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ISOMORPHISMS IN SWITCHING CLASSES OF GRAPHS

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Abstract

We investigate conditions on a permutation group G sufficient to ensure that G fixes a graph in any switching class of graphs that it stabilizes. Our main result gives a necessary and sufficient condition for a dihedral group G to have this property.

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1. Introduction

Let G be a permutation group stabilizing the switching class $\mathcal{S}(\Gamma)$ of a graph Γ. Although every element of G occurs in the automorphism group of some graph in $\mathcal{S}(\Gamma)$, the group G does not necessarily fix a graph in the class. If it does, we say that G is exposable in $\mathcal{S}(\Gamma)$. We consider the following problem: what conditions on G are sufficient to ensure that G is exposable in all switching classes that it stabilizes? It is implicit in the work of Mallows and Sloane (1975) that it is sufficient for G to be cyclic. Further known conditions are that G be of odd order, or of order $4k+2$. Our main result is Theorem 5.10, where we give a necessary and sufficient condition on the permutation representation of a dihedral group G such that G is exposable in all switching classes that it stabilizes.

We will present a general approach for studying isomorphisms in switching classes of graphs, which we then apply to obtain the above results.

2. Definitions and notation

We consider the collection $\mathcal{G}$ of labelled undirected graphs on n vertices, without loops and without multiple edges. Let Γ be a graph in $\mathcal{G}$. We label its vertices $1, 2, \ldots, n$, and call the set of labels $\Omega = \{1, 2, \ldots, n\}$.

A *switch* on Γ with respect to the vertex labelled i is a function s_i mapping Γ to the graph $s_i\Gamma$ which is obtained from Γ by deleting all edges in Γ which are

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incident to the vertex i and adding edges $\{i,k\}$, for all vertices k not adjacent to vertex i in Γ. Switching is a commutative operation: $s_i(s_j\Gamma) = s_j(s_i\Gamma)$ for all i, $i \in \Omega$. A switch s with respect to a set of vertices $\Phi = \{i_1, \ldots, i_r\}$ is defined to be the composition of functions $s = s_{i_1} \ldots s_{i_r}$. This switch transforms Γ into the graph $s\Gamma$, which is obtained from Γ by deleting all the edges in Γ that are incident to a vertex in Φ and a vertex in $\Omega \setminus \Phi$, and adding edges $\{i,k\}$ for all vertices k not adjacent to i in Γ, where $i \in \Phi$, $k \in \Omega \setminus \Phi$.

A switch s on Γ with respect to Φ is equal to a switch on Γ with respect to Φ' if and only if either $\Phi' = \Phi$ or $\Phi' = \Omega \setminus \Phi$. Clearly $s(s\Gamma) = s^2\Gamma = \Gamma$, and we write $s^2 = e$, where e is the switch with respect to the empty set, or equivalently with respect to Ω. The set of all switches on any graph in $\mathscr{G}$ forms an elementary Abelian group S with respect to the natural composition of switches. Its identity is e and its order is 2^{n-1}.

Suppose that s and s' are switches on Γ with respect to the subsets Φ and Φ' of Ω. Then the product ss' is a switch with respect to the symmetric difference of Φ and Φ' given by $\Phi \,\Delta\, \Phi' = (\Phi \cup \Phi') \setminus (\Phi \cap \Phi')$. The *switching class* $\mathscr{S}(\Gamma)$ is the set of 2^{n-1} graphs $\{s\Gamma \mid s \in S\}$.

Given a permutation π in Σ, the symmetric group on Ω, we define $\pi\Gamma$ to be the labelled graph, such that $\{\pi(i), \pi(j)\}$ is an edge in $\pi\Gamma$ if and only if $\{i,j\}$ is an edge in Γ. The *stabilizer* of the switching class $\mathscr{S}(\Gamma)$ is the group $\operatorname{Stab}\mathscr{S}(\Gamma)$ of all permutations in Σ that permute the members of $\mathscr{S}(\Gamma)$ among themselves; that is,

$$\operatorname{Stab}\mathscr{S}(\Gamma) = \{\pi \in \Sigma \mid \Gamma' \in \mathscr{S}(\Gamma) \Rightarrow \pi\Gamma' \in \mathscr{S}(\Gamma)\}.$$

An *automorphism* of a graph Γ is a permutation π in Σ such that $\pi\Gamma = \Gamma$. The set of all automorphisms of Γ is a group which we denote by $\operatorname{Aut}\Gamma$.

Our definitions can be presented in terms of the $(-1,1,0)$ adjacency matrix of Γ. (See, for example, Seidel (1976).) Let G be a subgroup of $\operatorname{Stab}\mathscr{S}(\Gamma)$. Two possibilities arise: either G is a subgroup of the automorphism group of some graph in $\mathscr{S}(\Gamma)$ or there is no graph in $\mathscr{S}(\Gamma)$ fixed by G. We say that G is *exposable* in $\mathscr{S}(\Gamma)$ in the first case and that G is *hidden* in $\mathscr{S}(\Gamma)$ in the second. We say that a permutation group G is *always exposable* if it is exposable in every switching class that it stabilizes.

3. Preliminary results

As in the previous section, Γ denotes a graph on n vertices.

LEMMA 3.1. *Given $\pi \in \Sigma$ and switch s with respect to $\Phi = \{i_1, \ldots, i_r\} \subseteq \Omega$, define switch ${}_{\pi}s$ with respect to $\Phi_\pi = \{\pi(i_1), \ldots, \pi(i_r)\}$.* Then

$$\pi(s\Gamma) = {}_{\pi}s(\pi\Gamma). \tag{3.2}$$

PROOF. Immediate.

We observe that the graphs in $\mathscr{G}$ are permuted by switches in S, by permutations in Σ and by compositions of these operations, which we call *switch-permutations*. Their totality forms a group W, where the law of composition of a switch and a permutation is given by (3.2). In view of our definition of left action on graphs, products of elements in W are evaluated from right to left. (Our notation ensures that ${}_{\sigma}({}_{\pi}s) = {}_{\sigma\pi}s$.)

We proceed to study the stabilizer of a switching class $\mathscr{S}(\Gamma)$. Our first result shows that a necessary and sufficient condition for a permutation to belong to the stabilizer of $\mathscr{S}(\Gamma)$ is that it maps any one graph in $\mathscr{S}(\Gamma)$ to a graph in this class.

LEMMA 3.3. *Let* $\pi \in \Sigma$. *Then* $\pi\Gamma \in \mathscr{S}(\Gamma)$ *if and only if* $\pi \in \operatorname{Stab}\mathscr{S}(\Gamma)$.

PROOF. Suppose that $\pi\Gamma \in \mathscr{S}(\Gamma)$. Then for some switch s, $\pi\Gamma = s\Gamma$. Now consider an arbitrary switch s'. Then, by Lemma 3.1,

$$\pi(s'\Gamma) = {}_{\pi}s'(\pi\Gamma) = {}_{\pi}s'(s\Gamma) = s^{*}\,\Gamma \in \mathscr{S}(\Gamma).$$

Therefore $\pi \in \operatorname{Stab}\mathscr{S}(\Gamma)$. The converse is true by definition.

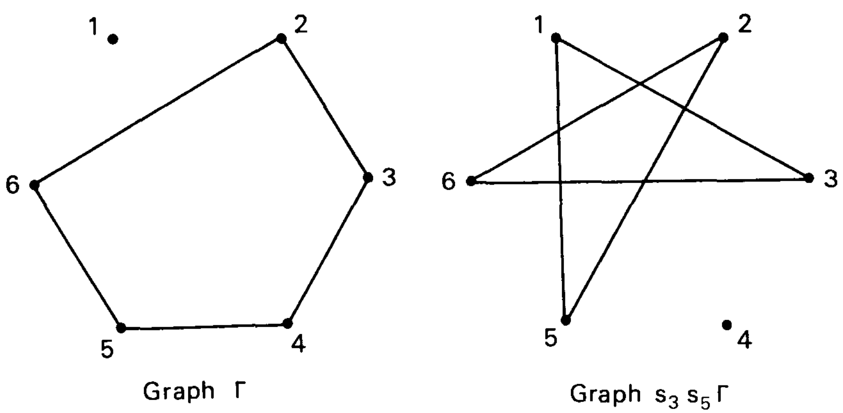

FIG. 1.

EXAMPLE 3.4. Let Γ be the labelled graph illustrated in Fig. 1. The stabilizer $\operatorname{Stab}\mathscr{S}(\Gamma)$ is a representation of degree 6 of the Alternating group A_5. It is generated by μ and π where

$$\mu = (143)(256), \quad \pi = (23456)(1).$$

As is clear from Fig. 1, the graphs Γ and $s_3 s_5 \Gamma$ are isomorphic, and $s_3 s_5 \Gamma = \mu\Gamma$. The cyclic group $\langle\mu\rangle$ lies in $\operatorname{Aut}(s_1 s_6 \Gamma)$. Next consider the dihedral subgroups of $\operatorname{Stab}\mathscr{S}(\Gamma)$

$$D = \langle (26)(35)(1)(4),\ (23456)(1)\rangle \quad \text{of order } 10$$

and

$$D' = \langle (26)(35)(1)(4),\ (14)(35)(2)(6)\rangle \quad \text{of order } 4.$$

The group D is equal to $\operatorname{Aut}\Gamma$; however, the group D' does not occur in the automorphism group of any graph in $\mathscr{S}(\Gamma)$ and so D' is hidden in $\mathscr{S}(\Gamma)$. The element $s_3 s_5 \mu$ fixes Γ, and no other $s\mu$ in W has this property. In contrast, there is no graph in $\mathscr{S}(\Gamma)$ fixed by $s_1\mu$. In fact there is no graph in the class $\mathscr{G}$ of all graphs on 6 vertices that is fixed by $s_1 \mu$.

The underlying theory of Example 3.4 will be explained in Section 5. The following well-known result provides a partial answer to our problem.

THEOREM 3.5. *A necessary and sufficient condition for a permutation group G to be exposable in a switching class $\mathscr{S}(\Gamma)$ is that it has an orbit on $\mathscr{S}(\Gamma)$ of odd length.*

PROOF. Suppose that $\{\Gamma_1, \ldots, \Gamma_r\}$ is an orbit of G on $\mathscr{S}(\Gamma)$ where r is odd. Let $s^{(i)}$ denote a switch such that

$$s^{(i)}\Gamma_1 = \Gamma_i, \quad i = 1, \ldots, r.$$

A permutation π in $\mathscr{G}$ permutes the graphs $\Gamma_1, \ldots, \Gamma_r$. Since $\pi \in \operatorname{Stab}\mathscr{S}(\Gamma)$ there exists a switch s such that $\pi\Gamma_1 = s\Gamma_1$. Then

$$\pi s^{(i)}\Gamma_1 = {}_{\pi}s^{(i)}\, s\Gamma_1 = s^{(j)}\Gamma_1 \quad \text{for some } j \in \{1, \ldots, r\}.$$

Put $s' = s^{(1)} \ldots s^{(r)}$. Since r is odd,

$${}_{\pi}s'\, s'\, \Gamma_1 = (s)^r\, \Gamma_1 = s\Gamma_1 = \pi\Gamma_1,$$

and hence $\pi(s'\Gamma_1) = s'\Gamma_1$. The choice of s' is independent of the choice of π in G, and so G fixes $s'\Gamma_1$, and G is exposable in $\mathscr{S}(\Gamma)$. The converse is immediate.

COROLLARY 3.6. *A group G is always exposable if it has an odd orbit on Ω. In particular, a permutation group on Ω, where $|\Omega|$ is odd, is always exposable.*

PROOF. Corresponding to an odd orbit $\{1, \ldots, r\}$ of G on Ω there is an orbit of G on $S(\Gamma)$ containing the graphs $s^{(i)}\Gamma$, $i = 1, \ldots, r$, where $s^{(i)}\Gamma$ denotes the (unique) graph in $\mathscr{S}(\Gamma)$ that has vertex i isolated. The number of graphs in this orbit is a divisor of r.

COROLLARY 3.7. *Let G be a permutation group containing a subgroup H that is always exposable. If the index r of H in G is odd, then G is always exposable.*

PROOF. Suppose that G stabilizes $\mathscr{S}(\Gamma)$. Then there is a graph Γ' in $\mathscr{S}(\Gamma)$ which is fixed by H. Then Γ' lies in an orbit of G on $\mathscr{S}(\Gamma)$ whose length divides r.

Towards further progress it is important to establish a criterion for the existence of a graph fixed by an element $s\pi$ of W, or more generally by a subgroup Q in W.

NOTATION. We introduce a convenient notation for the switch-permutations $w = s\pi$ of W: the permutation π is written as a product of disjoint cycles, and a bar is placed over each symbol that occurs in the set Φ switched by s. To illustrate, in Example 3.4

$$s_3 s_5 \mu = (14\bar{3})(2\bar{5}6).$$

Since $s_3 s_5 \mu = s_1 s_2 s_6 s_6 \mu$, we observe that

$$(14\bar{3})(2\bar{5}6) = (\bar{1}\bar{4}3)(\bar{2}5\bar{6}).$$

THEOREM 3.8. *A subgroup Q of W does not fix any graph Γ in G if and only if some element in Q involves either a switch-transposition $(\bar{i}j)$ or switch-1-cycles $(\bar{i})(j)\ldots$*

PROOF. In view of the action of permutations and switches on graphs, a necessary and sufficient condition for a switch-permutation $s\pi$ to fix a graph Γ is as follows: for all $p, q \in \Omega$, $\{p,q\}$ and $\{\pi(p), \pi(q)\}$ are both edges or both non-edges of Γ if and only if the set Φ switched by s contains both or neither of $\pi(p)$ and $\pi(q)$.

The construction of a graph fixed by Q will break down if and only if the stage is reached that an unordered pair $\{i,j\}$ represents both an edge and a non-edge. This will arise if and only if Q contains a switch-permutation $s\pi$ such that Φ contains exactly one of i and j and either (1) $\pi(i) = j$, $\pi(j) = i$, or (2) $\pi(i) = i$ and $\pi(j) = j$.

COROLLARY 3.9. *Suppose a group Q of switch-permutations fixes a graph.*

(i) *If $s\pi$ and $s'\pi$ belong to Q then $s' = s$. The set of permutations $\{\pi \in \Sigma | s\pi \in Q$ for some switch s (depending on π)$\}$ forms a group, which we call the permutation group associated with Q. Its order is $|Q|$.*

(ii) *Q fixes exactly 2^λ different graphs, where λ is the number of orbits of unordered pairs $\{i, j\}$, $i \neq j$, in $\Omega \times \Omega$ under the action of the permutation group associated with Q.*

PROOF. (i) $s\pi \in Q$ and $s'\pi \in Q \Rightarrow s\pi\, \pi^{-1} s' = ss' \in Q$. By Theorem 3.8, $ss' = e$.

(ii) A graph is specified by assigning in each orbit one pair to be an edge or a non-edge.

4. Cyclic subgroups of stabilizers

THEOREM 4.1. *A cyclic group is always exposable.*

To prove this result we require the following lemmas.

LEMMA 4.2. *Consider the r-cycle $\sigma = (1\ 2\ \ldots\ r)$ and the switch s with respect to $\Phi \subseteq \{1, 2, \ldots, r\}$. Then*

$$(s\sigma)^r = \begin{cases} (1)(2)\ldots(r), & \text{if } |\Phi| \text{ is even,} \\ (\bar{1})(\bar{2})\ldots(\bar{r}), & \text{if } |\Phi| \text{ is odd.} \end{cases}$$

Moreover, if $r = 2k$, then $(s\sigma)^k$ involves a switch-transposition $(\bar{i}j)$ if and only if $|\Phi|$ is odd.

PROOF. Apply the formula

$$(s\sigma)^m = s({}_{\sigma}s)\,({}_{\sigma^2}s)\ldots({}_{\sigma^{m-1}}s)\sigma^m.$$

DEFINITION 4.3. Let π be a permutation in Σ, and let $\Phi \subseteq \Omega$. We say that Φ is *compatible* with π if each cycle of π involves an even number of symbols of Φ, where π is expressed as the product of disjoint cycles (including 1-cycles).

LEMMA 4.4. *Let s be a switch with respect to $\Phi \subseteq \Omega$, and let $\pi \in \Sigma$. Then the switch permutation $s\pi$ fixes some graph if and only if either Φ or $\Omega \setminus \Phi$ is compatible with π.*

PROOF. This follows by applying Lemma 4.2 to Theorem 3.5.

LEMMA 4.5. *Let s be a switch with respect to $\Phi \subseteq \Omega$, and let $\pi \in \Sigma$. Then there exists a switch s' such that $s\pi = s'\pi s'$ if and only if either Φ or $\Omega \setminus \Phi$ is compatible with π.*

PROOF. If $s\pi = s'\pi s'$ then $s = s'\,{}_{\pi}s'$ and clearly Φ or $\Omega \setminus \Phi$ is compatible with π.

Conversely, suppose that Φ or $\Omega \setminus \Phi$ is compatible with π. We suppose without loss of generality that Φ involves an even number of symbols from each cycle of π. Consider a particular cycle of π, which we write $\sigma = (1\ 2\ \ldots\ r)$. If $\Phi \cap \operatorname{supp} \sigma$ is not empty then it is expressible in the form

$$\Phi \cap \operatorname{supp} \sigma = \{i_1, \ldots, i_{2k}\},$$

where $1 \leqslant i_1 < \ldots < i_{2k} \leqslant r$. Define the set

$$\Phi^* = \{i \mid i_{2q-1} \leqslant i < i_{2q},\ q = 1, \ldots, k\}.$$

Then

$$(\Phi^*)_\pi = \{i \mid i_{2q-1} < i \leqslant i_{2q},\ q = 1, \ldots, k\}$$

and, forming the symmetric difference, we obtain

$$(\Phi^*)\ \Delta\ (\Phi^*)_\pi = \{i_1, \ldots, i_{2k}\} = \Phi \cap \operatorname{supp} \sigma.$$

We define s′ to be the switch with respect to the set Φ' which is the union of the sets Φ^* constructed in the above manner and corresponding to all the cycles of π having common symbols with Φ. Then $s'{}_{\pi}s' = s$, and $s\pi = s'\pi s'$. This completes the proof.

PROOF OF THEOREM 4.1. Suppose a cyclic group G stabilizes $\mathscr{S}(\Gamma)$. If $G = \langle \pi \rangle$, then there is a switch s with respect to a set $\Phi \subseteq \Omega$ such that $s\pi$ fixes Γ. By Lemma 4.4, either Φ or $\Omega \setminus \Phi$ is compatible with π. By Lemma 4.5 there exists a switch s' such that $s'\pi s'$ is equal to $s\pi$. But then $s'\pi s'\,\Gamma = \Gamma$, and hence $\pi(s'\Gamma) = s'\Gamma$. Thus π fixes the graph $\Gamma' = s'\Gamma$, and G is exposable in $\mathscr{S}(\Gamma)$.

As an immediate application of Theorem 4.1 and Corollary 3.7 we have the following result.

THEOREM 4.6. *A group with cyclic Sylow 2-subgroup is always exposable. In particular, all groups of order $4k+2$ are always exposable.*

5. Dihedral subgroups of stabilizers

Our aim in this section is to classify the dihedral subgroups in Σ which are always exposable. We see from Example 3.4 that not all dihedral groups are always exposable.

Now let D be an arbitrary dihedral subgroup of Σ. Then D is generated by two involutions, α and β. The following lemma applies to dihedral groups as a special case.

LEMMA 5.1. *Suppose a subgroup G of Σ is generated by two permutations π and μ. If G stabilizes a switching class $\mathscr{S}(\Gamma)$ then G is associated with a group Q fixing a graph in $\mathscr{S}(\Gamma)$, such that Q is generated by switch-permutations $s\pi$ and $s\mu$ for some switch s.*

PROOF. By Theorem 4.1, there is a graph Γ' in $\mathscr{S}(\Gamma)$ which is fixed by $\mu^{-1}\pi$. So there is a switch s such that

$$\pi\Gamma' = \mu\Gamma' = s\Gamma',$$

and the switch-permutations $s\pi$ and $s\mu$ fix Γ'.

According to Lemma 5.1, in order to study the action of the dihedral group D on a switching class which it stabilizes, we can equivalently study subgroups Q of W that fix a graph, where Q is generated by switch permutations $s\alpha$ and $s\beta$. We next establish a criterion depending on s, α and β for the existence of a graph fixed by $Q = \langle s\alpha, s\beta\rangle$.

LEMMA 5.2. *Let α and β by involutions in Σ, and let s be a switch with respect to $\Phi \subseteq \Omega$. There exists a graph fixed by $Q = \langle s\alpha, s\beta\rangle$ if and only if either Φ or $\Omega\setminus\Phi$ is compatible with α, and either Φ or $\Omega\setminus\Phi$ is compatible with β.*

PROOF. If Q fixes a graph then the condition of Lemma 5.2 is satisfied, by Lemma 4.4. Conversely, suppose that the condition of Lemma 5.2 is satisfied. Then, since α and β are involutions,

$$_{\alpha}s = {}_{\beta}s = s. \tag{5.3}$$

We will show the existence of a graph fixed by Q by an application of Theorem 3.8. The elements of Q are of the form

$$(s\alpha)^k(s\beta s\alpha)^l = (s\alpha)^k(\beta\alpha)^l,$$

where $k = 0, 1$ and $l = 0, 1, 2, \ldots$, the last expression being obtained on applying

(5.3). We must show that the conditions in Theorem 3.8 for the non-existence of a graph do not arise. This is clear when $k = 0$. Consider next an element $w = s\alpha(\beta\alpha)^l$ of Q. If the permutation $\alpha(\beta\alpha)^l$ transposes two symbols then by (5.3) either Φ or $\Omega \setminus \Phi$ contains both these symbols. Finally, suppose that $\alpha(\beta\alpha)^l$ fixes two symbols i and j. If $l = 2m$, put

$$(\beta\alpha)^m(i) = p, \quad (\beta\alpha)^m(j) = q.$$

Then $\alpha(p) = \alpha(\beta\alpha)^m(i) = (\beta\alpha)^m(i) = p$ and similarly $\alpha(q) = q$. By our hypothesis, either Φ or $\Omega \setminus \Phi$ contains both of p and q and hence also both of $i = (\alpha\beta)^m(p)$ and $j = (\alpha\beta)^m(q)$. If $l = 2m+1$, a similar argument applies to the elements $\alpha(\beta\alpha)^m(i)$ and $\alpha(\beta\alpha)^m(j)$ which are fixed by β, using the hypothesis that either Φ or $\Omega \setminus \Phi$ is compatible with β.

LEMMA 5.4. *Let s be a switch with respect to $\Phi \subseteq \Omega$, where Φ is compatible with both the involutions α and β in Σ. Then there is a switch s' such that*

$$s\alpha = s'\alpha s' \quad and \quad s\beta = s'\beta s'. \tag{5.5}$$

PROOF. By our hypothesis on s, the set Φ is a union of orbits $\Phi_1, \ldots, \Phi_t$ of $D = \langle \alpha, \beta \rangle$ on Ω. Choose from each orbit Φ_r a symbol i_r, $r = 1, \ldots, t$. Then the switch s' is defined with respect to the set Φ', where

$$\Phi' = \{(\beta\alpha)^m(i_r),\ r = 1, \ldots, t,\ m = 0, 1, 2, \ldots\}.$$

We will show that s' satisfies relations (5.5), in other words, that $s = s'{}_\alpha s' = s'{}_\beta s'$. This follows from the observation that Φ' consists of precisely one symbol from each transposition in α and in β whose symbols lie in Φ. For if this is not the case then for some i_r in Φ' and some integer m,

$$(\beta\alpha)^m(i_r) = \alpha(i_r) \quad \text{or} \quad \beta(i_r).$$

In either case this leads to the conclusion (by a method used in the proof of Lemma 5.2) that either α or β fixes a symbol in Φ_r. This contradicts that Φ is compatible with both α and β, and the proof is complete.

COROLLARY 5.6. *Suppose that the graph Γ is fixed by $Q = \langle s\alpha, s\beta \rangle$, where α and β are involutions. If the set Φ switched by s is compatible with both α and β then there is a graph Γ' in $\mathscr{S}(\Gamma)$ which is fixed by the dihedral group $D = \langle \alpha, \beta \rangle$.*

PROOF. Apply Lemma 5.4, putting $\Gamma' = s'\Gamma$.

We must now consider the case of a switch s with respect to Φ, where Φ is compatible with α, and $\Omega \setminus \Phi$ is compatible with β (so that by Lemma 5.2 there exists a graph fixed by $s\alpha$ and $s\beta$), but neither Φ nor $\Omega \setminus \Phi$ is compatible with both

α and β. The following examples motivate our next lemma. The second of these examples provides a further illustration of a dihedral group stabilizing a class but fixing no graph in it.

EXAMPLES 5.7

(i) Consider the switch involutions

$$s\alpha = (1)(2)(\bar{3}\,\bar{4})(\bar{5}\,\bar{6})(\bar{7}\,\bar{8}),$$
$$s\beta = (\bar{1}\,\bar{2})(35)(46)(7)(8).$$

Here $\Phi = \{3,4,5,6,7,8\}$, and this is compatible with α and not with β, whereas $\Omega \setminus \Phi = \{1,2\}$ is compatible with β and not with α. There exists a switch s' such that $s\alpha = s'\alpha s'$ and $s\beta = s'\beta s'$. (Choose for example $\Phi' = \{1,3,5,7\}$ or $\{2,4,6,7\}$.) By Lemma 5.2 there exists a graph Γ fixed by $s\alpha$ and by $s\beta$. Let s' be the switch with respect to Φ'. The graph $s'\Gamma$ is fixed by $D = \langle \alpha, \beta \rangle$.

(ii) Put

$$s\alpha = (\bar{1}\,\bar{2})(\bar{3}\,\bar{4})(\bar{5}\,\bar{6})(\bar{7}\,\bar{8})(\bar{9}\,\overline{10})(11)(12),$$
$$s\beta = (1)(3)(24)(5\,10)(67)(89)(\overline{11}\,\overline{12}).$$

Here Φ is compatible with α and not β, and $\Omega \setminus \Phi$ is compatible with β and not α. There is no switch s' such that $s\alpha = s'\alpha s'$ and $s\beta = s'\beta s'$. Again by Lemma 5.2, there exists a graph Γ fixed by $s\alpha$ and by $s\beta$, but in this case there is no graph in $\mathscr{S}(\Gamma)$ fixed by $D = \langle \alpha, \beta \rangle$.

The essential difference between Examples 5.7(i) and (ii) lies in the length of the orbits of D on Ω, none of whose symbols is fixed by α or by β. In Example (i) the only such orbit is $\{3,4,5,6\}$, and in Example (ii) the only such orbit is $\{5,6,7,8,9,10\}$. As the next lemma shows, the length of these orbits is crucial to our analysis.

LEMMA 5.8. *Let D be the dihedral group generated by involutions α and β, and let s be a switch with respect to Φ. Suppose that Φ is compatible with α and not with β, and that $\Omega \setminus \Phi$ is compatible with β and not with α. Then there is a switch s' such that $s\alpha = s'\alpha s'$ and $s\beta = s'\beta s'$ if and only if every orbit of D on Ω, none of whose symbols is fixed by α or by β, has length divisible by four.*

PROOF. We partition the orbits of D on Ω into three classes:

(i) orbits containing a symbol fixed by α;
(ii) orbits containing a symbol fixed by β;
(iii) orbits none of whose symbols is fixed by α or by β.

The classes are disjoint, for suppose an orbit Θ is common to class (i) and class (ii). Then it contains a symbol fixed by α and a symbol fixed by β, and it follows from our hypothesis on Φ that this cannot happen.

First we note that Φ is a union of orbits of D. For if $i \in \Phi$ then $\alpha(i) \in \Phi$, since Φ is compatible with α, and $\beta(i) \in \Phi$ since $\Omega \setminus \Phi$ is compatible with β. It follows from this that if Θ is an orbit in class (i) then $\Theta \subseteq \Omega \setminus \Phi$ and that if Θ is an orbit in class (ii) then $\Theta \subseteq \Phi$.

Suppose now that every orbit of D in class (iii) has length divisible by four. We will construct a switch s' with respect to a set $\Phi' \subseteq \Omega$ such that $s\alpha = s'\alpha s'$ and $s\beta = s'\beta s'$, or equivalently $s = s'{}_{\alpha}s' = s'{}_{\beta}s'$. The set Φ' will be a union of subsets Φ^* constructed as follows.

First consider an orbit Θ in class (i). Then the symbols of Θ are involved in, say, k transpositions of β where $|\Theta| = 2k$, and α fixes at least two symbols of Θ. We claim that $\alpha\beta$ acts on Θ as a $2k$-cycle. To prove this, consider a symbol i in Θ fixed by α. Every element of D is expressible in the form $(\alpha\beta)^r$ or $(\alpha\beta)^r\alpha$ for some integer r. If $\alpha\beta$ were not a $2k$-cycle then, since $(\alpha\beta)^r\alpha(i) = (\alpha\beta)^r(i)$, the group D would not act transitively on Θ. Let the subset Φ^* of Θ consist of the k alternate symbols from the cycle $\alpha\beta$, so chosen as to include the symbol i. We calculate

$$\alpha(\alpha\beta)^r(i) = \alpha(\alpha\beta)^r\alpha(i) = (\beta\alpha)^r(i) = (\alpha\beta)^{2k-r}(i)$$

and

$$\beta(\alpha\beta)^r(i) = \beta(\alpha\beta)^r\alpha(i) = (\beta\alpha)^{r+1}(i) = (\alpha\beta)^{2k-r-1}(i).$$

From this we see that α fixes Φ^* setwise, and β maps Φ^* onto $\Theta \setminus \Phi^*$. Hence $\Phi^* \Delta \Phi^*_{\alpha}$ is empty and $\Phi^* \Delta \Phi^*_{\beta} = \Theta$. By reversing the roles of α and β or an orbit Θ in class (ii) we obtain similarly a set Φ^* such that $\Phi^* \Delta \Phi^*_{\alpha} = \Theta$ and $\Phi^* \Delta \Phi^*_{\beta}$ is empty.

Finally consider an orbit Θ in class (iii). Then either $\Theta \subseteq \Phi$ or $\Theta \subseteq \Omega \setminus \Phi$. In either case $|\Theta|$ is even, $|\Theta| = 2k$, say. Choose an arbitrary symbol i in Θ. We will show that the sets

$$\{(\alpha\beta)^r(i),\ r = 1, \ldots, k\} \quad \text{and} \quad \{(\alpha\beta)^r\alpha(i),\ r = 1, \ldots, k\}$$

are disjoint. For if not, then there are integers b and c such that

$$(\alpha\beta)^b(i) = (\alpha\beta)^c\alpha(i), \quad \text{giving } \alpha(\beta\alpha)^{c-b}(i) = i.$$

This implies, as in the proof of Lemma 5.2, that α or β fixes a symbol in Θ, depending on the parity of $c-b$.

Since $|\Theta| = 2k$, it now follows that $\alpha\beta$ acts on Θ as the product of two k-cycles. In the case that $\Theta \subseteq \Phi$ we choose Φ^* as the subset of Θ consisting of (a) alternate symbols including i in the cycle of $\alpha\beta$ that contains i, and (b) alternate symbols in the other cycle of $\alpha\beta$ not including the symbol $\alpha(i)$. (It is at this stage that we require k to be even and hence $|\Theta|$ to be a multiple of four.) It can be shown by a method similar to that used for class (i) orbits that $\Phi^* \Delta \Phi^*_{\alpha} = \Theta$ and that $\Phi^* \Delta \Phi^*_{\beta}$ is empty. In the case that $\Theta \subseteq \Omega \setminus \Phi$ we choose Φ^* as above but with the roles of α and β reversed. Then $\Phi^* \Delta \Phi^*_{\alpha}$ is empty and $\Phi^* \Delta \Phi^*_{\beta} = \Theta$.

We now define s' to be the switch with respect to the set Φ', where Φ' is the union of the sets Φ^* constructed in the above manner, one for each orbit. Then $\Phi' \Delta \Phi'_\alpha = \Phi$ and $\Phi' \Delta \Phi'_\beta = \Omega \setminus \Phi$, and so $s = s'_\alpha s' = s'_\beta s'$.

Conversely, suppose that there is a switch s' with respect to a set Φ' such that $s\alpha = s' \alpha s'$ and $s\beta = s' \beta s'$, or equivalently $s = s'_\alpha s' = s'_\beta s'$. Since $\Omega \setminus \Phi$ is not compatible with α, $\Phi = \Phi' \Delta \Phi'_\alpha$, and since Φ is not compatible with β, $\Omega \setminus \Phi = \Phi' \Delta \Phi'_\beta$.

Let Θ be an orbit of class (iii), and assume by way of contradiction that $|\Theta| = 2+4k$ for some integer k. Then α and β each contain the symbols of Θ in $1+2k$ transpositions. Now either $\Theta \subseteq \Phi$ or $\Theta \subseteq \Omega \setminus \Phi$. In the first case Φ' must contain exactly one symbol from each of these transpositions that occur in α, which is $1+2k$ symbols in all from Θ. But also, $(\Omega \setminus \Phi) \cap \Theta$ is empty and $\Omega \setminus \Phi = \Phi' \Delta \Phi'_\beta$, and this means that Φ' contains either both or neither of the symbols in each transposition in β that involves Θ. So Φ' contains an even number of symbols from Θ, which is a contradiction. The case $\Theta \subseteq \Omega \setminus \Phi$ is treated similarly. Hence $|\Theta| = 4k$ for some integer k, and the proof is complete.

COROLLARY 5.9. *Let s be a switch with respect to $\Phi \subseteq \Omega$ and let α and β be involutions in Σ. Suppose that the graph Γ is fixed by $\langle s\alpha, s\beta \rangle$. If Φ is compatible with α but not with β and $\Omega \setminus \Phi$ is compatible with β but not with α then the dihedral group $D = \langle \alpha, \beta \rangle$ is exposable in $\mathscr{S}(\Gamma)$ if and only if every orbit of D on Ω containing no symbol fixed by α or by β has length divisible by four.*

It is clear that a dihedral group $D = \langle \alpha, \beta \rangle$ can stabilize many switching classes. Provided that a switch s is chosen to satisfy the conditions of Lemma 5.2, a switching class $\mathscr{S}(\Gamma)$ stabilized by D can be constructed by applying Theorem 3.8 to the group Q generated by the switch-permutations $s\alpha$ and $s\beta$. Our next result gives a necessary and sufficient condition on a dihedral group D in a permutation representation to be always exposable.

THEOREM 5.10. *A dihedral group D, represented as a permutation group on Ω, and generated by involutions α and β, is always exposable if and only if at least one of the following three conditions is satisfied.*

(1) *At least one of α and β fixes no symbol in Ω.*

(2) *Some orbit of D contains a symbol fixed by α and a symbol fixed by β.*

(3) (i) *α and β both fix symbols.* (ii) *The orbits containing symbols fixed by α contain no symbols fixed by β.* (iii) *Every orbit of D, none of whose symbols is fixed by α or by β has length divisible by four.*

PROOF. Suppose D satisfies at least one of conditions (1), (2) and (3), and stabilizes a switching class $\mathscr{S}(\Gamma)$. Then by Lemma 5.1 there is a switch s with respect to a set Φ such that $Q = \langle s\alpha, s\beta \rangle$ fixes a graph in $\mathscr{S}(\Gamma)$. By Lemma 5.2 we may suppose that either Φ is compatible with both α and β or Φ is compatible with α and not with β and $\Omega \setminus \Phi$ is compatible with β and not with α. If D satisfies conditions (1) or (2) then the first case arises and, by Corollary 5.6, D fixes a graph in $\mathscr{S}(\Gamma)$. If D satisfies condition (3) either case may arise, the first being dealt with by Corollary 5.6 and the second by Corollary 5.9. Hence D is always exposable.

Conversely, if D does not satisfy any of conditions (1), (2) and (3), then (i) α and β both fix symbols; (ii) the orbits containing symbols fixed by α contain no symbols fixed by β; (iii) there is an orbit of D none of whose symbols is fixed by α or by β and whose length is of the form $2+4k$. Let Φ be the union of the orbits containing symbols fixed by β. Then Φ is compatible with α and not β and $\Omega \setminus \Phi$ is compatible with β and not α. Let s be the switch with respect to Φ. By Lemma 5.2, $Q = \langle s\alpha, s\beta \rangle$ fixes some graph, Γ say. By Corollary 5.9, D is not exposable in $\mathscr{S}(\Gamma)$. This completes the proof.

COROLLARY 5.11. *A dihedral group D is always exposable if D on Ω has fewer than three orbits. In particular, all transitive dihedral groups are always exposable.*

PROOF. If D is transitive on Ω then condition (1) or (2) of Theorem 5.10 must hold. If D on Ω has two orbits and if conditions (1) and (2) do not hold then α fixes symbols in the first but not the second orbit and β fixes symbols in the second but not the first orbit. Then condition (3) holds, for (3)(iii) is vacuously satisfied.

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