J. Aust. Math. Soc. 83 (2007), 125–134

A GENERALIZATION OF THE MATRIX FORM OF THE BRUNN-MINKOWSKI INEQUALITY

JUN YUAN and GANGSONG LENG✉

(Received 4 March 2005; revised 24 March 2006)

Communicated by A. Rubinov

Abstract

In this paper, we establish an extension of the matrix form of the Brunn-Minkowski inequality. As applications, we give generalizations on the metric addition inequality of Alexander.

2000 *Mathematics subject classification*: primary 52A40.
Keywords and phrases: Brunn-Minkowski inequality, metric addition inequality, Ky Fan inequality, Bellman's inequality, matrix, simplex.

1. Introduction

The Brunn-Minkowski inequality is one of the most important geometric inequalities. There is a vast amount of work on its generalizations and on its connections with other areas, (see [2, 5–13, 21, 22]). An excellent survey on this inequality is provided by Gardner (see [12]). The matrix form of the Brunn-Minkowski inequality (see [14, 15]) asserts that if A and B are two positive definite matrices of order n, then

$$|A+B|^{1/n} \geq |A|^{1/n} + |B|^{1/n}, \tag{1.1}$$

with equality if and only if $A = cB (c \geq 0)$, where $|A|$ denotes the determinant of A.

In [4], Bergström proved the following interesting inequality, which is analogous to (1.1).

If A and B are positive definite matrices of order n, and $A_{(i)}$, $B_{(i)}$ denote the sub-matrices obtained by deleting the i-th row and column, then

$$\frac{|A+B|}{|A_{(i)}+B_{(i)}|} \geq \frac{|A|}{|A_{(i)}|} + \frac{|B|}{|B_{(i)}|}. \tag{1.2}$$

Supported in part by the National Natural Science Foundation of China (Grant No. 10671117).

https://doi.org/10.1017/S1446788700036429 Published online by Cambridge University Press

In [9], Ky Fan gave a simultaneous generalization of (1.1) and (1.2). He established the following elegant inequality.

Let A_k denote the principal sub-matrix of A formed by taking the first k rows and columns of A. If $C = A + B$, where A and B are positive definite matrices of order n, then

$$(1.3)\qquad \left(\frac{|C|}{|C_k|}\right)^{1/(n-k)} \geq \left(\frac{|A|}{|A_k|}\right)^{1/(n-k)} + \left(\frac{|B|}{|B_k|}\right)^{1/(n-k)}$$

In this paper, a new generalization of the matrix form of the Brunn-Minkowski inequality is presented, which is an extension of (1.3) also.

Let I_{n-k} denote the unit matrix of order $n-k$, $(0 \leq k < n)$. One of our main results is the following theorem.

THEOREM 1.1. *Let A and B be positive definite matrices of order n, and let a and b be two nonnegative real numbers such that $A > aI_n$ and $B > bI_n$. If $C = A + B$, then*

$$(1.4)\qquad \left(\frac{|C|}{|C_k|} - |(a+b)I_{n-k}|\right)^{1/(n-k)} \geq \left(\frac{|A|}{|A_k|} - |aI_{n-k}|\right)^{1/(n-k)} + \left(\frac{|B|}{|B_k|} - |bI_{n-k}|\right)^{1/(n-k)}$$

with equality if and only if $a^{-1}A = b^{-1}B$.

The other aim of this paper is to provide a generalization of the metric addition inequality of Alexander. The concept of metric addition began with Oppenheim in [20], and was first explicitly defined and named by Alexander in [1].

Let $\Omega_1 = \{P_0^{(1)}, \ldots, P_n^{(1)}\}$ and $\Omega_2 = \{P_0^{(2)}, \ldots, P_n^{(2)}\}$ denote two simplices in the n-dimensional Euclidean space R^n with vertices $P_0^{(1)}, \ldots, P_n^{(1)}$ and $P_0^{(2)}, \ldots, P_n^{(2)}$, respectively. If there exists a set of points $\Omega_3 = \{P_0^{(3)}, \ldots, P_n^{(3)}\}$, such that

$$(1.5)\qquad \left|P_i^{(3)} - P_j^{(3)}\right|^2 = \left|P_i^{(1)} - P_j^{(1)}\right|^2 + \left|P_i^{(2)} - P_j^{(2)}\right|^2,$$

then Ω_3 is called *metric addition* of Ω_1 and Ω_2, and is denoted by

$$(1.6)\qquad \Omega_3 = \Omega_1 + \Omega_2.$$

It can be proved that the set of points Ω_3 exists and is an n-dimensional simplex (see [1]). Alexander conjectured the following inequality:

$$(1.7)\qquad V^2(\Omega_3) \geq V^2(\Omega_1) + V^2(\Omega_2).$$

However in [23], Yang and Zhang proved that (1.7) is not true, and gave the following correct form

$$V^{2/n}(\Omega_3) \geq V^{2/n}(\Omega_1) + V^{2/n}(\Omega_2), \tag{1.8}$$

with equality if and only if Ω_1 and Ω_2 are similar.

As an application of Theorem 1.1, we establish the following theorem, which is a special case of Theorem 4.1 of this paper.

THEOREM 1.2. *Let simplex Ω_3 be a metric addition of simplex Ω_1 and simplex Ω_2. Let D_1 and D_2 be compact domains in R^n and $D_1 \subset \Omega_1$, $D_2 \subset \Omega_2$. Then*

$$\begin{aligned} &\left[V^2(\Omega_3) - \left(V^{2/n}(D_1) + V^{2/n}(D_2)\right)^n\right]^{1/n} \\ &\qquad \geq \left[V^2(\Omega_1) - V^2(D_1)\right]^{1/n} + \left[V^2(\Omega_2) - V^2(D_2)\right]^{1/n}. \end{aligned} \tag{1.9}$$

The equality holds if and only if Ω_1 and Ω_2 are similar and $(V(\Omega_1), V(\Omega_2)) = \mu(V(D_1), V(D_2))$, where μ is a constant.

REMARK 1.3. Taking $D_1 = D_2 = \varnothing$ or taking $D_1 = \Omega_1$, $D_2 = \Omega_2$ in Theorem 1.2, we can obtain (1.8). Hence (1.9) is a generalization of (1.8).

2. Definitions and lemmas

Let $S_n(R)$ denote the set of $n \times n$ real symmetric matrices. Let I_n denote the $n \times n$ unit matrix. We use the notation $A > 0$ ($A \geq 0$) if A is a positive definite (positive semi-definite) matrix, and A^T denotes the transpose of A. Let $A, B \in S_n(R)$. Then $A > B$ ($A \geq B$) if and only if $A - B > 0$ ($A - B \geq 0$). Let k_n denote the volume of the unit ball in R^n.

DEFINITION 2.1. Let $A = \left[\begin{smallmatrix} A_k & A_{12} \\ A_{21} & A_{22} \end{smallmatrix}\right]$ be a matrix of order n, and let A_k denote the principal sub-matrix of A formed by taking the first k rows and columns of A. If A_k is nonsingular, then $A_{22} - A_{21}A_k^{-1}A_{12}$ is called a *Schur complement* of A, with respect to A_k, which is denoted by A/A_k.

Obviously, if A_k is a matrix of order 0, then $A/A_k = A$.

LEMMA 2.2. *Let $A \in S_n(R)$, $A > 0$, and A_k be its k-th order principal minor. Then*

$$A/A_k > 0 \quad \text{and} \quad |A/A_k| = \frac{|A|}{|A_k|}. \tag{2.1}$$

The proof of Lemma 2.2 can be found in [17, page 22].

https://doi.org/10.1017/S1446788700036429 Published online by Cambridge University Press

LEMMA 2.3 ([10, 16]). *Let $A, B \in S_n(R)$, $A > 0$, $B > 0$, and A_k and B_k be k-th order principal minors of A and B, respectively. Then*

$$(A + B)/(A_k + B_k) \geq A/A_k + B/B_k. \tag{2.2}$$

LEMMA 2.4. *Let $A, B \in S_n(R)$, $A > B > 0$. Then*

$$|A| > |B|. \tag{2.3}$$

The proof of Lemma 2.4 can be found in [17, page 472].

LEMMA 2.5 ([19]). *Let $A, B \in S_n(R)$, $A \geq B > 0$, A_k, B_k be k-th order principal minors of A and B, respectively. Then*

$$|A/A_k| \geq |B/B_k|. \tag{2.4}$$

LEMMA 2.6. *Let $A, B \in S_n(R)$, $A > 0$, $B > 0$. Then there exists an invertible matrix P satisfying $|P^T P| = 1$ such that $P^T A P = \mathrm{diag}(a_1, \ldots, a_n)$ and $P^T B P = \mathrm{diag}(b_1, \ldots, b_n)$.*

LEMMA 2.7. *Let $x_i \geq 0$, $y_i \geq 0$ $(i = 1, \ldots, n)$. Then*

$$\left(\prod_{i=1}^{n} x_i\right)^{1/n} + \left(\prod_{i=1}^{n} y_i\right)^{1/n} \leq \left(\prod_{i=1}^{n}(x_i + y_i)\right)^{1/n}, \tag{2.5}$$

with equality if and only if $x_i = \nu y_i$, where ν is a constant.

This is a special case of Maclaurin's inequality.

LEMMA 2.8 (Bellman's inequality). *Suppose that $a = \{a_1, \ldots, a_n\}$ and $b = \{b_1, \ldots, b_n\}$ are two n-tuples of positive real numbers, and $p > 1$ such that*

$$a_1^p - \sum_{i=2}^{n} a_i^p > 0 \quad \text{and} \quad b_1^p - \sum_{i=2}^{n} b_i^p > 0.$$

Then

$$\left((a_1 + b_1)^p - \sum_{i=2}^{n}(a_i + b_i)^p\right)^{1/p} \geq \left(a_1^p - \sum_{i=2}^{n} a_i^p\right)^{1/p} + \left(b_1^p - \sum_{i=2}^{n} b_i^p\right)^{1/p}, \tag{2.6}$$

with equality if and only if $a = \nu b$, where ν is a constant.

The proof of Lemma 2.8 can be found in [3, page 38].

https://doi.org/10.1017/S1446788700036429 Published online by Cambridge University Press

3. Proof of Theorem 1.1

PROOF. According to Lemma 2.2 and Lemma 2.3, we have

$$|A/A_k| = \frac{|A|}{|A_k|}, \quad |B/B_k| = \frac{|B|}{|B_k|},$$

and

$$\frac{|A+B|}{|A_k+B_k|} = |(A+B)/(A_k+B_k)| \geq |(A/A_k)+(B/B_k)|. \tag{3.1}$$

So

$$\left(\frac{|A+B|}{|A_k+B_k|} - |(a+b)I_{n-k}|\right)^{1/(n-k)} \geq \left(|(A/A_k)+(B/B_k)| - |(a+b)I_{n-k}|\right)^{1/(n-k)}. \tag{3.2}$$

Let $\widetilde{A} = A/A_k > 0$ and $\widetilde{B} = B/B_k > 0$. Then to prove (1.4), we need only to prove the following inequality

$$\left(|\widetilde{A}+\widetilde{B}| - |(a+b)I_{n-k}|\right)^{1/(n-k)} \geq \left(|\widetilde{A}| - |aI_{n-k}|\right)^{1/(n-k)} + \left(|\widetilde{B}| - |bI_{n-k}|\right)^{1/(n-k)}. \tag{3.3}$$

Notice that $\widetilde{A}$ and $\widetilde{B}$ are matrices of order $n-k$. By condition $A > aI_n$, $B > bI_n$ and Lemma 2.5, we have $\widetilde{A} > aI_{n-k}$, $\widetilde{B} > bI_{n-k}$. By Lemma 2.6, there is an invertible matrix P such that $|P^TP| = 1$, and

$$P^T\widetilde{A}P = \operatorname{diag}(a_1, \dots, a_{n-k}), \quad P^T\widetilde{B}P = \operatorname{diag}(b_1, \dots, b_{n-k}).$$

So

$$|\widetilde{A}| = |P^T\widetilde{A}P| = \prod_{i=1}^{n-k} a_i, \quad |\widetilde{B}| = |P^T\widetilde{B}P| = \prod_{i=1}^{n-k} b_i, \quad \text{and} \quad |\widetilde{A}+\widetilde{B}| = \prod_{i=1}^{n-k}(a_i+b_i).$$

It is straightforward to see that (3.3) holds if and only if

$$\left(\prod_{i=1}^{n-k}(a_i+b_i) - (a+b)^{n-k}\right)^{1/(n-k)} \geq \left(\prod_{i=1}^{n-k} a_i - a^{n-k}\right)^{1/(n-k)} + \left(\prod_{i=1}^{n-k} b_i - b^{n-k}\right)^{1/(n-k)}. \tag{3.4}$$

Now we prove (3.4). Put $X^{n-k} = \prod_{i=1}^{n-k} a_i - a^{n-k}$ and $Y^{n-k} = \prod_{i=1}^{n-k} b_i - b^{n-k}$. Then

$$X^{n-k} + a^{n-k} = \prod_{i=1}^{n-k} a_i, \quad Y^{n-k} + b^{n-k} = \prod_{i=1}^{n-k} b_i.$$

Applying Minkowski inequality, we have

$$\left((X+Y)^{n-k}+(a+b)^{n-k}\right)^{1/(n-k)} \leq \left(X^{n-k}+a^{n-k}\right)^{1/(n-k)}+\left(Y^{n-k}+b^{n-k}\right)^{1/(n-k)} = \left(\prod_{i=1}^{n-k} a_i\right)^{1/(n-k)}+\left(\prod_{i=1}^{n-k} b_i\right)^{1/(n-k)}.$$

Applying Lemma 2.7 to the right of the above inequality, we obtain

$$\left((X+Y)^{n-k}+(a+b)^{n-k}\right)^{1/(n-k)} \leq \left(\prod_{i=1}^{n-k}(a_i+b_i)\right)^{1/(n-k)},$$

which implies that $(X+Y)^{n-k} \leq \prod_{i=1}^{n-k}(a_i+b_i)-(a+b)^{n-k}$. It follows that

$$X+Y \leq \left(\prod_{i=1}^{n-k}(a_i+b_i)-(a+b)^{n-k}\right)^{1/(n-k)},$$

which is just inequality (3.4). □

REMARK 3.1. Let $a=b=0$ in Theorem 1.1. Then we get the Ky Fan inequality (1.3). Let $k=0$ in Theorem 1.1, and we obtain

$$(3.5)\qquad \left(|A+B|-|(a+b)I_n|\right)^{1/n} \geq \left(|A|-|aI_n|\right)^{1/n}+\left(|B|-|bI_n|\right)^{1/n},$$

with equality if and only if $a^{-1}A=b^{-1}B$.

This is [18, Equation (23)], so Theorem 1.1 is a generalization of the Ky Fan inequality (1.3) and (3.5).

Replacing A and B by λA and μB, and at the same time replacing a and b by λa and μb in Theorem 1.1, yields the following corollary.

COROLLARY 3.2. *Let $A, B \in S_n(R)$, and A_k and B_k be k-th order principal minors of A and B respectively. Let $C=\lambda A+\mu B$, $a \geq 0$, $b \geq 0$. If $A > aI_n$, $B > bI_n$, then*

$$(3.6)\qquad \left(\frac{|C|}{|C_k|}-|(\lambda a+\mu b)I_{n-k}|\right)^{1/(n-k)} \geq \lambda\left(\frac{|A|}{|A_k|}-|aI_{n-k}|\right)^{1/(n-k)}+\mu\left(\frac{|B|}{|B_k|}-|bI_{n-k}|\right)^{1/(n-k)}$$

for all $\lambda > 0$, $\mu > 0$, with equality if and only if $a^{-1}A=b^{-1}B$.

By induction, we infer the following.

COROLLARY 3.3. *Let $A_i \in S_n(R)$, $a_i \geq 0$, $\lambda_i > 0$, $A_i > a_i I_n$, and $A_{i(k)}$ be k-th order principal minors of A_i, $i = 1, \dots, m$. Then*

$$\left(\frac{\left|\sum_{i=1}^{m}\lambda_i A_i\right|}{\left|\sum_{i=1}^{m}\lambda_i A_{i(k)}\right|} - \left|\sum_{i=1}^{m}\lambda_i a_i I_{n-k}\right|\right)^{1/(n-k)} \geq \sum_{i=1}^{m}\lambda_i\left(\frac{|A_i|}{|A_{i(k)}|} - |a_i I_{n-k}|\right)^{1/(n-k)}, \tag{3.7}$$

with equality if and only if $a_1^{-1}A_1 = \cdots = a_m^{-1}A_m$.

Applying the generalized arithmetic-geometric mean inequality to the right side of (3.7), we get the following inequality.

COROLLARY 3.4. *Let $A_i \in S_n(R)$, $a_i \geq 0$, $\lambda_i > 0$, and $\sum_{i=1}^{m}\lambda_i = 1$, $A_i > a_i I_n$, and $A_{i(k)}$ be k-th order principal minors of A_i, $i = 1, \dots, m$. Then*

$$\frac{\left|\sum_{i=1}^{m}\lambda_i A_i\right|}{\left|\sum_{i=1}^{m}\lambda_i A_{i(k)}\right|} - \left|\sum_{i=1}^{m}\lambda_i a_i I_{n-k}\right| \geq \prod_{i=1}^{m}\left(\frac{|A_i|}{|A_{i(k)}|} - |a_i I_{n-k}|\right)^{\lambda_i}. \tag{3.8}$$

When $a_1 = \cdots = a_m = 0$, the equality holds in (3.8) *if and only if $A_1, \dots, A_m$ are equal.*

Taking $i = 2$, $k = 0$ in Corollary 3.4, we obtain a generalization of the Ky Fan concave theorem as follows.

COROLLARY 3.5. *Let $A_i \in S_n(R)$, $a_i \geq 0$, $A_i > a_i I_n$ $(i = 1, 2)$. Then*

$$|\lambda A_1 + (1-\lambda)A_2| - [\lambda a_1 + (1-\lambda)a_2]^n \geq (|A_1| - a_1^n)^{\lambda}(|A_2| - a_2^n)^{1-\lambda}, \tag{3.9}$$

where $0 \leq \lambda \leq 1$.

4. Inequalities for metric addition

Let $\Omega_l = \{P_0^{(l)}, \dots, P_n^{(l)}\}$ $(1 \leq l \leq m)$ be a simplex in R^n. For any $\lambda_l > 0$ $(1 \leq l \leq m)$, there exists a unique simplex $\Omega_{m+1} = \{P_0^{(m+1)}, \dots, P_n^{(m+1)}\}$ such that

$$\left|P_i^{(m+1)} - P_j^{(m+1)}\right|^2 = \sum_{l=1}^{m}\lambda_l\left|P_i^{(l)} - P_j^{(l)}\right|^2. \tag{4.1}$$

Then Ω_{m+1} is called the *weighted metric addition* of $\Omega_1, \dots, \Omega_m$, denoted by

$$\Omega_{m+1} = \sum_{l=1}^{m}\lambda_l\Omega_l \tag{4.2}$$

(see [1, 23]). On the weighted metric addition, we have the following theorem.

https://doi.org/10.1017/S1446788700036429 Published online by Cambridge University Press

THEOREM 4.1. *Let Ω_i be an n-dimensional simplex in R^n $(1 \leq i \leq m)$. If $\Omega_{m+1} = \sum_{i=1}^{m} \lambda_i \Omega_i$ and compact domains $D_i \subset \Omega_i$, then*

$$\left[V^2(\Omega_{m+1})) - \left(\sum_{i=1}^{m} \lambda_i V^{2/n}(D_i)\right)^n\right]^{1/n} \geq \sum_{l=1}^{m} \lambda_i \left[V^2(\Omega_i)) - V^2(D_i)\right]^{1/n}. \tag{4.3}$$

The equality holds if and only if $\Omega_1, \dots, \Omega_m$ are similar and $(V(\Omega_1), \dots, V(\Omega_m))$ $= \mu(V(D_1), \dots, V(D_m))$, where μ is a constant.

PROOF. Let $a_{ij}^{(l)}$ be the distance between $P_i^{(l)}$ and $P_j^{(l)}$. Let

$$\rho_{ij}^{(l)} = \left(a_{i0}^{(l)}\right)^2 + \left(a_{0j}^{(l)}\right)^2 - \left(a_{ij}^{(l)}\right)^2, \quad (0 \leq i, j \leq n).$$

Then the matrix $A^{(l)} = (\rho_{ij}^{(l)})_{n\times n}$ is a positive definite matrix.

It is straightforward to verify that

$$A^{(m+1)} = \sum_{i=1}^{m} \lambda_i A^{(i)}, \tag{4.4}$$

then by the volume formula of a simplex, we have

$$|A^{(i)}| = 2^n n!^2 V^2(\Omega_i), \tag{4.5}$$

where $1 \leq i \leq m+1$.

Let

$$a_i^n = 2^n n!^2 V^2(D_i), \quad (1 \leq i \leq m). \tag{4.6}$$

Since $D_i \subset \Omega_i$, $1 \leq i \leq m$, then $|A_i| > a_i^n$, $1 \leq i \leq m$. Setting $k = 0$ in Corollary 3.3, we have

$$\left[\left|\sum_{i=1}^{m} \lambda_i A_i\right| - \left(\sum_{i=1}^{m} \lambda_i a_i\right)^n\right]^{1/n} \geq \sum_{i=1}^{m} \lambda_i \left(|A_i| - a_i^n\right)^{1/n}, \tag{4.7}$$

with equality if and only if $a_1^{-1} A^{(1)} = \cdots = a_m^{-1} A^{(m)}$.

By (4.4), we have

$$\left[|A^{(m+1)}| - \left(\sum_{i=1}^{m} \lambda_i a_i\right)^n\right]^{1/n} \geq \sum_{i=1}^{m} \lambda_i \left(|A^{(i)}| - a_i^n\right)^{1/n}. \tag{4.8}$$

Substituting (4.5) and (4.6) into (4.7) and rearranging, we obtain (4.3). □

Let D_1 and D_2 be two closed balls with radii r_1 and r_2, respectively. We infer the following.

COROLLARY 4.2. *Let $\Omega_3 = \Omega_1 + \Omega_2$, and $r(\Omega_1)$ and $r(\Omega_2)$ be the radii of simplex Ω_1 and Ω_2, respectively. If $0 \le r_1 \le r(\Omega_1)$, $0 \le r_2 \le r(\Omega_2)$, then*

$$(4.9)\quad (V^2(\Omega_3) - (r_1^2 + r_2^2)^n k_n^2)^{1/n} \ge (V^2(\Omega_1) - r_1^{2n} k_n^2)^{1/n} + (V^2(\Omega_2) - r_2^{2n} k_n^2)^{1/n}.$$

When $r_1 = r_2 = 0$, there is equality if and only if Ω_1 and Ω_2 are similar; when $r_1 \ne 0$ and $r_2 \ne 0$, equality holds if and only if Ω_1 and Ω_2 are similar and $r_1^n / r_2^n = V(\Omega_1)/V(\Omega_2)$.

PROOF. From (1.8) and applying Bellman's inequality (2.6), we have

$$\begin{aligned}(V^2(\Omega_3) - (r_1^2 + r_2^2)^n k_n^2)^{1/n} &\ge \left((V^{2/n}(\Omega_1) + V^{2/n}(\Omega_2))^n - (r_1^2 + r_2^2)^n k_n^2\right)^{1/n} \\ &\ge \left(V^2(\Omega_1) - r_1^{2n} k_n^2\right)^{1/n} + \left(V^2(\Omega_2) - r_2^{2n} k_n^2\right)^{1/n}.\end{aligned}$$

The proof is complete. □

Acknowledgment

The authors are most grateful to the referee for his valuable suggestions.

References

[1] R. Alexander, *The geometry of metric and linear space* (Springer-Verlag, Berlin, 1975) pp. 57–65.
[2] I. J. Bakelman, *Convex analysis and nonlinear geometric elliptic equations* (Springer, Berlin, 1994).
[3] E. F. Beckenbach and R. Bellman, *Inequalities* (Springer, Berlin, 1961).
[4] H. Bergström, *A triangle inequality for matrices* (Den Elfte Skandiaviski Matematiker-kongress, Trondheim, 1949).
[5] C. Borell, 'The Brunn-Minkowski inequality in Gauss space', *Invent Math.* **30** (1975), 202–216.
[6] ———, 'Capacitary inequality of the Brunn-Minkowski inequality type', *Math. Ann.* **263** (1993), 179–184.
[7] Y. D. Burago and V. A. Zalgaller, *Geometric inequalities*, (Translated from Russian: Springer Series in Soviet Mathematics) (Springer, New York, 1988).
[8] K. Fan, 'Problem 4786', *Amer. Math. Monthly* **65** (1958), 289.
[9] ———, 'Some inequalities concerning positive-definite Hermitian matrices', *Proc. Cambridge Phil. Soc.* **51** (1958), 414–421.
[10] M. Fiedler and T. Markham, 'Some results on the Bergström and Minkowski inequalities', *Linear Algebra Appl.* **232** (1996), 199–211.

[11] R. J. Gardner, *Geometric tomography*, Encyclopædia of Mathematics and its Applications 58 (Cambridge University Press, Cambridge, 1995).
[12] ———, 'The Brunn-Minkowski inequality', *Bull. Amer. Math. Soc. (N.S.)* **39** (2002), 355–405.
[13] R. J. Gardner and P. Gronchi, 'A Brunn-Minkowski inequality for the integer lattice', *Trans. Amer. Math. Soc.* **353** (2001), 3995–4024.
[14] E. V. Haynesworth, 'Note on bounds for certain determinants', *Duke Math. J.* **24** (1957), 313–320.
[15] ———, 'Bounds for determinants with positive diagonals', *Trans. Amer. Math. Soc.* **96** (1960), 395–413.
[16] E. V. Haynsworth, 'Applications of an inequality for the Schur complement', *Proc. Amer. Math. Soc.* **24** (1970), 512–516.
[17] R. Horn and C. R. Johnson, *Matrix analysis* (Cambridge University Press, Cambridge, 1985).
[18] G. S. Leng, 'The Brunn-Minkowski inequality for volume differences', *Adv. in Appl. Math.* **32** (2004), 615–624.
[19] C. K. Li and R. Mathias, 'Extremal characterizations of Schur complement and resulting inequalities', *SIAM Rev.* **42** (2000), 233–246.
[20] A. Oppenheim, 'Advanced problems 5092', *Amer. Math. Monthly* **701** (1963), 444.
[21] R. Osserman, 'The Brunn-Minkowski inequality for multiplictities', *Invent. Math.* **125** (1996), 405–411.
[22] R. Schneider, *Convex bodies: The Brunn-Minkowski theory* (Cambridge University Press, Cambridge, 1993).
[23] L. Yang and J. Z. Zhang, 'On Alexander's conjecture', *Chinese Sci. Bull.* **27** (1982), 1–3.

School of Mathematics and Computer Science
Nanjing Normal University
Nanjing 210097
P.R. China
e-mail: yuanjun@graduate.shu.edu.cn

Department of Mathematics
Shanghai University
Shanghai 200444
P. R. China
e-mail: gleng@staff.shu.edu.cn

https://doi.org/10.1017/S1446788700036429 Published online by Cambridge University Press