A NOTE ON EXTENDING PARTIAL AUTOMORPHISMS OF ABELIAN GROUPS

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To Bernhard Hermann Neumann on his 60th birthday

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1. Introduction

Given a group G and a partial automorphism μ of G, i.e. an isomorphism mapping a subgroup A of G onto another subgroup B of G, then it is known [3] that μ can always be extended to a total automorphism, in fact an inner one, of a supergroup of G; that is there exists a group $G^* \supseteqq G$ with an inner automorphism μ^* whose effect on the elements of A is the same as that of μ. Also any number of partial automorphisms μ_σ, where σ ranges over some index set Σ can be simultaneously extended to inner automorphisms of one and the same group [3, Theorem II].

In a previous paper [1], the first of us derived conditions which are sufficient for extending two partial automorphisms μ and ν of a group G to total automorphisms μ^* and ν^* of a supergroup $G^* \supseteqq G$ such that μ^* and ν^* commute. The technique there applied leads necessarily to a non-abelian group G^* even though the group G were abelian. The question then arises whether these conditions are also sufficient if we start with an abelian group G and require that the extension group G^* be also abelian. It is the purpose of this paper to answer this question in the affirmative; our main tool will be the direct product of two groups with one amalgamated subgroup.

2. Some preliminary results and definitions

Let G_α be a system of groups defined for every α in an index set I, such that for every $\alpha \in I$, G_α contains a subgroup H_α isomorphic to a fixed group H. Let ρ_α be a fixed isomorphism between H_α and H: $H_\alpha \rho_\alpha = H$ for every $\alpha \in I$. Denote the direct product of the G_α by D,

$$D = \prod_{\alpha \in I}^{\times} G_\alpha,$$

and let K be the set consisting of all elements of the form $h_\alpha h_\beta^{-1}$, $h_\alpha \in H_\alpha$, $h_\beta \in H_\beta$ where α, $\beta(\neq \alpha)$ run over I and h_α, h_β correspond under the isomorphism resulting between H_α and H_β from ρ_α and ρ_β. If K is a normal subgroup of D, the factor group D/K is called the direct product of the groups G_α with the subgroup H amalgamated and it is denoted by

$$P = \prod_{\alpha \in I}^{\times} \{G_\alpha ; H\}.$$

For the existence of P it is necessary and sufficient that H lies in the centre of each G_α, $\alpha \in I$, [2].

Later on we shall restrict ourselves to the direct product of two groups with an amalgamated subgroup which in case the constituent groups are abelian coincides with their free abelian product.

The following lemmas whose proofs run over the same lines as the corresponding lemmas in [1, § 1] will be needed.

LEMMA 1. *Let* $P = \{G_1 \times G_2 ; H\}$ *and let* A_1, A_2 *be subgroups of* G_1 *and* G_2 *respectively which have the same intersection* B *with* H. *If* $Q = \{A_1, A_2\}$ *then* $Q \cap G_1 = A_1$ *and* $Q \cap G_2 = A_2$.

The lemma could be proved using the normal form for elements of a direct product with one amalgamated subgroup which is analogous to that in a free product with one amalgamated subgroup.

LEMMA 2. *Let* G_1 *and* G_2 *be two groups with* $U = G_1 \cap G_2$ *and let* μ_i *map* G_i *isomorphically onto* H_i $(i = 1, 2)$. *Suppose that*

$$U\mu_1 = U\mu_2 = V$$

and that, more precisely, $u\mu_1 = u\mu_2$ *for all* $u \in U$; *then there exists an isomorphic mapping of*

$$P_1 = \{G_1 \times G_2 ; U\}$$

onto

$$P_2 = \{H_1 \times H_2 ; V\}$$

which extends μ_1 *and* μ_2 *simultaneously.*

3. First step of the construction

Let G be an abelian group which contains the subgroups A, B, C and D and two partial automorphisms μ and ν that map A isomorphically onto B and C isomorphically onto D respectively. Assume that μ commutes with ν, i.e. that

(1) $g\mu\nu = g\nu\mu$, whenever $g\mu$, $g\nu$, $(g\mu)\nu$, $(g\nu)\mu$ are defined;
and moreover that

$$(A \cap C)\mu = B \cap C, \tag{2}$$

$$(A \cap D)\mu = B \cap D, \tag{3}$$

$$(A \cap C)\nu = A \cap D. \tag{4}$$

Define for each i in J, the set of all integers, a group G_i isomorphic to G under a fixed isomorphism $\gamma_i : G\gamma_i = G_i$. Thus each G_i contains subgroups A_i, B_i, C_i, D_i which are the images of A, B, C, D under γ_i and there exist isomorphisms $\mu_i = \gamma_i^{-1}\mu\gamma_i$ and $\nu_i = \gamma_i^{-1}\nu\gamma_i$ mapping A_i onto B_i and C_i onto D_i respectively. A_i, B_i, C_i, D_i and μ_i, ν_i satisfy the conditions that correspond to (1)–(4).

Now we define a sequence of groups $P_{i,j}$ for all $i, j \in J$ and $i < j$ as follows: We first form the direct product of G_i and G_{i+1} amalgamating $B_i \subseteqq G_i$ with $A_{i+1} \subseteqq G_{i+1}$ according to the isomorphism $\gamma_i^{-1}\mu^{-1}\gamma_{i+1}$. Call this direct product $P_{i,i+1}$:

$$P_{i,i+1} = \{G_i \times G_{i+1};\ B_i = A_{i+1}\}.$$

Then define $P_{i,j}$ inductively to be the direct product

$$P_{i,j} = \{P_{i,j-1} \times G_j;\ B_{j-1} = A_j\}$$

amalgamating $B_{j-1} \subseteqq P_{i,j-1}$ with $A_j \subseteqq G_j$ according to the isomorphism $\gamma_{j-1}^{-1}\mu^{-1}\gamma_j$.

If we form

$$H_1 = \bigcup_{n=1}^{\infty} P_{-n,n}$$

then H_1 is evidently abelian.

Using lemmas (1) and (2) and through steps similar to those in [1, lemmas 5–8] we can prove that H_1 possesses an automorphism θ_1 that extends each μ_i and a partial automorphism ϕ_1 mapping the subgroup

$$\Gamma = \{\cdots, C_{-1}, C_0, C_1, \cdots\} \subseteqq H_1$$

onto the subgroup

$$\Delta = \{\cdots, D_{-1}, D_0, D_1, \cdots\} \subseteqq H_1$$

such that ϕ_1 extends each ν_i and the automorphism θ_1 maps Γ onto itself and Δ onto itself.

4. Second step of the construction

LEMMA 3. *If we replace* G; A, B, C, D; μ, ν *by* H_1; Γ, Δ, H_1, H_1; ϕ_1, θ_1 *respectively then the conditions that correspond to* (1)–(4) *will be satisfied.*

The proof that ϕ_1 commutes with θ_1 is the same as that of lemma 9 in [1]. Conditions (2)–(4) translate respectively into

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$$(\Gamma \cap H_1)\,\phi_1 = \Delta \cap H_1,$$
$$(\Gamma \cap H_1)\,\phi_1 = \Delta \cap H_1,$$
$$(\Gamma \cap H_1)\,\theta_1 = \Gamma \cap H_1$$

which are obviously true.

Thus we can repeat the procedure in § 3, this time embedding H_1 in an abelian group H_2 which possesses an automorphism ϕ_2 that extends ϕ_1 and a partial automorphism θ_2 extending θ_1 such that θ_2 and ϕ_2 commute.

We carry on indefinitely, thus when H_{n-1} is formed we embed it in the abelian group H_n that possesses two mappings θ_n and ϕ_n one of which is a total and the other is a partial automorphism such that θ_n commutes with ϕ_n.

Finally we form the group

$$H = \bigcup_{n=1}^{\infty} H_n,$$

which is abelian and define the two mappings θ and ϕ as follows: For any $h \in H$, $h \in H_i$ for some suitable i, and we put

$$h\theta = h\theta_i, \quad h\phi = h\phi_i.$$

Thus θ and ϕ are total automorphisms of H which extend each θ_i and each ϕ_i respectively and hence extend μ and ν.

LEMMA 4. *For any* $h \in H$, $h\theta\phi = h\phi\theta$.

PROOF. $h \in H_i$ for some suitable i. Thus

$$h\theta = h\theta_i, \quad h\phi = h\phi_i.$$

We distinguish two cases:

(i) If $(h\theta_i)\phi_i$ is defined then

$$h\theta\phi = (h\theta_i)\phi_i = h\phi_i\theta_i = h\phi\theta.$$

(ii) If $(h\theta_i)\phi_i$ is not defined, in which case ϕ_i is a partial automorphism of H_i then ϕ_{i+1} is a total automorphism of H_{i+1} and $(h\theta_i)\phi_{i+1}$ is defined. Thus

$$h\theta\phi = (h\theta_i)\phi_{i+1} = h\theta_{i+1}\phi_{i+1} = h\phi_{i+1}\theta_{i+1} = h\phi\theta.$$

This completes the proof of the lemma and hence the proof of the following theorem.

THEOREM. *Conditions* (1)–(4) *are sufficient for extending two partial automorphisms* μ *and* ν *of an abelian group* G *to total commutative automorphisms* θ *and* ϕ *of an abelian group* $H \supseteqq G$.

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Added in proof. The argument in [1] was carried out under similar assumptions to those of section 3, except that G was not necessarily abelian and there was a fifth condition

$$(B \cap C)\nu = B \cap D. \tag{5}$$

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