beta_k}\rho \times \nu^{\alpha+n}\rho \times \nu^{\alpha+n-1}\rho \times \nu^{\beta_{k+1}}\rho \times \cdots \times \nu^{\beta_{n-1}}\rho \rtimes \sigma\\
&\quad \ncong \nu^{\beta_1}\rho \times \cdots \times \nu^{\beta_k}\rho \times \nu^{\alpha+n-1}\rho \times \nu^{\alpha+n}\rho \times \nu^{\beta_{k+1}}\rho \times \cdots \times \nu^{\beta_{n-1}}\rho \rtimes \sigma.
\end{aligned}$$

The inequivalence follows from Lemma 3.1 for the regular representation $\nu^{\alpha+n}\rho \otimes \cdots \otimes \nu^{\alpha}\rho \otimes \sigma$. We also need to use the fact that $\nu^{\alpha+n}\rho \times \nu^{\alpha+n-1}\rho$ is reducible, which follows from [Z]. Furthermore,

$$\begin{aligned}
&\nu^{\beta_1}\rho \times \cdots \times \nu^{\beta_k}\rho \times \nu^{\alpha+n-1}\rho \times \nu^{\alpha+n}\rho \times \nu^{\beta_{k+1}}\rho \times \cdots \times \nu^{\beta_{n-1}}\rho \rtimes \sigma\\
&\quad \cong \nu^{\beta_1}\rho \times \cdots \times \nu^{\beta_k}\rho \times \nu^{\alpha+n-1}\rho \times \nu^{\beta_{k+1}}\rho \times \nu^{\alpha+n}\rho \times \cdots \times \nu^{\beta_{n-1}}\rho \rtimes \sigma\\
&\quad \vdots\\
&\quad \cong \nu^{\beta_1}\rho \times \cdots \times \nu^{\beta_k}\rho \times \nu^{\alpha+n-1}\rho \times \nu^{\beta_{k+1}}\rho \times \cdots \times \nu^{\beta_{n-1}}\rho \times \nu^{\alpha+n} \rtimes \sigma\\
&\quad \cong \nu^{\beta_1}\rho \times \cdots \times \nu^{\beta_k}\rho \times \nu^{\alpha+n-1}\rho \times \nu^{\beta_{k+1}}\rho \times \cdots \times \nu^{\beta_{n-1}}\rho \times \nu^{-\alpha-n}\rho \rtimes \sigma\\
&\quad \vdots\\
&\quad \cong \nu^{-\alpha-n}\rho \times \nu^{\beta_1}\rho \times \cdots \times \nu^{\beta_k}\rho \times \nu^{\alpha+n-1}\rho \times \nu^{\beta_{k+1}}\rho \times \cdots \times \nu^{\beta_{n-1}}\rho \rtimes \sigma.
\end{aligned}$$

Lemma 3.1 tells us that $\big(\mathrm{Sh}_{(1,n+1)} \cup \mathrm{Sh}_{(1,n+1)}(-1, 1_n)\big)\,(\nu^{\alpha+n}\rho \otimes w_0\tau)$ belongs to Jacquet modules of two irreducible subquotients of $\nu^{\alpha+n}\rho \times \cdots \times \nu^{\alpha}\rho \rtimes \sigma$, denote them by p_1 and p_2. Note that p_1 and p_2 are subquotients of $\nu^{\alpha+n}\rho \rtimes q$. Further, for $w \in S$, we have

$$\begin{aligned}
\nu^{\alpha+n}\rho \rtimes i_{N,M'}(w\tau) &\cong \nu^{\alpha+n}\rho \rtimes i_{N,M'}(w_0\tau)\\
\nu^{-\alpha-n}\rho \rtimes i_{N,M'}(w\tau) &\cong \nu^{-\alpha-n}\rho \rtimes i_{N,M'}(w_0\tau),
\end{aligned}$$

so, using Lemma 3.1, we obtain

$$\big(\mathrm{Sh}_{(1,n+1)} \cup \mathrm{Sh}_{(1,n+1)}(-1, 1_n)\big)\,\big(\nu^{\alpha+n}\rho \otimes r_{M',N}(q)\big) \subseteq r_{M,G}(p_1) + r_{M,G}(p_2).$$

It follows that $\nu^{\alpha+n}\rho \rtimes q$ has exactly two irreducible subquotients, p_1 and p_2. By the inductive assumption, $\nu^{\alpha+n-1}\rho \times \cdots \times \nu^{\alpha}\rho \rtimes \sigma$ has 2^n irreducible subquotients. We conclude that the length of $\nu^{\alpha+n}\rho \times \cdots \times \nu^{\alpha}\rho \rtimes \sigma$ is $2 \cdot 2^n = 2^{n+1}$.

(3) Write

$$\tau = \nu^{\alpha+n}\rho \otimes \cdots \otimes \nu^{\alpha}\rho \otimes \sigma.$$

Let q be an irreducible subquotient of $\nu^{\alpha+n}\rho \times \cdots \times \nu^{\alpha}\rho \rtimes \sigma$, $q \neq \delta([\nu^{\alpha}\rho, \nu^{\alpha+n}\rho], \sigma)$. Take $w \in W$ such that $w\tau \leq r_{M,G}(q)$. Then $q \hookrightarrow i_{G,M}(w\tau)$. We can write

$$w\tau = \nu^{\epsilon_1\beta_1}\rho \otimes \cdots \otimes \nu^{\epsilon_{n+1}\beta_{n+1}} \otimes \sigma,$$

where $\epsilon_i = \pm 1$ and $(\beta_1, \ldots, \beta_{n+1})$ is a permutation of $(\alpha, \alpha+1, \ldots, \alpha+n)$. Suppose that there exists $1 \leq k \leq n+1$ such that $\epsilon_1 = \cdots = \epsilon_{k-1} = 1$, $\epsilon_k = -1$. Then

$$\begin{aligned} i_{G,M}(w\tau) &= \nu^{\beta_1}\rho \times \cdots \times \nu^{\beta_{k-1}}\rho \times \nu^{-\beta_k}\rho \times \nu^{\epsilon_{k+1}\beta_{k+1}}\rho \times \cdots \times \nu^{\epsilon_{n+1}\beta_{n+1}}\rho \rtimes \sigma \\ &\cong \nu^{-\beta_k}\rho \times \nu^{\beta_1}\rho \times \cdots \times \nu^{\beta_{k-1}}\rho \times \nu^{\epsilon_{k+1}\beta_{k+1}}\rho \times \cdots \times \nu^{\epsilon_{n+1}\beta_{n+1}}\rho \rtimes \sigma. \end{aligned}$$

According to Frobenius reciprocity,

$$\nu^{-\beta_k}\rho \otimes \nu^{\beta_1}\rho \otimes \cdots \otimes \nu^{\beta_{k-1}}\rho \otimes \nu^{\epsilon_{k+1}\beta_{k+1}}\rho \otimes \cdots \otimes \nu^{\epsilon_{n+1}\beta_{n+1}}\rho \otimes \sigma \leq r_{M,G}(q),$$

so q is not square integrable.

Now, suppose that $\epsilon_1 = \cdots = \epsilon_{n+1} = 1$. Since $q \neq \delta([\nu^{\alpha}\rho, \nu^{\alpha+n}\rho], \sigma)$, we have $w\tau \neq \nu^{\alpha+n}\rho \otimes \cdots \otimes \nu^{\alpha}\rho \otimes \sigma$. There exists $k \in \{1, \ldots, n+1\}$ such that $\beta_k \neq \alpha+n-k+1$, $\beta_{k+1} = \alpha+n-k, \ldots, \beta_{n+1} = \alpha$. Then $\beta_k \neq \alpha$, so $\nu^{\beta_k}\rho \rtimes \sigma$ is irreducible and $\nu^{\beta_k}\rho \rtimes \sigma \cong \nu^{-\beta_k}\rho \rtimes \sigma$. Also, $\nu^{\beta_k}\rho \times \nu^{\beta_l}\rho \cong \nu^{\beta_l}\rho \times \nu^{\beta_k}\rho$ for every $l = k+1, \ldots, n+1$. We have

$$\begin{aligned} i_{G,M}(w\tau) &= \nu^{\beta_1}\rho \times \cdots \times \nu^{\beta_{k-1}}\rho \times \nu^{\beta_k}\rho \times \nu^{\beta_{k+1}}\rho \times \cdots \times \nu^{\beta_{n+1}}\rho \rtimes \sigma \\ &\cong \nu^{\beta_1}\rho \times \cdots \times \nu^{\beta_{k-1}}\rho \times \nu^{\beta_{k+1}}\rho \times \cdots \times \nu^{\beta_{n+1}}\rho \times \nu^{\beta_k}\rho \rtimes \sigma \\ &\cong \nu^{\beta_1}\rho \times \cdots \times \nu^{\beta_{k-1}}\rho \times \nu^{\beta_{k+1}}\rho \times \cdots \times \nu^{\beta_{n+1}}\rho \times \nu^{-\beta_k}\rho \rtimes \sigma \\ &\cong \nu^{-\beta_k}\rho \times \nu^{\beta_1}\rho \times \cdots \times \nu^{\beta_{k-1}}\rho \times \nu^{\beta_{k+1}}\rho \times \cdots \times \nu^{\beta_{n+1}}\rho \rtimes \sigma. \end{aligned}$$

It follows

$$\nu^{-\beta_k}\rho \otimes \nu^{\beta_1}\rho \otimes \cdots \otimes \nu^{\beta_{k-1}}\rho \otimes \nu^{\beta_{k+1}}\rho \otimes \cdots \otimes \nu^{\beta_{n+1}}\rho \otimes \sigma \leq r_{M,G}(q)$$

so q is not square integrable. ■

Proposition 7.3 *Let ρ be an irreducible supercuspidal representation of $\mathrm{GL}(k, F)$, k odd. Suppose that $\rho \cong \tilde{\rho}$. Let $n \geq 1$. Then the representation*

$$\nu^n\rho \times \cdots \times \nu\rho \times \rho \rtimes 1$$

of $\mathrm{SO}\big(2(n+1)k, F\big)$ contains a unique irreducible subrepresentation, denote it by $\delta([\rho, \nu^n\rho], 1)$. This representation is square integrable. We have

$$r_{M,G}\big(\delta([\rho, \nu^n\rho], 1)\big) = \nu^n\rho \otimes \cdots \otimes \rho \otimes 1.$$

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Note that for $k > 1$, $\rho \rtimes 1$ is irreducible, tempered, but not square integrable. If ρ is the trivial representation of $GL(1,F)$, then $\rho \rtimes 1$ is the trivial representation of $SO(2,F) \cong F^\times$ which is square integrable.

Proof Set

$$\tau = \nu^n\rho \otimes \cdots \otimes \rho \otimes 1.$$

The representation τ is regular, so $i_{G,M}(\tau)$ contains a unique irreducible subrepresentation.

First, we will consider the case $n = 1$. Then $\tau = \nu\rho \otimes \rho \otimes 1$. Write

$$\nu\rho \times \rho = p_1 + p_2,$$

where p_1 (resp., p_2) is the unique irreducible subrepresentation (resp., quotient) of $\nu\rho \times \rho$. If we set $M = GL(k,F) \times GL(k,F)$, $N = GL(2k,F)$, then

$$r_{M,N}(p_1) = \nu\rho \otimes \rho, \quad r_{M,N}(p_2) = \rho \otimes \nu\rho.$$

We have

$$\begin{aligned} s.s.\big(i_{G,M}(\tau)\big) &= i_{G,N}(p_1) + i_{G,N}(p_2), \\ r_{M,G} \circ i_{G,N}(p_1) &= \nu\rho \otimes \rho \otimes 1 + \rho \otimes \nu^{-1}\rho \otimes 1, \\ r_{M,G} \circ i_{G,N}(p_1) &= \rho \otimes \nu\rho \otimes 1 + \nu^{-1}\rho \otimes \rho \otimes 1. \end{aligned}$$

We want to prove that $i_{G,N}(p_1)$ and $i_{G,N}(p_2)$ are reducible. We will show it using the Langlands classification for $SO(2m,F)$ in the subrepresentational setting [J4].

Denote by $\delta(\nu^{\frac{1}{2}}\rho \times \nu^{-\frac{1}{2}}\rho)$ the unique irreducible subrepresentation of $\nu^{\frac{1}{2}}\rho \times \nu^{-\frac{1}{2}}\rho$. Then $\delta(\nu^{\frac{1}{2}}\rho \times \nu^{-\frac{1}{2}}\rho)$ is square integrable [Z]. We consider the standard parabolic subgroup with Levi factor $sNs^{-1} = s(N)$ (s as in Section 6 (b)). Then $s\big(\delta(\nu^{\frac{1}{2}}\rho \times \nu^{-\frac{1}{2}}\rho) \otimes 1\big)$ is a representation of $s(N)$ and $i_{G,s(N)}s\big(\nu^{-\frac{1}{2}}\delta(\nu^{\frac{1}{2}}\rho \times \nu^{-\frac{1}{2}}\rho) \otimes 1\big)$ contains a unique Langlands subrepresentation, denote it by

$$L\Big(i_{G,s(N)}s\big(\nu^{-\frac{1}{2}}\delta(\nu^{\frac{1}{2}}\rho \times \nu^{-\frac{1}{2}}\rho) \otimes 1\big)\Big).$$

We have

$$r_{M,G} \circ i_{G,s(N)}s\big(\nu^{-\frac{1}{2}}\delta(\nu^{\frac{1}{2}}\rho \times \nu^{-\frac{1}{2}}\rho) \otimes 1\big) = \rho \otimes \nu\rho \otimes 1 + \nu^{-1}\rho \otimes \rho \otimes 1.$$

By the Langlands classification and regularity, $\nu^{-1}\rho \otimes \rho \otimes 1$ must have been contributed by the Jacquet module of $L(\nu^{-1}\rho \times \rho \rtimes 1)$. So $i_{G,s(N)}s\big(\nu^{-\frac{1}{2}}\delta(\nu^{\frac{1}{2}}\rho \times \nu^{-\frac{1}{2}}\rho) \otimes 1\big)$ has two components, $L\Big(i_{G,s(N)}s\big(\nu^{-\frac{1}{2}}\delta(\nu^{\frac{1}{2}}\rho \times \nu^{-\frac{1}{2}}\rho) \otimes 1\big)\Big)$ and $L(\nu^{-1}\rho \times \rho \rtimes 1)$. This implies that $i_{G,N}(p_2)$ is reducible.

To show that $i_{G,N}(p_1)$ is reducible, we will apply the Aubert involution [A] (denote it by $\hat{\ }$). Since $\hat{p_1} = p_2$ and the Aubert involution commutes with parabolic induction, we have

$$i_{G,N}(p_1) \cong i_{G,N}(\hat{p_2}) \cong \widehat{i_{G,N}(p_2)}.$$

Hence, $i_{G,N}(p_1)$ is reducible. It has two irreducible components and for the unique irreducible subrepresentation $\delta([\rho, \nu\rho], 1)$, we have

$$r_{M,G}\big(\delta([\rho, \nu\rho], 1)\big) = \nu\rho \otimes \rho \otimes 1.$$

According to [Ba2], we have the following

$$s(\nu\rho \times \rho \rtimes 1) \cong \nu\rho \times s(\rho \rtimes 1) \cong \nu\rho \times \rho \rtimes 1.$$

Also, for any irreducible smooth representation π of $\mathrm{SO}(2m, F)$,

$$\pi \text{ is square integrable} \Longleftrightarrow s(\pi) \text{ is square integrable.}$$

Since $\delta([\rho, \nu\rho], 1)$ is the unique square integrable subquotient of $\nu\rho \times \rho \rtimes 1$, it follows $s\big(\delta([\rho, \nu\rho], 1)\big) \cong \delta([\rho, \nu\rho], 1)$. Using Proposition 3.3, we obtain

$$\begin{aligned} r_{s(M),G}\big(\delta([\rho, \nu\rho], 1)\big) &\cong r_{s(M),G}s\big(\delta([\rho, \nu\rho], 1)\big) \cong s\Big(r_{M,G}\big(\delta([\rho, \nu\rho], 1)\big)\Big) \\ &= s(\nu\rho \otimes \rho \otimes 1). \end{aligned}$$

Now, suppose that $n \geq 2$. The proof is similar to that of Proposition 7.1, 3. Let

$$N = \mathrm{GL}\big((n-1)k, F\big) \times \mathrm{SO}(4k, F).$$

We have

$$\begin{aligned} r_{M,N}\big(\delta([\nu^2\rho, \nu^n\rho]) \otimes \delta([\rho, \nu\rho], 1)\big) &= \nu^n\rho \otimes \cdots \otimes \rho \otimes 1 = \tau, \\ r_{s(M),N}\big(\delta([\nu^2\rho, \nu^n\rho]) \otimes \delta([\rho, \nu\rho], 1)\big) &= s(\tau). \end{aligned}$$

Using Remark 5.1 and Lemma 6.2, we obtain

$$\begin{aligned} & r_{M,G}\big(\delta([\nu^2\rho, \nu^n\rho]) \rtimes \delta([\rho, \nu\rho], 1)\big) \\ & \quad = \sum_{i=0}^{n-1} \mathrm{Sh}_{(n-i-1,n-1,n+1)}\, z_{(n-i,n-1)}(1_{n-i-1}, -1_i, 1_2)\tau. \end{aligned}$$

On the other hand,

$$r_{M,G}\big(\delta([\rho, \nu^n\rho]) \rtimes 1\big) = \sum_{\substack{i=0 \\ i \text{ even}}}^{n-1} \mathrm{Sh}_{(n-i+1,n+1)}\, z_{(n-i+2,n+1)}(1_{n-i+1}, -1_i)\tau.$$

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The only common factor for these two sums is τ. Since both representations have $\delta([\rho, \nu^n\rho], 1)$ as a subrepresentation, we conclude that

$$r_{M,G}\big(\delta([\rho, \nu^n\rho], 1)\big) = \tau. \qquad \blacksquare$$

Remark 7.1 Let G be a connected reductive p-adic group, $P = MU$ a standard parabolic subgroup. Let σ be an irreducible supercuspidal regular representation of G. If $i_{G,M}(\sigma)$ has a square integrable subquotient p such that $r_{M,G}(p)$ is irreducible, we call p a generalized Steinberg representation. For $G = \mathrm{Sp}(n, F)$, $\mathrm{SO}(2n+1, F)$ and $\mathrm{SO}(2n, F)$, all generalized Steinberg representations are described by Proposition 7.1 (3) $\big(\delta([\nu^\alpha\rho, \nu^{\alpha+n}\rho], \sigma)\big)$ and Proposition 7.3 $\big(\delta([\rho, \nu^n\rho], 1)\big)$.

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