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ARTICLE

Simulating multiple equilibria in rational expectations models with occasionally-binding constraints: An algorithm and a policy application

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Abstract
This paper presents an algorithm for simulating multiple equilibria in otherwise-linear dynamic models with occasionally-binding constraints. Our algorithm extends the guess-and-verify approach of Guerrieri and Iacoviello (2015) to detect and simulate *multiple* perfect foresight equilibria, and allows arbitrary "news shocks" up to a finite horizon. When there are multiple equilibria, we show how to compute expected paths using a "prior probabilities" approach and we provide an approach for running stochastic simulations with switching between equilibria on the simulated path. A policy application studies a New Keynesian model with a zero lower bound on nominal interest rates and multiple equilibria, including a "bad" solution based on self-fulfilling pessimistic expectations. A price-level targeting rule does not always eliminate the bad solution, but it shrinks the indeterminacy region substantially and improves stabilization and welfare relative to more conventional interest rate rules or forward guidance.

Keywords: occasionally-binding constraints; indeterminacy; multiple equilibria; rational expectations; interest rate rules

1. Introduction

Occasionally-binding constraints, such as borrowing limits and the lower bound on nominal interest rates, introduce a stark non-linearity in economic models. As a result, standard solution methods for linear rational expectations models (Blanchard and Kahn, 1980; Binder and Pesaran, 1997; Uhlig, 1999; Sims, 2002), which assume a time-invariant structure, must be adapted to cope with such constraints. An important contribution to the literature was made by Guerrieri and Iacoviello (2015). They show how otherwise-linear rational expectations models with occasionally-binding constraints and many state variables can be solved using a guess and verify method, and they also provide a toolkit (OccBin) that implements the solution algorithm in the popular software package Dynare. When a solution exists, their algorithm finds one solution under perfect foresight and assuming zero anticipated future shocks. However, it is known that models with occasionally-binding constraints may have *multiple* perfect foresight equilibria (see Holden, 2023), and neglecting these additional solutions may have non-trivial quantitative or policy implications.

In this paper, we therefore extend the Guerrieri and Iacoviello (2015) solution method so that *multiple* perfect foresight equilibria can be detected and simulated. Using this approach, we show how researchers can compute expected outcomes (such as welfare measures) when there are multiple perfect foresight solutions, and we also show how to run stochastic simulations with switching between equilibria on the simulated path. Our algorithm also extends the solution of Guerrieri and Iacoviello (2015) to allow non-zero "news shocks" while preserving computational tractability.[1] We demonstrate our approach using a running example (Fisherian model) and a *policy application* that studies conventional versus unconventional interest rate rules in a New Keynesian model with a zero lower bound and multiple equilibria.

The modern literature on occasionally-binding constraints began with Eggertsson and Woodford (2003) and Jung et al. (2005), who study the benchmark New Keynesian model with a zero lower bound on nominal interest rates, the former in a version of the model with a two-state Markov process and the latter under perfect foresight. The papers most relevant to the current paper are those which study *perfect foresight* solutions to models with occasionally-binding constraints, including the computational papers of Guerrieri and Iacoviello (2015), Holden (2016), Boehl (2022), and the theory paper by Holden (2023). Most of these papers focus on dynamic models which are linear aside from occasionally-binding constraints (as here), and only Holden (2016, 2023) considers multiple equilibria.

The present paper makes two methodological contributions. First, relative to Guerrieri and Iacoviello (2015), we extend their solution method (based on undetermined coefficients) to detect and simulate *multiple* perfect foresight equilibria, and also to allow non-zero anticipated news, such as "forward guidance" shocks. Our algorithm allows a much wider range of economic scenarios, as additional solutions and simulations paths are not neglected, while the addition of "news shocks" means that announcements or news events can be studied. Second, as argued by Farmer et al. (2015, 17), "a model with an indeterminate set of equilibria is an incomplete model." The incompleteness problem can be resolved by drawing a sunspot that picks an equilibrium (i.e. a *particular* perfect foresight path). On the one hand, a unique *realized* equilibrium path is selected and can be studied, but as a result one particular solution path is given precedence over others. We attempt to speak to both views by providing an approach for computing *expected* paths that summarize "average outcomes" given multiple solutions (an approach not used in Holden (2023)), and we also show how to run stochastic simulations with switching between equilibria on the simulated path in response to unanticipated shocks (here we use an extended path method).

Our paper also contributes to the literature by providing *policy applications*. First, in the methodology section we include a simple "running example", namely a Fisherian model with multiple equilibria: both a high-inflation and low-inflation solution exist for the same initial conditions. We use this simple example to illustrate our expected outcomes approach based on "prior probabilities" and the extension to stochastic simulation with switching between multiple equilibria. We also show how these approaches can be used for *policy analysis*.

Our main policy application studies a New Keynesian model with a zero lower bound on nominal interest rates and multiple equilibria for some parameter values (Brendon et al. 2013). Here we show that our algorithm replicates their finding of two perfect foresight equilibria: there is a "good" solution for which the lower bound is not hit, and a "bad" solution based on *self-fulfilling* pessimistic expectations, for which interest rates spend some time at the bound and inflation and the output gap are strongly negative and persistent. Multiplicity occurs under a conventional "growth-based" interest rate rule that includes an inflation target and a response to the first-difference in the output gap – the "speed limit." Such speed-limit policies are considered attractive by some policymakers and have some known stabilization advantages, including robustness to output gap measurement error;[2] however, by following such rules, policymakers may inadvertently bring about indeterminacy. A natural question is then: would *unconventional* monetary policy rules restore determinacy and stabilize the economy while retaining the potential advantages of a speed limit?

We answer this question by studying two unconventional policies—price-level targeting and forward guidance—which provide stimulus at the lower bound and may be useful to rule out (or mitigate) multiplicity of equilibria based on past work. For forward guidance, we consider a large number of announcements promising low interest rates and find that multiplicity generally prevails and that the "good" and "bad" solutions have inflation and output gaps *exacerbated* relative to a more conventional interest rate rule. By comparison, an interest rate rule that replaces the inflation target with a *price-level* target shrinks but does *not* eliminate the indeterminacy region,

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and when both solutions exist the "bad" solution is often comparatively tame in terms of inflation and output gap deviations.

These results shed new light on a conclusion in Holden (2023). In particular, while he shows that Taylor-type rules with a price-level target and an output gap *level* target ensure uniqueness in a range of New Keynesian models, our "speed limit" application shows that an interest rate rule with a price-level response is *not* (in general) sufficient to ensure determinacy in New Keynesian models with a zero lower bound on nominal interest rates. Furthermore, we find that a price-level targeting rule outperforms the other interest rate rules we study in terms of social welfare.

The paper proceeds as follows. Section 2 outlines the solution method and describes how we extend the benchmark algorithm to study multiple equilibria. Section 3 provides details of our "prior probabilities" approach to simulating multiple equilibria, including expected outcomes and the construction of stochastic simulations. Section 4 presents our policy application in a New Keynesian model. Finally, Section 5 concludes.

2. Methodology

Consider a multivariate rational expectations model with *perfect foresight*. The model is linear aside from multiple possible regimes due to occasionally-binding constraints, and time is discrete: $t \in \mathbb{N}_+$. As in Guerrieri and Iacoviello (2015), we focus for exposition purposes on the case of a single occasionally-binding constraint, so there are two regimes.[3] The reference regime (slack) is described by (1), and the alternative regime (bind) by (2):

$$\overline{B}_1 x_t = \overline{B}_2 E_t x_{t+1} + \overline{B}_3 x_{t-1} + \overline{B}_4 e_t + \overline{B}_5 \tag{1}$$

$$\tilde{B}_1 x_t = \tilde{B}_2 E_t x_{t+1} + \tilde{B}_3 x_{t-1} + \tilde{B}_4 e_t + \tilde{B}_5 \tag{2}$$

where x_t is an $n \times 1$ vector of endogenous state and jump variables, E_t is the conditional expectations operator, and e_t is an $m \times 1$ vector of exogenous "news shocks" whose values are *known*. Note that serially correlated exogenous processes can be included in the vector x_t.

Matrices $\overline{B}_i, \tilde{B}_i, i \in [5]$, contain the model parameters. The $\overline{B}_i, \tilde{B}_i, i \in \{1, 2, 3\}$, are $n \times n$ matrices, $\overline{B}_4, \tilde{B}_4$ are $n \times m$ matrices, and $\overline{B}_5, \tilde{B}_5$ are $n \times 1$ vectors of intercepts. As shown in Binder and Pesaran (1997), the above formulation is quite general as it can accommodate multiple leads and lags of the endogenous variables through an appropriate definition of x_t.

The first variable $x_{1,t}$ is subject to a lower bound constraint in all periods:

$$x_{1,t} = \max\{\underline{x}_1, x^*_{1,t}\}, \qquad x^*_{1,t} := F \begin{bmatrix} x_t \\ E_t x_{t+1} \\ x_{t-1} \end{bmatrix} + G e_t + H \tag{3}$$

where $\underline{x}_1 \in \mathbb{R}$ is the lower bound and $x^*_{1,t}$ is the "shadow value" of the constrained variable. Here, F is $1 \times 3n$ vector with $f_{11} = 0$; G is a $1 \times m$ vector; and $H \in \mathbb{R}$ is a scalar.

The specification in (3) allows the constrained variable to depend on the news shocks and on contemporaneous, past or future values of the endogenous variables; note that an upper bound constraint can easily be accommodated.[4] The vectors F, G, H are given by the equation that describes the bounded variable when the constraint is slack; for example, with a lower bound on nominal interest rates, this equation is typically a Taylor(-type) rule.

In typical applications, one of the intercept matrices may be zero, as DSGE models are often log-linearized around a non-stochastic steady state (see Uhlig, 1999). Given mutually exclusive regimes, we introduce an *indicator variable* $\mathbb{1}_{\{x^*_{1,t} > \underline{x}_1\}}$ that is equal to 1 if $x^*_{1,t} > \underline{x}_1$ and 0 otherwise (i.e. if $x^*_{1,t} \leq \underline{x}_1$). Our model based on (1)–(3) is then

$$\begin{aligned} B_{1,t} x_t &= B_{2,t} E_t x_{t+1} + B_{3,t} x_{t-1} + B_{4,t} e_t + B_{5,t}, \quad \forall t \geq 1 \\ B_{i,t} &= \mathbb{1}_{\{x^*_{1,t} > \underline{x}_1\}} \overline{B}_i + (1 - \mathbb{1}_{\{x^*_{1,t} > \underline{x}_1\}}) \tilde{B}_i, \quad \forall i \in [5] \end{aligned} \tag{4}$$

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where $x_0 \in \mathbb{R}^n$ (given) and all news shocks $e_1, e_2, \ldots \in \mathbb{R}^m$ are known.

The information set at time t includes all current, past and future values of the endogenous and exogenous variables; note that the indicator $\mathbb{1}_{\{x^*_{1,t} > \underline{x}_1\}}$ is *endogenous*. As in Guerrieri and Iacoviello (2015) and Holden (2023), we assume the model returns to the reference regime forever after some finite date $T \geq 1$ (i.e. $\mathbb{1}_{\{x^*_{1,t} > \underline{x}_1\}} = 1 \; \forall t > T$).[5] Following Guerrieri and Iacoviello (2015), we find $(\mathbb{1}_{\{x^*_{1,t} > \underline{x}_1\}})_{t=1}^{T}$ using a guess-verify method. That is, we guess a sequence of regimes and date T, and accept the resulting time path $(x_t)_{t \geq 1}$ as a solution only if the guessed sequence of regimes is verified by the shadow variable $x^*_{1,t}$.

2.1 Preliminaries

Definition 1. *A perfect foresight solution to model (4) is a path* $x_t = f(x_{t-1}, (e_s)_{s=t}^{\infty})$ *such that the system in (4) holds for all* $t \geq 1$ *given a known sequence of news shocks* $(e_t)_{t=1}^{\infty}$.

An alternative way of characterizing a solution is in terms of a set of matrices $\{\Omega_t, \Gamma_t, \Psi_t\}_{t=1}^{\infty}$ that generalize the constant-coefficient decision rules of a linear rational expectations model:

$$x_t = \Omega_t x_{t-1} + \Gamma_t e_t + \Psi_t \tag{5}$$

where Ω_t is an $n \times n$ matrix, Γ_t is an $n \times m$ matrix, Ψ_t is an $n \times 1$ vector, and the t subscript indicates that the matrices are in general time-varying.

Following Guerrieri and Iacoviello (2015) and Kulish and Pagan (2017), the matrices $\Omega_t, \Gamma_t, \Psi_t$ are determined *recursively* using simple formulas. Our perfect foresight assumption implies that the solution x_t will generally depend on both current shocks e_t and anticipated future shocks $e_{t+1}, e_{t+2}, \ldots$, which enter the solution via the "intercept" matrix Ψ_t.

There are three key requirements for the application of our algorithm:

(i) Existence of a rational expectations solution at the reference regime (terminal solution).

(ii) A series of regularity conditions $\det[B_{1,t} - B_{2,t}\Omega_{t+1}] \neq 0$ must hold for $t = 1, \ldots, T$, where $T + 1$ is a date from which the terminal solution applies in perpetuity.

(iii) The solution path x_t must satisfy (4) and $x_{1,t} > \underline{x}_1$ for all $t > T$ (terminal condition).

Requirements (i)–(iii) also apply in Guerrieri and Iacoviello (2015). The terminal solution in (i) can be found using standard methods, such as Blanchard and Kahn (1980), Binder and Pesaran (1997), Sims (2002) or Dynare (Adjemian et al. 2011), which can check if the solution is unique. Requirement (i) is necessary but not sufficient for existence of a solution; the regularity conditions in (ii) must hold, and the perfect foresight path must satisfy the occasionally-binding constraint and the terminal condition in (iii), as in Holden (2023).

We assume the Blanchard-Kahn conditions for uniqueness and stability are satisfied and that the terminal solution is away from the lower bound for all $t > T$. Formally, we have:

Assumption 1. *We assume* $\det[\overline{B}_1 - \overline{B}_2 - \overline{B}_3] \neq 0$, *such that there exists a unique steady state* $\overline{x} = (\overline{B}_1 - \overline{B}_2 - \overline{B}_3)^{-1}\overline{B}_5$ *at the reference regime. This steady state satisfies* $\overline{x}_1 > \underline{x}_1$. □

Assumption 2. *For any initial value, there is a unique stable terminal solution at the reference regime* $x_t = \overline{\Omega} x_{t-1} + \overline{\Psi}, \forall t > T$, *where* $\overline{\Psi} = (\overline{B}_1 - \overline{B}_2\overline{\Omega})^{-1}(\overline{B}_2\overline{\Psi} + \overline{B}_5) = (I_n - \overline{\Omega})\overline{x}$, *and* $\overline{\Omega} = (\overline{B}_1 - \overline{B}_2\overline{\Omega})^{-1}\overline{B}_3$ *has eigenvalues in the unit circle, so* $x_t \to \overline{x}$ *as* $t \to \infty$. □

Assumption 3. *Agents know all future shocks* $(e_t)_{t=1}^{\infty}$, *and* $e_t = 0_{m \times 1}$ *for all* $t > T$. □

Assumptions 1–3 are analogous to the assumptions in Holden (2023, Supp. Appendix). Assumption 1 restricts attention to models with a unique steady state $\overline{x}$ at the reference regime that does not violate the lower bound constraint. Assumption 2 ensures that the terminal solution

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at the reference regime converges to this steady state. Lastly, Assumption 3 states that agents have *perfect foresight* and that news shocks "die out" after date T.

2.2 Solution algorithm

Recall that date $t = 1$ is the first period. Given perfect foresight, expectations coincide with future values: $E_t[x_{t+1}] = x_{t+1}$ for all $t \geq 1$. The system to be solved is therefore:

$$\begin{cases} B_{1,t}x_t = B_{2,t}x_{t+1} + B_{3,t}x_{t-1} + B_{4,t}e_t + B_{5,t}, & 1 \leq t \leq T \\ \overline{B}_1 x_t = \overline{B}_2 x_{t+1} + \overline{B}_3 x_{t-1} + \overline{B}_5, & \forall t > T \end{cases} \tag{6}$$

where $B_{i,t} = \mathbb{1}_{\{x^*_{1,t} > \underline{x}_1\}}\overline{B}_i + (1 - \mathbb{1}_{\{x^*_{1,t} > \underline{x}_1\}})\tilde{B}_i \ \forall i \in [5]$; see (4).

By assumption, the reference regime holds for all $t > T$, and the terminal solution $x_t = \overline{\Omega}x_{t-1} + \overline{\Psi}$ is away from the bound. Thus, agents can use backward induction from the terminal solution $x_{T+1} = \overline{\Omega}x_T + \overline{\Psi}$ in period T, giving the following solution algorithm which uses a "guess and verify" approach. Note that *guesses* on the indicator variable are denoted by $\mathbb{1}_t \in \{0, 1\}$ because some guesses on the sequence of regimes may not be verified.

1. Pick a $T \geq 1$ and a simulation length $T_s > T$. Guess a sequence $(\mathbb{1}_t)_{t=1}^T$ of 0s and 1s, starting with all 1s (slack in all periods) as an initial guess. Note: $\mathbb{1}_t = 1$ for $t > T$.
2. Find the structural matrices (or "regimes") implied by the guess:
$$B_{i,t} = \mathbb{1}_t\overline{B}_i + (1 - \mathbb{1}_t)\tilde{B}_i, \quad i \in [5], \quad \text{in periods } t = 1, \ldots, T_s.$$
3. Compute $(x_t)_{t=1}^{T_s}$ and the shadow value of the bounded variable $(x^*_{1,t})_{t=1}^{T_s}$ via
$$x_t = \begin{cases} \Omega_t x_{t-1} + \Gamma_t e_t + \Psi_t & \text{for } 1 \leq t \leq T \\ \overline{\Omega}x_{t-1} + \overline{\Psi} & \text{for } t > T \end{cases}, \quad x^*_{1,t} = F\left[x'_t \ x'_{t+1} \ x'_{t-1}\right]' + Ge_t + H$$
where, for $t = 1, \ldots, T$ and initial matrices $\Omega_{T+1} = \overline{\Omega}$, $\Psi_{T+1} = \overline{\Psi}$, $\Gamma_{T+1} = 0_{n\times m}$,
$$\Omega_t = (B_{1,t} - B_{2,t}\Omega_{t+1})^{-1}B_{3,t}, \qquad \Gamma_t = (B_{1,t} - B_{2,t}\Omega_{t+1})^{-1}B_{4,t}$$
$$\Psi_t = (B_{1,t} - B_{2,t}\Omega_{t+1})^{-1}(B_{2,t}(\Psi_{t+1} + \Gamma_{t+1}e_{t+1}) + B_{5,t}).$$
4. If $x_{1,t} = \max\{\underline{x}_1, x^*_{1,t}\}$ for $t = 1, \ldots, T$ and $x_{1,t} > \underline{x}_1 \ \forall t > T$, accept the guess and store the solution $(x_t)_{t=1}^{T_s}$; else reject. Return to Step 1 and repeat for a new guess.

The above algorithm has two additions relative to Guerrieri and Iacoviello (2015). First, it allows for *multiple* perfect foresight solutions. If a guessed structure is verified (Step 4), then the resulting time path is accepted as a perfect foresight solution and is stored for later use, along with any *additional* solutions found by repeating Steps 1–4 with new guesses. Second, the algorithm permits the inclusion of "news shocks" up to a finite horizon, whereas the original algorithm sets future shocks at *zero*. Conveniently, our solution has the same general form as in Guerrieri and Iacoviello (2015) because the intercept matrix Ψ_t (Step 3) incorporates the news shocks, and is determined by a recursive formula (it depends on its own future value). Hence, this generalization—which can be used to model pre-announced policies or other news events—comes at essentially zero computational cost.[6]

In Section 3 below, we present a method for performing model simulations when there are multiple equilibria. As argued by Farmer et al. (2015, p. 17), "a model with an indeterminate set of equilibria is an incomplete model" and will need to be *closed* by some means in order to simulate or estimate the model.[7] In general, a model with multiple equilibria may be closed either by using some "selection criterion" to rule out all equilibria but one on some economic grounds (such as

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learnability), or by drawing an exogenous "sunspot" (a non-fundamental shock from outside the model) that determines which equilibrium agents' expectations coordinate on. In the absence of a generic selection criterion, we take the latter sunspot approach. Note that this approach does not contradict the perfect foresight assumption (which implies that risk is absent) because the exogenous sunspot is *not* part of the solution path itself from date 1 onwards, but is only a means of *initially determining* (at date 1) which of the perfect foresight solutions will "play out."

We start out by formalizing our approach in which the solutions are treated as categorical and a sunspot determines which perfect foresight path is realized (see Remark 1). We give the sunspot a simple structure (uniform random variable) such that the probability that agents will coordinate their expectations on a particular equilibrium can be interpreted in terms of "prior beliefs." Hence, if there were a bank run equilibrium and a no-run equilibrium, then the researcher may assign a low probability to the "run equilibrium" if they view it as somewhat implausible. At the other extreme, a researcher may take an agnostic approach by assigning equal probability to each equilibrium—"flat priors"—and in this case the agents will have equal probability of coordinating on any given equilibrium path.

With this simple structure in hand, we show how researchers or policymakers can compute *expected* outcomes, such as average paths or expected welfare under the "veil of ignorance" (see Section 3.2).[8] As a result, this approach allows welfare evaluation to be conducted in the face of multiple solutions, as well as allowing robustness analysis that checks sensitivity to changes in the underlying probabilities. Both exercises may be useful to policymakers or researchers who would like to summarize economic outcomes when there are multiple solutions which cannot be ruled out *a priori* or assigned indisputable probabilities.

We also provide an extension for stochastic simulation in Section 3.3. We thereby generalize the stochastic simulation approach in Guerrieri and Iacoviello (2015) to the more difficult case of *multiple equilibria* and non-zero expected future shocks. In particular, we relax the perfect foresight assumption by allowing *unanticipated shocks* after period 1 and solve the model using an extended path method with risk ignored by agents when they form expectations.[9] Along such simulation paths, switching between different equilibria can occur and these switches can contribute substantially to macroeconomic fluctuations.

2.3 Implementing the algorithm

Our "guess and verify" algorithm builds on Guerrieri and Iacoviello (2015) and uses the simple idea that continuing the guess-verify procedure after one solution has been found may yield additional solutions to the linear complementarity problem.[10] As in Guerrieri and Iacoviello (2015), finding a solution requires inversions of the matrix $(B_{1,t} - B_{2,t}\Omega_{t+1})$ for $t = 1, \ldots, T$, as in Step 3 of the Algorithm above. In cases of non-invertibility, our algorithm automatically abandons the current guess and starts a new one so that computation time is not wasted. Our algorithm is written in MATLAB and codes are available at the author's GitHub page.[11]

An important issue when using guess-verify is how the guessed sequences of regimes, $(\mathbb{1}_t)_{t=1}^{T}$, are determined. We first try as an initial guess the case where the constraint is slack in all periods (see Step 1) and then guesses which involve a "single spell" at the lower bound (i.e. guesses differ in the *duration* of the spell at the bound or the *start date*), followed by "double spells" at the bound. We provide codes that enumerate the single and double spells for a given T and use these as guesses in a sequential manner; we also provide a code covering cases of triple spells at the bound.[12] Note that if not all columns of the "guesses matrix" are filled, the remaining columns are filled with random guesses of 0 s and 1 s. As an example of a model with *multiple spells* at the bound, we study a multiplier-accelerator model with forward-looking expectations in the *Supplementary Appendix* (final example); in this model, there are recurrent fluctuations in GDP not driven by structural (news) shocks.[13]

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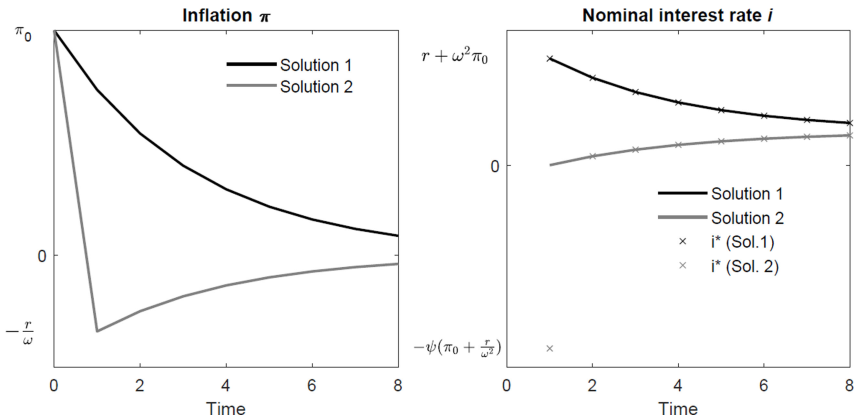

Figure 1. The two solutions when $\pi_0 > 0$.

A second important issue is when to *terminate* the guess-verify procedure. As shown by Holden (2023), if a square matrix M of impulse responses of the bounded variable to the "news shocks" has the property of being a P-matrix, there is a *unique solution* to the linear-complementarity problem for all initial conditions. In this case, it makes sense to terminate the guess-verify procedure when a solution is found (there can be only one). We therefore check if M is a P-matrix using our algorithm, and if so, we terminate the search procedure immediately after finding a solution. Note that matrix M is a P-matrix if and only if all its principal minors are positive; if not, there may be *multiple* solutions or no solution.

It is generally computationally-intensive to check if a matrix $M \in \mathbb{R}^{T\times T}$ is a P-matrix, and computation time increases sharply with the matrix dimension, T. In our algorithm, we rely on the recursive test for P-matrices in Tsatsomeros and Li (2000), for which the time complexity is exponential in T with base 2; however, to avoid running the test at times when this is not necessary, we have also built some pre-checks into our algorithm.[14]

2.4 Fisherian example

Following Holden (2023, Example 2) suppose that for all $t \geq 1$ our model consists of a Taylor-type rule subject to a zero lower bound and the Fisher equation:

$$i_t = \max\{0, r + \phi\pi_t - \psi\pi_{t-1}\} \tag{7}$$

$$i_t = r + E_t\pi_{t+1} \tag{8}$$

where $\phi - \psi > 1$, $\psi > 0$, $\pi_0 \in \mathbb{R}$ and $r > 0$ is a fixed real interest rate. To simplify presentation, we set $\phi = 2$. The results are not specific to this case.

There are two solutions to the model (7)–(8). The first solution is away from the bound in all periods and given by $\pi_t = \omega\pi_{t-1}$, $i_t = r + \omega\pi_t$ for all $t \geq 1$, where $\omega = 1 - \sqrt{1-\psi} \in (0, 1)$. This solution is stable (inflation converges to 0 and nominal rates to r), does not violate the lower bound in period 1 if $r + \phi\pi_1 - \psi\pi_0 \geq 0$ and is away from the bound for all $t > 1$ if $r + \phi\pi_t - \psi\pi_{t-1} > 0$; hence this solution exists for $\pi_0 \geq -\frac{r}{\omega^2}$.[15] The second solution has the constraint binding only in period 1, i.e. $i_1 = 0$ and $\pi_t = \omega\pi_{t-1}$, $i_t = r + \omega^2\pi_{t-1}$ for all $t > 1$. Note that $i_1 = 0$ implies that $\pi_2 = -r$ by (8), so $\pi_1 = -r/\omega$ and $\pi_t = \omega\pi_{t-1} = \omega^{t-2}(-r)$ for all $t > 1$ (deflation). The constraint binds in period 1 provided $r + \phi\pi_1 - \psi\pi_0 \leq 0$ and is escaped thereafter if $r + \phi\pi_t - \psi\pi_{t-1} > 0$ for all $t > 1$, so we again require $\pi_0 \geq -\frac{r}{\omega^2}$.[16] Hence, for $\pi_0 \geq -\frac{r}{\omega^2}$ both solutions exist, and for $\pi_0 < -\frac{r}{\omega^2}$ there is no solution.

Our Algorithm finds both these solutions (see *Supplementary Appendix*, Section 2). The two solutions are plotted in Figure 1, along with the shadow interest rate i_t^* in both cases. Note that Solution 1 has a positive shadow rate in all periods (which coincides with the actual interest rate); hence this solution is verified. By comparison, Solution 2 hits the bound in period 1 and has a negative shadow rate in this period (so the constraint binds); hence this solution is also verified. If we assign values to all the parameters (as in our codes), then numerical analysis using our algorithm indicates that the matrix M of impulse responses is *not* a P-matrix; note that this makes sense given the results above: we saw that there are multiple solutions (if initial inflation π_0 is high enough) or no solution (if π_0 is too low).

3. Simulating multiple equilibria

As argued by Holden (2023), multiple equilibria are a robust feature of otherwise-linear models with occasionally-binding constraints. Therefore, it is important that solution algorithms do not neglect multiplicity. In this section we provide our approach to simulating multiple equilibria; we therefore assume throughout this section that there are multiple solutions.

3.1 Determining an equilibrium path

Suppose $K \geq 2$ perfect foresight solutions are found using our Algorithm. To resolve the indeterminacy problem (incompleteness), we consider a "sunspot" approach rather than trying to "purge" solutions based on a selection criterion. We assume the researcher or policymaker has some "prior probabilities" $p_1, \ldots, p_K \in [0, 1]$, $\sum_{k=1}^{K} p_k = 1$, over the different perfect foresight solutions, where p_k is the probability that agents will coordinate expectations on equilibrium k at date 1 (the *first period* in which expectations are formed).

The simple first step is to draw a sunspot at date 1 that represents this coordination and will thereby ensure that a particular solution (i.e. perfect foresight path) is realized. We index the K perfect foresight solutions by $(x_t^k)_{t=1}^{\infty}$ for $k = 1, \ldots, K$, and let $u_1 \sim \mathcal{U}_{(0,1)}$ be a sunspot that is drawn from the uniform distribution on the interval (0, 1) at date $t = 1$. Given all this, we have the following rule for determining which equilibrium is realized given the sunspot u_1, which is analogous to how the values of categorical random variables, such as realizations of K-state Markov processes or outcomes of rolling a die, are determined.

Remark 1. *Suppose there are $K \geq 2$ perfect foresight solutions. Given probabilities $p_1, \ldots, p_K$ and a random draw $u_1 \sim \mathcal{U}_{(0,1)}$, the realized perfect foresight path at date 1 is*

$$(x_t)_{t=1}^{\infty} = \begin{cases} (x_t^1)_{t=1}^{\infty} & \text{if } u_1 \in (0, p_1] \\ (x_t^2)_{t=1}^{\infty} & \text{if } u_1 \in (p_1, p_1 + p_2] \\ \vdots & \\ (x_t^K)_{t=1}^{\infty} & \text{if } u_1 > p_1 + \ldots + p_{K-1} \end{cases} \tag{9}$$

i.e. for $u_1 \in \left(\sum_{k=0}^{k^*-1} p_k, \sum_{k=0}^{k^*} p_k\right]$, where $k^* \in \{1, \ldots, K\}$ and $p_0 := 0$, the unique (realized) perfect foresight solution is $(x_t)_{t=1}^{\infty} = (x_t^{k^*})_{t=1}^{\infty}$.

Remark 1 simply gives a way to choose a *realized* perfect foresight path when there are multiple candidates. It applies to any finite set of perfect foresight solutions and is flexible due to the free specification of probabilities. For instance, if the researcher thinks some solution(s) somewhat "unrealistic" they may attach low probability to those solution paths. On the other hand, a perfectly agnostic researcher would use "flat priors" $p_k = 1/K$ for all k.

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As noted, our Algorithm stores all (found) perfect foresight solutions. Given some specified probabilities $p_1, \ldots, p_K$, Remark 1 will then choose one solution as the realized equilibrium, akin to a lottery among perfect foresight paths (we give an example in Section 3.3). With this addition, our algorithm can be used to automate equilibrium determination (to some extent) while taking account of prior beliefs and their economic implications.[17]

Further, we will show that combining Remark 1 with some additional timing assumptions allows the computation of expected outcomes as a probability-weighted average of perfect foresight paths, such that expected welfare or other outcomes may be evaluated under the "veil of ignorance" – i.e. before the sunspot is realized. We do this in the next section.

3.2 Expected outcomes

Suppose now that *before* expectations are formed in period 1, there is some initial state, period 0, in which the initial conditions $x_0, (e_t)_{t=1}^{\infty}$ are *known*, but the sunspot u_1 and thus the expectations $x_1, x_2, \ldots,$ are *not*. To motivate this, note that if the sunspot u_1 were determined at date 0, multiplicity could *not* arise since expectations would *already have* coordinated on a particular equilibrium. Viewed this way, the sunspot u_1 coordinates agents' expectations on a particular equilibrium in period 1 and so "stands in" for some psychological process or "animal spirits" that are extraneous to the economy.[18] By contrast, the news shocks $(e_t)_{t=1}^{\infty}$ are structural and we take them as a *predetermined* aspect of the economic environment.

Thus, we think of date 0 as an initial position that is subject to the "veil of ignorance" in that the *expectations* of agents—an aspect of their *behavior*—are *un*known and *not* pinned down. Given this view, an expected path can be defined as follows.

Definition 2. *An expected path is a linear combination of perfect foresight solutions in which the weights are the probabilities of each solution,* $p_1, \ldots, p_K$.

Based on Definition 2, the expected path of the vector of endogenous variables x_t is:

$$(E_0[x_1], \quad E_0[x_2], \quad \ldots \quad E_0[x_s], \quad \ldots \ ) \tag{10}$$

where the expected values are

$$E_0[x_t] := \sum_{k=1}^{K} p_k x_t^k \tag{11}$$

i.e. a linear combination of the date-t points of the perfect foresight solutions $1, \ldots, K$.

Similarly, one may compute expected welfare, for example, based on a quadratic loss function or some other approximation to a social welfare function. If $W_k \in \mathbb{R}$ is the welfare associated with solution k, then analogous to (11) the expected welfare is

$$E_0[W] := \sum_{k=1}^{K} p_k W_k. \tag{12}$$

It should be emphasized that expressions such as (11) and (12) also provide a basis for robustness-type analysis in the sense that the impact of different "prior beliefs" can easily be studied and best and worst-case scenarios identified. We now give a simple example.

Example 1. *Consider again the Fisherian model with two solutions (Figure 1). A policymaker assigns probabilities* p_1 *to Solution 1 (slack) and* $p_2 = 1 - p_1$ *to Solution 2 (bind), and has an ad hoc loss function that penalizes squared deviations from a zero inflation target:*

$$L = \sum_{t=1}^{\infty} \beta^{t-1} \pi_t^2.$$

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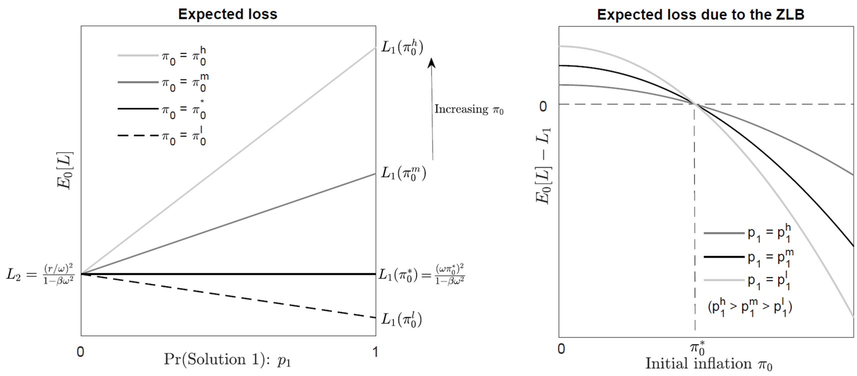

Figure 2. Expected loss $E_0[L]$ as p_1 is increased for various π_0 (left panel) and the expected loss due to the *zero lower bound* $E_0[L] - L_1$ as π_0 is increased (right panel). The initial values in the left panel satisfy $\pi_0^h > \pi_0^m > \pi_0^* > \pi_0^l$, where π_0^* is the positive initial inflation that makes the loss under Solution 1, $L_1(\pi_0)$, equal to the loss L_2 under Solution 2.

By (12), the expected loss is $E_0[L] = p_1L_1 + p_2L_2$, *where* $L_1 = \sum_{t=1}^{\infty} \beta^{t-1}(\pi_t^1)^2 = \frac{\omega^2}{1-\beta\omega^2}\pi_0^2$, $L_2 = \sum_{t=1}^{\infty} \beta^{t-1}(\pi_t^2)^2 = \frac{(r/\omega)^2}{1-\beta\omega^2}$, *and* π_t^1, π_t^2 *are the inflation solutions at date t. Hence,*

$$E_0[L] = \frac{p_1\omega^2\pi_0^2 + (1-p_1)(r/\omega)^2}{1-\beta\omega^2}$$

which is linear in p_1 *and quadratic in the initial inflation* π_0.

In Figure 2 (left panel), we plot the expected loss as the probability of Solution 1, p_1, *is increased from 0 to 1; we do this for several values of* $\pi_0 > 0$, *including the initial inflation* $\pi_0^* = r/\omega^2$ *for which the losses are equal, i.e.* $L_1 = L_2$.[19] *For inflation rates above* π_0^*, *the expected loss increases with* p_1, *since inflation declines geometrically from its initial value* π_0 *under the "good"solution (see Figure 1). Going in the other direction, reducing initial inflation below* π_0^* *makes inflation under Solution 1 "closer"to the zero inflation target, so loss* L_2 *is higher. For Solution 2, where the constraint binds in period 1, the loss* L_2 *is independent of the initial inflation* π_0, *as is clear from Figure 1 (and the equation above).*

Figure 2 (right panel) plots the expected loss relative to Solution 1, $E_0[L] - L_1$, *which can be interpreted as the extra loss attributable to the lower bound friction, which makes Solution 2 (binding constraint) a possible equilibrium path. The expected loss attributed to the lower bound falls as initial inflation is increased, since this raises the Solution 1 loss* L_1 *by more than it raises the expected loss* $E_0[L]$. *Increasing the probability of Solution 1 "flattens"the relationship between* $E_0[L] - L_1$ *and* π_0 *since the expected loss given the lower bound,* $E_0[L_1]$, *gets closer to* L_1 *the higher the probability attached to Solution 1 (right panel).*

The above exercise can be motivated by thinking of an asset like central bank digital currency that could potentially eliminate the lower bound. From a policy perspective, we would like to know ex ante whether it is worth providing such a currency (at some cost); this will depend partly on the expected benefit of eliminating the lower bound on nominal rates, shown in the right panel. While this is a simple example, the results are suggestive of the policy and robustness-type analysis possible using our expected outcomes approach.

3.3 Stochastic simulations

Lastly, we consider stochastic simulations with switching between multiple equilibria. For such simulations, we assume that the agents think they know all future shocks but are mistaken; as result, they have *im*perfect foresight and ignore risk, such that their expectations are not strictly rational, in contrast to the perfect foresight solutions studied thus far.

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We use this approach to construct stochastic simulations in which agents form expectations under certainty equivalence. Both Dynare (Adjemian et al. 2011) and OccBin (Guerrieri and Iacoviello, 2015) have built-in options for such simulations using an extended path method, which is widely used, for example, at policy institutions such as central banks.[20] While the extended path approach is typically applied to occasionally-binding constraint models with a *unique* solution (or assuming uniqueness), we provide an approach that allows simulations with *multiple equilibria*, which should speak to a wide audience.

Our approach draws on Remark 1, but it uses that approach to determine an equilibrium in every period t in which the model is simulated; note that this is necessary because while the entire solution path is known as of date 1 under perfect foresight, this is *not* true here since the actual (realized) path will generally differ from the one that agents expected. Hence, equilibrium determination (via a sunspot) arises every period in response to the new (unanticipated) initial conditions x_{t-1}, e_t with which agents are confronted.

To make this concrete, suppose that the shock vector e_t is drawn from some distribution. Then given e_t, x_{t-1} and agents' beliefs about future shocks, we can find the solution(s) for these initial conditions using our Algorithm, and one can be selected by a sunspot (akin to Remark 1). The same procedure is then repeated in period $t+1$, given x_t, e_{t+1} and the agents' expected future shocks. Provided a solution exists for all simulated t, we can construct a stochastic simulation path of desired length as follows.

1. Choose some systematic rule for assigning probabilities to different equilibria at each date, e.g. the "flat priors" approach, $p_k = \frac{1}{\text{no. of equilibria}}$ for each solution k.[21]
2. Given x_0, e_1 and expected shocks $e_2^a, \ldots, e_T^a$, use the Algorithm to find the solution paths $(x_t^k)_{t=1}^T$ for $k = 1, \ldots, K_1$, where K_1 is the number of solutions at date 1. Using the approach in Remark 1, select one of these solutions, $(x_t^{k_1^*})_{t=1}^T$, where $k_1^* \in \{1, \ldots, K_1\}$. Set $x_1 = x_1^{k_1^*}$ (our first simulated point) and move to period 2.
3. Draw vector e_2 from some distribution. Given x_1, e_2 and expected shocks $e_3^a, \ldots, e_{T+1}^a$, use the Algorithm to find the solution paths $(x_t^k)_{t=2}^{T+1}$ for $k = 1, \ldots, K_2$, where K_2 is the number of solutions in period 2. Use the approach in Remark 1 to select one of these solutions, $(x_t^{k_2^*})_{t=2}^{T+1}$, where $k_2^* \in \{1, \ldots, K_2\}$. Set $x_2 = x_2^{k_2^*}$ (second simulated point).
4. Repeat in periods 3, 4 etc. to get a simulation path of the desired length, say $(x_t)_{t=1}^{T^*}$.

We now show stochastic simulation in action using our running example (Fisherian model).

Example 2 (Ex. 1 cont'd). *We continue the Fisherian example (Section 2.4 and Example 1) but we add a shock* $e_t \in \mathbb{R}$ *in the Taylor-type rule, so* $i_t = \max\{0, r + \phi\pi_t - \psi\pi_{t-1} + e_t\}$ *as in Holden (2023, Section 2.3). We set parameter values:* $r = 0.01$, $\phi = 2$, $\psi = 0.93$, $\pi_0 = 0.02$, *along with date-1 anticipated shocks* $e_1 = e_2^a = -0.001$. *The probability of choosing Solution 1 (away from the zero bound) is set at* $p_1 = 0.95$, *so there is a 5% probability of choosing the Solution 2 that hits the bound. At dates* $t > 1$ *we solve for* i_t, π_t *conditional on the inherited state* π_{t-1} *and fresh draws for the monetary policy shocks* e_t, e_{t+1} *which are drawn from a normal distribution with a mean of zero and a standard deviation* σ_e *(see Figure 3).*

We found two solutions in all simulated periods; these are versions of the "high"and the "low"inflation solutions in Figure 1. To resolve the indeterminacy of two solutions, at each date t we drew a sunspot $u_t \sim \mathcal{U}_{(0,1)}$ *that selects either Solution 1 (away from bound) or Solution 2 (hits bound) in period t. Given our assumption that* $p_1 = 0.95$, *Solution 1 (Solution 2) is selected at date t if and only if* $u_t \in (0, 0.95]$ $(u_t > 0.95)$*; see Remark 1.*

In the upper panel of Figure 3, the standard deviation of the policy shock is very small to isolate the impact of the sunspot, i.e. switching between the two equilibria. Of five simulations, three hit the

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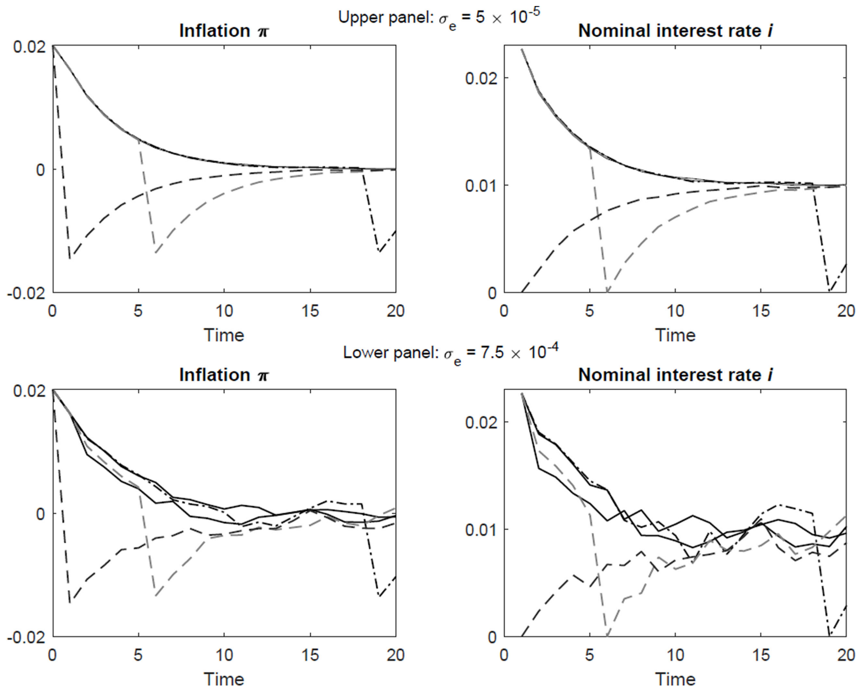

Figure 3. Five stochastic simulations: $p_1 = 0.95$ and initial values $\pi_0 = 0.02$, $e_1, e_2 = -0.001$.

zero lower bound in some period (see dashed lines), giving strong deflation (cf. Figure 1), in contrast to the solution paths that always remain away from the bound.

In the lower panel, the shock variance is large enough to make each individual stochastic simulation path discernible, but the main variations in inflation and interest rates come from switching between equilibria rather than from disturbances to the policy rule. Note that due to switching between equilibria, the average simulated values of inflation and nominal rates, in a long simulation, may differ somewhat from those at the terminal solution steady-state.

4. Policy application

We now consider an application to policy rules with multiple equilibria for some parameters. We make use of several concepts discussed so far (sunspots, expected outcomes, M matrix) and implementation details are discussed in the *Supplementary Appendix*. Since we restrict attention to perfect foresight solutions, we do not use stochastic simulation (Section 3.3).[22]

4.1 A New Keynesian model

We consider the New Keynesian model studied in Brendon et al. (2013). Besides the zero lower bound, the only other departure from the benchmark model is a policy response to the *change* in the output gap, similar to the "speed limit" policies considered by Walsh (2003):

$$i_t = \max\{\underline{i},\ i_t^*\} \tag{13}$$

$$i_t^* = \rho_i i_{t-1}^* + (1 - \rho_i)(\theta_\pi \pi_t + \theta_{\Delta y}(y_t - y_{t-1})) \tag{14}$$

$$y_t = E_t y_{t+1} - \frac{1}{\sigma}(i_t - E_t \pi_{t+1}) + e_t \tag{15}$$

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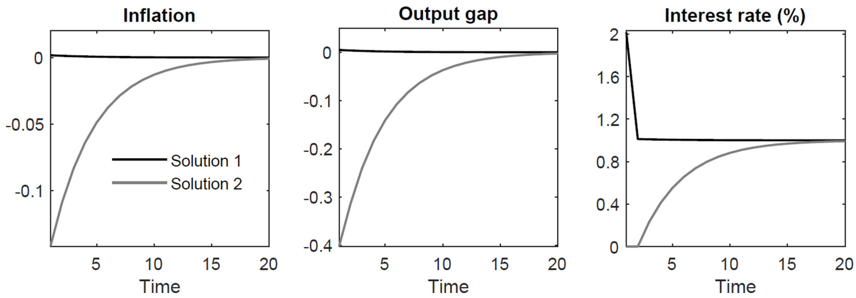

Figure 4. Multiple equilibria in the Brendon et al., model: $e_1 = 0.01$ and $i_0^* = y_0 = \rho_i = 0$.

$$\pi_t = \beta E_t \pi_{t+1} + \kappa y_t \tag{16}$$

where $\theta_\pi > 1$, $\beta \in (0, 1)$, $\theta_{\Delta y}, \kappa, \sigma > 0$, $\rho_i \in [0, 1)$, $\underline{i} = \beta - 1$ and all values of e_t are known.

We start by setting parameters at $\beta = 0.99$, $\sigma = 1$, $\rho_i = 0$ (no interest rate smoothing) and $\kappa = \frac{(1-0.85)(1-0.85\beta)}{0.85}(2 + \sigma)$ as in Brendon et al. (2013); additionally, we set $\theta_\pi = 1.5$ and $\theta_{\Delta y} = 1.6$ to initially replicate the exercise in Holden (2023, Appendix E). Starting at steady state, we consider a 1% demand shock at date 1 (i.e. $e_1 = 0.01$, $e_t = 0$ for $t > 1$) and search for solutions to model (13)–(16) using our algorithm. We found two perfect foresight solutions as expected and the solutions match the ones reported by Holden (see Figure 4).

There are two perfect foresight solutions in Figure 4: one where the lower bound is never hit and inflation and the output gap rise only marginally above their steady-state values; and a second solution where interest rates are at the lower bound in the first two periods and there is strong and persistent deflation and large negative output gaps (Figure 4, all panels). This "bad" solution is clearly inferior in terms of stabilization of inflation and output gaps and arises due to pessimistic *self-fulfilling* expectations: if agents expect low inflation, then the rise in the real rate lowers the output gap and inflation, validating the expectations.

Having a *growth-based* interest rate rule—for which the shadow rate responds to inflation and the *change* in the output gap—is important for the multiplicity result, as there is a unique perfect foresight solution if the shadow rate follows a Taylor rule with a response to the output gap in *levels* (Holden, 2023, Section 4.3). In addition, multiplicity occurs only when the response to the *change* in the output gap—or "speed limit"—is strong enough, as is clear from Figure 5 below. Such speed limit policies have been studied in the literature, with encouraging results from both theoretical and practical perspectives (Walsh, 2003; Orphanides, 2003; Yetman, 2006) and some empirical works suggest that central bank behavior is consistent with a speed limit rule (e.g. Mehra, 2002).[23]

Given the potential negative consequences of the interest rate rule (14) in this model, we consider some alternative policy rules, to see if they can restore uniqueness by eliminating the "bad" solution. Before doing so, we first confirm that multiplicity is a robust feature of this model by studying the properties of the M-matrix (discussed in Section 2.3 above).

In Figure 5, we show some parameter regions for which the M matrix of impulse responses is a P matrix and is *not* a P matrix. Recall that if the M matrix is a P-matrix, then there is a unique solution for all initial conditions. We set $T = 16$ and plot the regions for which the M matrix is a P-matrix (determinacy region, white), and is not a P-matrix (black region). We consider different combinations of the response coefficients θ_π, $\theta_{\Delta y}$ in the interest rate rule and in each panel we vary the inverse elasticity of intertemporal substitution, σ.

Figure 5 shows there is a unique solution if the response to the change in the output gap, $\theta_{\Delta y}$, is not too strong relative to the inflation response θ_π (see the white regions). In the first panel, which uses the baseline value of $\sigma = 1$, we see that M is a P-matrix only if the response to the change in the output gap is *smaller* than the response coefficient on inflation. Note that the parameter

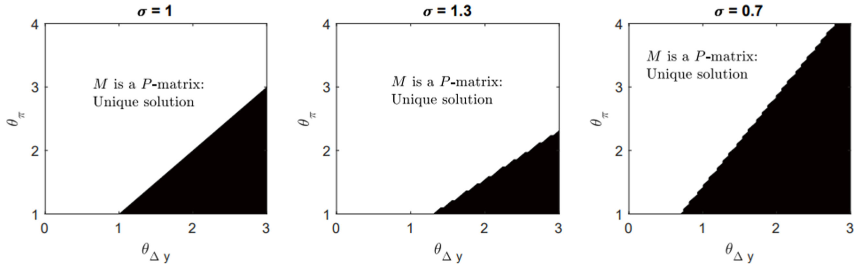

Figure 5. Regions in which M is not a P-matrix (black) when $T = 16$ (Case: $\rho_i = 0$).

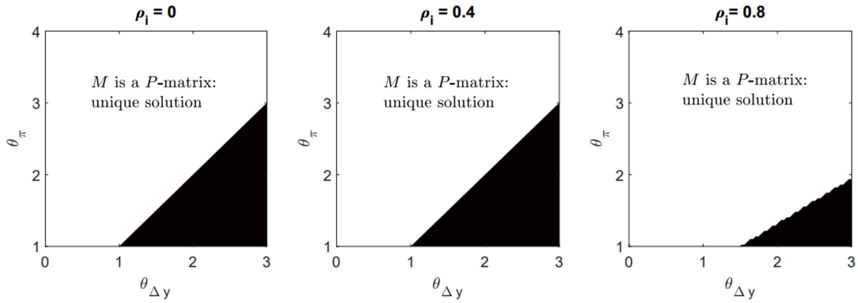

Figure 6. Regions in which M is not a P-matrix (black) for $T = 16$ and various ρ_i.

combination used in Figure 4 ($\theta_\pi = 1.5, \theta_{\Delta y} = 1.6$), where we see two solutions, lies in the black region as expected. In fact, Brendon et al. (2013, Proposition 1) show that the model (13)–(16) has multiple perfect foresight equilibria if and only if $\theta_{\Delta y} > \sigma\theta_\pi$, and Figure 5 is consistent with this conclusion.

In summary, multiplicity is a robust outcome and this raises the question of whether alternative monetary policies could restore uniqueness by eliminating the bad solution. We investigate this below while retaining the "speed limit" aspect of the policy rule, which may have theoretical and practical advantages as argued by Walsh (2003) and others. We start with interest rate smoothing before turning to unconventional monetary policy rules.

4.2 Interest rate smoothing

We first ask whether policymakers could achieve a better outcome by smoothing the shadow interest rate in (14) by setting $\rho_i \in (0, 1)$. We started out by checking the regions where the M matrix is a P-matrix for $T = 16$, similar to Figure 5 but with three ρ_i values and $\sigma = 1$.

The P-matrix regions in Figure 6 indicate that the determinacy region expands once the interest rate smoothing parameter ρ_i is large enough. For $\rho_i = 0.8$, the parameter combination $\theta_\pi = 1.5$, $\theta_{\Delta y} = 1.6$ is now in the determinacy region (white), so there is a unique solution at these parameter values (plotted in Figure 7 below). However, indeterminacy remains if the response to the speed limit is strong enough (Figure 6, right) and as noted by Holden (2023, Appendix E), the P-matrix regions under interest rate smoothing tend to those in the model *without* any smoothing as T is increased, such that *multiplicity remains a widespread problem* and there are both "good" and "bad" equilibria.[24]

In Figure 7, we present a numerical simulation. We set $\theta_\pi = 1.5$, $\theta_{\Delta y} = 1.6$ and $e_1 = 0.01$ as in the baseline simulation in Figure 4; the only difference is that the interest rate smoothing parameter is set at either $\rho_i = 0.40$ (weak smoothing) or a high value $\rho_i = 0.80$ (strong smoothing). With moderate interest rate smoothing ($\rho_i = 0.40$) there are two solutions and the "bad" solution is

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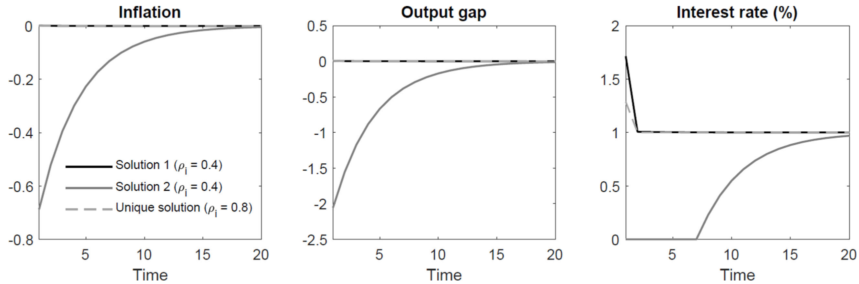

Figure 7. Perfect foresight solutions with interest rate smoothing when $e_1 = 0.01$, $i_0^* = y_0 = 0$, $\sigma = 1$ and $\theta_\pi = 1.5$, $\theta_{\Delta y} = 1.6$: two different values of ρ_i ($\rho_i = 0.4$, solid; $\rho_i = 0.8$, dashed).

exacerbated relative to Figure 4: inflation and the output gap fall by around *5 times* as much initially and interest rates spend *7 periods* at the lower bound, rather than 2. Intuitively, it is easier to induce a lengthy spell at the lower bound when the shadow rate is persistent, provided ρ_i is not large enough to eliminate the bad solution. In the case of $\rho_i = 0.8$, there is a unique solution that is away from the bound in all periods (dashed gray line)—though as Figure 6 shows, uniqueness is *not* a general result.

For the parameters $\theta_\pi = 1.5$, $\theta_{\Delta y} = 1.6$, the determinacy result seems to hold for smoothing of around $\rho_i = 0.8$ or higher.[25] Some intuition can be gained by scaling θ_π, $\theta_{\Delta y}$ in Eq. (14) by $\frac{1}{1-\rho_i}$ and letting $\rho_i \to 1$. In this case, the shadow interest rate tends to $\Delta i_t^* = \theta_\pi \pi_t + \theta_{\Delta y} \Delta y_t$, which is consistent with any rule of the form $i_t^* = constant + \theta_\pi p_t + \theta_{\Delta y} y_t$, where $p_t = \pi_t + p_{t-1}$ is the log price level. The latter is a price-level targeting rule *without* a speed limit term. Given that Holden (2023, Appendix E) finds that a *levels* rule restores determinacy, it is intuitive that sufficiently high values of ρ_i lead to the same conclusion.

In short, interest rate smoothing does *not*, in general, prevent the occurrence of multiple equilibria, and when the "bad" solution is present we see that a smoothing rule *worsens* destabilization of inflation and the output gap somewhat. At the same time, however, we saw that highly inertial interest rate rules may eliminate the bad solution.

4.3 Forward guidance

Since interest rate smoothing is *not* a robust solution to indeterminacy and destabilization, we now consider *forward guidance*. We are motivated here by the observation that forward guidance promises an extended period of expansionary monetary policy, such that self-fulfilling pessimistic expectations of inflation and the output gap—as under the "bad" solution—might not be rational. This unconventional policy is *not* studied by Holden (2023).

To model forward guidance, we consider expansionary "news shocks" to the shadow interest rate, such that (14) becomes

$$i_t^* = \rho_i i_{t-1}^* + (1 - \rho_i)(\theta_\pi \pi_t + \theta_{\Delta y}(y_t - y_{t-1})) + e_t^{FG} \tag{17}$$

where $e_t^{FG} \leq 0$ is a forward guidance "news shock" and $e_1^{FG} = 0$.

We assume that $e_t^{FG} < 0$ only in periods when forward guidance is in place. Our question is whether such a policy eliminates the "bad" solution or better stabilizes inflation and the output gap. Table 1 records the percentage of initial conditions (out of 800) for which both a "good" and "bad" solution exist at different forward guidance *horizons*; here, different initial conditions refers to varying the size of the news shocks e_t^{FG} within a range and we consider 800 such variations (i.e. cases) for each forward guidance horizon; see the Table 1 notes. By forward guidance *horizon* we mean the number of periods in which there are "expansionary" news shocks $e_t^{FG} < 0$ and we

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Table 1. Determinacy at various forward guidance horizons (800 cases, $\rho_i = 0$)

FG Horizon	Unique	Indeterminacy	Time at Bound (Max,Min)
Period 2 only	0%	100%	Mean: 2 periods (2,2)
Periods 2–3	0%	100%	Mean: 3 periods (3,3)
Periods 2–4	0%	100%	Mean: 4 periods (4,4)
Periods 2–5	0%	100%	Mean: 5 periods (5,5)
Periods 2–6	0%	100%	Mean: 3.6 periods (6,1)

Note: Forward guidance is modeled via news shocks $e_t^{FG} = -0.01 - Uniform(0, 0.01)$ which start in period 2 and last to the stated date. Each row: 800 cases of different news shocks.

consider only *consecutive* periods of such shocks. All other parameters and initial conditions are kept fixed at the baseline values used in Figure 4.

The results in Table 1 show that forward guidance is *not* effective in eliminating indeterminacy. Up to a 5-period forward guidance horizon (rows 1–5), *all* 800 sets of news shocks led to multiple solutions (i.e. 100% of cases).[26] If the forward guidance horizon is 5 periods, the length of spells at the zero lower bound is no longer the same across all 800 cases of news shocks (see Table 1, final column) and there can be multiple spells at the lower bound (see Figure 8, bottom right for an example); however, neither a longer horizon nor interest rate smoothing seem to alter the conclusion of widespread multiplicity.

Figure 8 shows that the “good” solutions under forward guidance have inflation and output “overstimulated” (upper panel), while the “bad” solutions have very poor stabilization relative to the original policy given by the solid black line ("Baseline," lower panel). Importantly, stabilization under the “bad” solutions is worse when forward guidance is present and stabilization of inflation and the output gap deteriorates as forward guidance becomes more aggressive (i.e. when it delivers a similar “dose” for a larger number of periods). Hence, while some previous works suggest that forward guidance could be stabilizing at the zero lower bound (e.g. Eggertsson et al. 2021), our results point in the opposite direction.

4.4 Price-level targeting

Lastly, we consider price-level targeting. A key motivation is work showing that price-level targeting interest rate rules can mitigate or resolve indeterminacy in New Keynesian models (Giannoni, 2014; Holden, 2023). We ask whether a response to the price level is *sufficient* to restore determinacy when retaining the “speed limit” term in the interest rate rule. We retain the speed limit because policymakers may find such policies attractive (Walsh, 2003), but inadvertently bring about multiplicity (Figures 4–8); this is a problem we want to solve.

Accordingly, we assume the shadow interest rate under price-level targeting is

$$i_t^* = \rho_i i_{t-1}^* + (1 - \rho_i)\left(\theta_p p_t + \theta_{\Delta y}(y_t - y_{t-1})\right) \tag{18}$$

where $\theta_p > 0$ and $p_t := \pi_t + p_{t-1}$ is the log price level.

Differently from the rule in Holden (2023, Appendix E), the shadow interest rate still responds to the *change* in output gap. We now consider implications for determinacy.

Figure 9 plots the regions in which M is a P-matrix, again for $T = 16$ as in Figure 5. The indeterminacy region shrinks substantially, but we see that indeterminacy is *not* ruled out when a strong response to the “speed limit” is combined with a relatively weak response to the price level (black regions). Thus, only a sufficiently strong interest rate response to the price level—a large enough

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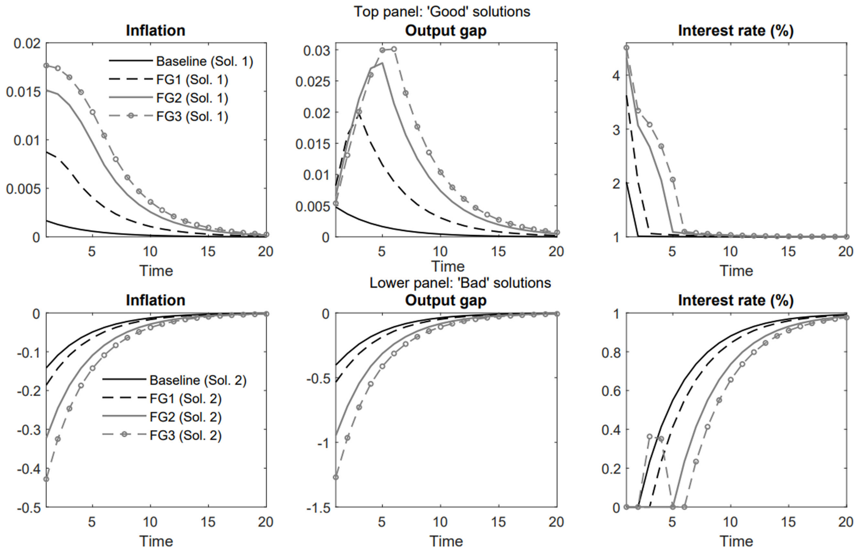

Figure 8. "Good" and "bad" solutions with forward guidance: $e_1 = 0.01$, $i_0^* = y_0 = 0$, $\theta_\pi = 1.5$, $\theta_{\Delta y} = 1.6$, $\rho_i = 0$, $\sigma = 1$. Forward guidance news shocks $e_t^{FG} = -0.015$ for 2 periods (FG1), 4 periods (FG2), 5 periods (FG3). Start: $t = 2$ and $e_1^{FG} = 0$ in all cases. Baseline: no FG.

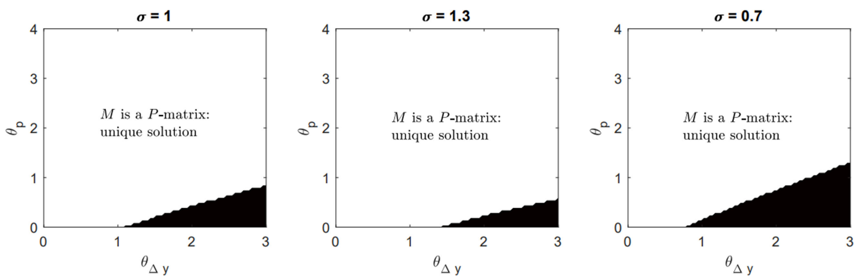

Figure 9. Regions in which M is not a P-matrix (black): price-level targeting when $T = 16$. Note that θ_p is the response coefficient on the log price level and $\rho_i = 0$.

θ_p in (18)—ensures determinacy. Intuitively, this says that a long-lasting, aggressive expansionary policy to restore the price level to target is sufficient to rule out the "bad" equilibrium based on pessimistic expectations. Perfect foresight simulations suggest that the "good solution" under price-level targeting (with the nominal rate away from the bound) has good performance in terms of stabilization, and allowing interest rate smoothing in the rule reinforces this conclusion (see *Supplementary Appendix*, Section 3.4).

Interestingly, if the reaction to the price level θ_p is small enough for *both* solutions to exist, then a very weak response to the price level works well from a stabilization perspective. In this case, the "bad" solution under price-level targeting looks somewhat "better" than with the inflation targeting, interest rate smoothing or forward guidance rules (see Figure 10). We see that inflation and the output gap drop far less initially (i.e. in period 1), and these variables and interest rates then oscillate around the "good" solution, but within quite a narrow range and deviations "die out" quite quickly. It should be noted, however, that not all "bad" solutions under price-level targeting are "tamer" than for the original rule.[27]

Table 2. Welfare losses and policy rules: good and bad solutions ($\lambda = 0.1$)

Policy Rule	Loss L_1 (good)	Loss L_2 (bad)
IT1	1	7,256
IT2	0.7	171,600
FG1	31.1	12,508
FG2	107.7	37,654
PLT1	0.3	-
PLT2	3.5	384.7

Note: IT1 is the baseline case in Figure 4. IT2 adds interest rate smoothing with $\rho_i = 0.4$ (Figure 7). FG1 and FG2 shown and described in Figure 8. PLT1 (PLT2) sets $\theta_p = 1.5$ ($\theta_p = 0.015$, see Figure 10). Reported losses are computed as the ratio of the loss relative to the 'good' loss L_1 for rule IT1.

Notably, our conclusions on price-level targeting are less positive than in Holden (2023), where analytical and numerical results are used to show that a price-level targeting rule ensures determinacy in a range of New Keynesian models. What our results highlight is that the assumption of a response to the *level* of the output gap, rather than its *first-difference*, is crucial. In the present model, where the "speed limit" $y_t - y_{t-1}$ enters the interest rate rule, determinacy is *not* guaranteed but instead requires a *sufficiently strong* response to the price level *relative to* the response to the speed limit, as shown in Figure 9. A particular example of multiplicity when the response to the price level is "too weak" can be seen in Figure 10.

4.5 Welfare analysis

Finally, we consider some welfare implications of multiplicity, an issue *not* considered in Holden (2023). We make use here of the "expected outcomes" approach in Section 3.2. In the above model, the microfounded social loss function has the form:

$$L = \sum_{t=1}^{\infty} \beta^{t-1}(\pi_t^2 + \lambda y_t^2) \tag{19}$$

where $\lambda > 0$ is the relative weight on output gap variations.[28]

As we have seen, there are two perfect foresight solutions for some parameter values. In Table 2, we report the losses under each policy rule based on (19), for both the "good" solution (Solution 1) and the "bad" solution (Solution 2, when it exists), i.e.

$$L_k = \sum_{t=1}^{\infty} \beta^{t-1}(\pi_{k,t}^2 + \lambda y_{k,t}^2), \qquad k \in \{1, 2\} \tag{20}$$

where $k = 1$ denotes the good solution and $k = 2$ denotes the bad solution.

For each type of policy rule we consider two parameterizations and other parameters and initial conditions are set at their baseline values as in the exercises in Figures 4, 7, 8, 10. We see that the price-level targeting rule PLT1 ($\theta_p = 1.5$) gives the lowest loss among the rules shown and restores determinacy (penultimate row); it also outperforms an interest rate smoothing rule with $\rho_i = 0.8$ (Figure 7) or higher.[29] Even when the price level response is very small ($\theta_p = 0.015$, Figure 10)—such that multiple solutions arise—the loss under the good solution is much smaller than under the forward guidance rules (at 3.5 times the loss under the *baseline* "good solution"), while the bad

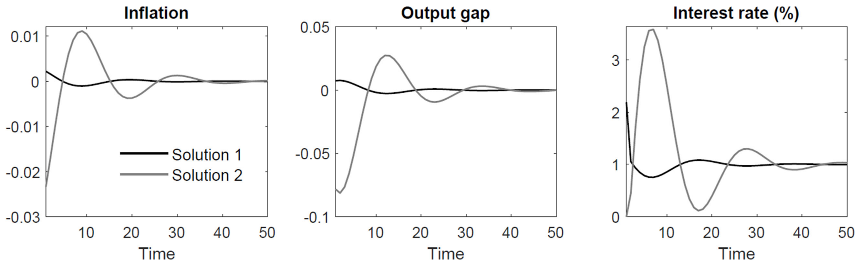

Figure 10. "Good" and "bad" solutions under price-level targeting for $e_1 = 0.01$, $i_0^* = y_0 = 0$ and a "weak" response to the price level $\theta_p = 0.015$ when $\sigma = 1$, $\theta_{\Delta y} = 1.6$ and $\rho_i = 0$.

solution imposes a relatively small welfare loss compared to the alternatives. These main conclusions are not sensitive to the relative weight λ on output gap deviations in the loss function (which is set at 0.1 in Table 2).[30]

We now consider *expected* social welfare losses in the above model. To do so, we follow the "expected outcomes" approach of Section 3.2 that computes a weighted average in which the weights are the probabilities that expectations will coordinate on each solution. The latter can be viewed as an expected outcome from some initial position or state (period 0) in which it is not yet known which *perfect foresight* solution agents' expectations will coordinate on. Therefore, we can think of this welfare exercise as an assessment made by policymakers (or researchers) who are viewing the economy from under the "veil of ignorance."

We stick with the same policy rule (and parameter values) as in Table 2. Given (20) and a probability p_1 of Solution 1, the expected loss under a given rule is

$$E_0[L] = p_1 L_1 + (1 - p_1) L_2 \tag{21}$$

where $L_1 = L_2$ in the case of rules for which there is a unique solution.

In Figure 11, we plot the expected welfare losses under each rule. While the price-level targeting rule PLT1 ($\theta_p = 1.5$) performs best as expected, there is no clear ranking among the other rules because the expected losses are sensitive to the probability p_1 of Solution 1 (both panels). The "weak response" price-level targeting rule PLT2 ($\theta_p = 0.015$) is ranked second for all values of p_1 up to approx. 0.9998 (right panel), but is then outperformed by the inflation targeting rules IT1 (no smoothing), and as the probability p_1 approaches 1, the inflation targeting rule IT2 (smoothing, $\rho_i = 0.4$) outperforms both the PLT2 rule and IT1, though this requires that the bad solution is a *near-zero* probability event.

Overall, the performance of price-level targeting is quite robust compared to the other policies for which losses are very sensitive to the probability that agents coordinate expectations on the "good" solution, 1. If the "bad equilibria" have non-trivial probability, price-level targeting looks highly attractive; however, if the such equilibria are considered extremely unlikely, then the potential benefits of price-level targeting look smaller and might not be convincing to policymakers. Thus, we see the potential value of assessing robustness of expected welfare to the probability of the good solution, as in Figure 11.

Finally, a word of caution is in order. It should be noted that since the "good" and "bad" solutions differ under each policy rule, the probabilities of each solution could also differ in these cases; for example, the "tame" bad solution under price-level targeting with $\theta_p = 0.015$ (Figure 10) might be considered more plausible than the highly destabilizing bad solution under inflation targeting with moderate interest rate smoothing (Figure 7, $\rho_i = 0.4$). If so, then an analysis like that in Figure 11 is still useful, but with the caveat that the expected welfare losses under each rule should be computed *conditional* on the different probabilities of each solution that the researcher or policymaker has in mind.[31]

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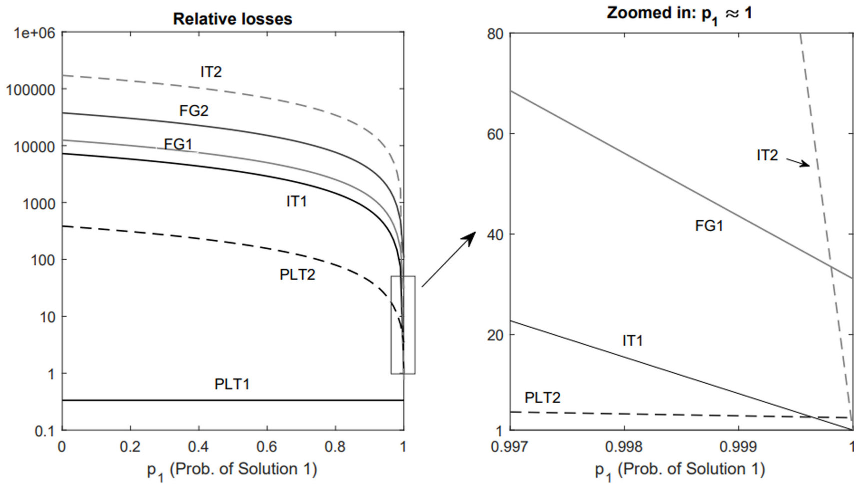

Figure 11. Expected welfare losses as the probability of Solution 1, p_1, is varied. The different interest rate rules are the same ones as in Table 2. The left panel uses a log scale.

5. Conclusion

In this paper we have extended the guess-verify algorithm in Guerrieri and Iacoviello (2015) to detect and simulate multiple *perfect foresight* equilibria of otherwise-linear dynamic models with news shocks. We showed how to compute expected paths in models with multiple equilibria using a "prior probabilities" approach, and also how to run stochastic simulations in which there is switching between equilibria on the simulated path.

We illustrated our algorithm—and the above extensions—using a simple Fisherian model with "high" and "low" inflation solutions for a range of initial values. We also presented a *policy application* based on a New Keynesian model with a zero lower bound, a policy response to the *change* in the output gap (or "speed limit") rather than its level, and multiple equilibria for some parameter values. One of these equilibria is a "bad" solution for which self-fulfilling pessimistic expectations drive down inflation and the output gap, and interest rates spend some time at the lower bound. Multiplicity arises for a wide range of parameter values with an inflation targeting rule, and rules with interest rate smoothing or forward guidance do *not* provide robust solutions to multiplicity and existence of a bad solution.

However, replacing an inflation target in the interest rate rule with a price-level target shrinks the indeterminacy region substantially. Our results suggest a simple rule-of-thumb: a strong enough response of interest rates to the price level avoids indeterminacy by eliminating the bad solution. Further, a price-level targeting rule is quite robust: when the "bad solution" does exist, it is not highly deflationary as under the other policies provided the price level response is small enough. Price-level targeting also performs well against the alternatives in terms of welfare, including a measure of expected welfare under the veil of ignorance.

The above results highlight some uses of our algorithm, such as policy analysis and simulation, and suggest some interesting avenues for future research.

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Supplementary material. To view supplementary material for this article, please visit https://doi.org/10.1017/S1365100525000021

Notes

1 In particular, news shocks enter via a solution matrix of "intercepts" which, conveniently, is recursive.
2 See, for example, the papers by Walsh (2003) and Orphanides (2003) and the references therein.
3 We discuss multiple occasionally-binding constraints in Section 6 of the Supplementary Appendix.
4 For example, with a simple borrowing constraint $b_t = \min\{\bar{b}, b_t^*\}$, we have $-b_t = \max\{-\bar{b}, -b_t^*\}$, so we can replace b_t with $-x_{1,t}$, where $x_{1,t} = \max\{\underline{x}_1, x_{1,t}^*\}$, with $\underline{x}_1 = -\bar{b}$ and $x_{1,t}^* = -b_t^*$.
5 For models with the constraint binding at steady-state, see Section 5 of the *Supplementary Appendix*.
6 The solution matrices $\Omega_t, \Gamma_t, \Psi_t$ are derived for exogenous structural change in Hatcher (2022) and differ slightly relative to those in the backward-forward algorithm of Kulish and Pagan (2017); in Guerrieri and Iacoviello (2015) all future news is *zero*. For further details, see the *Supplementary Appendix*, Section 4.
7 Farmer et al. (2015) attribute the incompleteness argument to McCallum (1983). A recent re-statement of the point (in the context of dynamic lower-bound models) is given by Ascari and Mavroeidis (2022, p. 1).
8 Our interpretation of "veil of ignorance" is in the spirit of Rawls' "original position." In particular, outcomes are viewed by policymakers or researchers at some initial date before the start of calendar time.
9 Adjemian and Juillard (2013) provide a stochastic extended path approach that takes risk into account; however, this approach is computationally intensive for DSGE models with many shocks.
10 As our algorithm uses guess-verify, it can find a *finite* number of solutions, but not infinitely many.
11 If the invertibility conditions are not satisfied, computing a pseudo-inverse, as in Chen et al. (2012), would arbitrarily select a solution path. We do not follow this approach. Codes at: https://github.com/MCHatcher.
12 Our code enumerates triple spells that end in a spell of length l, and allows users to run a loop over l.
13 13. Further, we find some solutions with multiple spells at the bound in our policy application in Section 4; see, in particular, Figure 8 (bottom right, forward guidance) and Figure 4 in the *Supplementary Appendix*.
14 E.g. M is a P-matrix if $M + M'$ is positive definite, and is *not* if any diagonal entry is non-positive.
15 Here, we use $\phi\omega - \psi = \omega^2$ and $r + \phi\pi_t - \psi\pi_{t-1} > 0$ for all $t > 1$ when $\pi_0 \geq -\frac{r}{\omega^2}$ (since $\omega \in (0, 1)$).
16 16. Note that when $i_1 = 0$ (as under Solution 2) we have $i_t = r + \omega^2\pi_{t-1} = [1 - (\omega)^{t-1}]r > 0$ for all $t > 1$.
17 We say "automate. . .(to some extent)" as researchers may find it beneficial to first view the solution paths and then assign probabilities. We return to this point below when discussing stochastic simulation.
18 For an interpretation of sunspots in terms of animal spirits, see e.g. Farmer and Guo (1994).
19 In general, there are two initial inflation rates that make the losses L_1 (slack) and L_2 (bind) equal: $\pi_0^* = \pm\frac{r}{\omega^2}$. However, we confine our attention to positive initial inflation in this example.
20 Occbin allows users to find a perfect foresight solution from date 1 onwards under the assumption of zero future shocks, giving an initial solution x_1. The process is then repeated in period 2 (and later periods) but with shocks e_t drawn at each t; this is implemented via the "stoch simul" option in Dynare. An application that uses an extended path approach (in a non-linear setting) is Christiano et al. (2015, Section IV.D).
21 In general, assigning probabilities in stochastic simulations is not easy as the number (and nature) of equilibria is not known *a priori*. Researchers may either "find solutions, then assign probabilities" or follow simple rules such as "flat priors" or "assign low (or high) probability to solutions where the bound is hit."
22 See https://github.com/MCHatcher/OBC-multiple-equilibria for the replication codes and other files.
23 Walsh (2003) and Yetman (2006) highlight some theoretical advantages of speed-limit policies in New Keynesian models, while Orphanides (2003) emphasizes that real-time measurement error is substantial for the output gap in *levels* but is somewhat smaller for the *first-difference* of the output gap (as in rule (14)).
24 Intuitively, persistence in the shadow interest rate makes it harder to induce *short-lived* spells at the lower bound, but increasing T amounts to a relaxation of this requirement by allowing longer spells.
25 The computational burden of checking whether M is a P-matrix for very large T means that our results are suggestive rather than conclusive. As a sanity check, we also studied some individual perfect foresight simulations for uniqueness (with affirmative results) for values of T up to 100 and several $\rho_i \geq 0.8$.
26 Note that "Unique: 0%, Indeterminacy: 100%" refers to the percentage of outcomes for 800 particular sets of initial conditions which differ only in terms of the magnitude of the forward guidance news shocks.
27 For several additional examples, see Section 3.4 of the *Supplementary Appendix*. For instance, for $\theta_p = 0.2$ there are multiple solutions, and the "bad" solution has worse destabilization than under the original rule. The relative welfare loss comes out at $L_2 = 17,593$—cf. Table 2 below—while the "good" loss is $L_1 = 1.2$.
28 See Walsh (2017, Ch. 8) for a derivation and Vestin (2006) for an application. The target output gap is zero when fiscal policy offsets steady-state distortions. The weight λ depends e.g. on κ, but the elasticity of substitution of differentiated goods can be set to increase or decrease λ while holding everything else fixed.
29 If Rule IT2 instead had $\rho_i = 0.8$, there is a unique solution (see Figure 7) but the loss L_1 is larger than under PLT1 at around 0.6. Higher ρ_i do *not* seem to change the conclusion that the loss is larger.
30 We set a relatively small value for λ because typical parameter values imply a rather low weight on output gap variations in the loss function, as is standard in the New Keynesian literature.
31 In particular, the expected loss in (21) would be computed for each policy at different probabilities p_1.

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