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ON THE POINTS OF INFLECTION OF BESSEL FUNCTIONS OF POSITIVE ORDER, II

Dedicated to the 75th birthday of Professor L. Lorch.

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1. **Introduction.** Let $j_{\nu,1}, j_{\nu,2}, \ldots$ denote the positive zeros of the Bessel function $J_\nu(x)$, and similarly, let $j'_{\nu,1}, j'_{\nu,2}, \ldots$ denote the positive zeros of $J'_\nu(x)$, which are the positive critical points of $J_\nu(x)$. It is well-known that when ν is positive, both $j_{\nu,k}$ and $j'_{\nu,k}$ are increasing functions of ν; see, e.g., [12, pp. 246 and 248]. Recently, Lorch and Szego [6] have attempted to show that the same is true for the positive zeros $j''_{\nu,1}, j''_{\nu,2}, \ldots$ of $J''_\nu(x)$, which are the positive inflection points of $J_\nu(x)$. They have succeeded in proving that this statement holds for $k = 1$, but for $k = 2, 3, \ldots$, they have proved only that it is true when $0 < \nu \leq 3838$. Their method is based on an integral representation for $dj''_{\nu,k}/d\nu$, and they have shown that the monotonicity of $j''_{\nu,k}$ is determined by the sign of

$$G(x) = \int_0^x \frac{J_\nu^2(t)}{t}\,dt - J_\nu^2(x) \tag{1.1}$$

when $x = j''_{\nu,k}$.

The purpose of this paper is to demonstrate that $G(j''_{\nu,k}) > 0$ for $\nu \geq 10$ and $k = 2, 3, \ldots$. This, together with the result obtained by Lorch and Szego, will establish the fact that $j''_{\nu,k}$ increases in the entire interval $0 < \nu < \infty$ for $k = 1, 2, \ldots$. Our method is based on asymptotic approximations with delicate error estimates. We first prove that for $\nu \geq 10$,

$$0 < J_\nu^2(j''_{\nu,k}) \leq \frac{\mu_k}{\nu^2}, \qquad k = 2, 3, \ldots, \tag{1.2}$$

where $\{\mu_k\}$ is a decreasing sequence. From (1.1) and (1.2), it is evident that for our purpose, it suffices to show

$$\int_0^{j''_{\nu,2}} \frac{J_\nu^2(t)}{t}\,dt - \frac{\mu_2}{\nu^2} > 0 \qquad \text{for } \nu \geq 10. \tag{1.3}$$

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Next we derive an asymptotic expansion for the integral

$$I(\lambda) = \int_0^\infty f(t) Ai^2(-\lambda t)\, dt \tag{1.4}$$

complete with error bounds, where the large positive parameter is $\lambda = \nu^{2/3}$ and $f(t)$ is a C^∞-function in $0 < t < \infty$. Finally, we consider the function

$$F(x) = \int_x^\infty \frac{J_\nu^2(t)}{t}\, dt, \tag{1.5}$$

and we use the result for $I(\lambda)$ to obtain the asymptotic approximation

$$F(j''_{\nu,2}) = \frac{1}{2\nu} - \frac{0.813}{\nu^{4/3}} + \frac{\varepsilon(\nu)}{\nu^2}, \tag{1.6}$$

where $|\varepsilon(\nu)| \leq 2.08$ for $\nu \geq 10$. Since $\mu_2 = 0.056$ and

$$\int_0^\infty \frac{J_\nu^2(t)}{t}\, dt = \frac{1}{2\nu}, \tag{1.7}$$

the validity of (1.3) follows immediately. Here and throughout the paper, the last significant figure in decimal numbers is the result of rounding to the nearest digit except for numbers in inequalities, which are rounded to obtain the weakest inequality.

2. **Some results of Olver.** It is well-known that the Bessel function $J_\nu(x)$ has the uniform asymptotic expansion

$$\begin{aligned} J_\nu(\nu x) \sim \frac{\varphi(\zeta)}{\nu^{1/3}} \Bigg\{ & Ai(\nu^{2/3}\zeta) \left[1 + \frac{A_1(\zeta)}{\nu^2} + \frac{A_2(\zeta)}{\nu^4} + \cdots \right] \\ & + \frac{Ai'(\nu^{2/3}\zeta)}{\nu^{4/3}} \left[B_0(\zeta) + \frac{B_1(\zeta)}{\nu^2} + \cdots \right] \Bigg\}, \end{aligned} \tag{2.1}$$

valid when $\nu > 0$ and $x > 0$, where ζ and x are related in a one-to-one manner by the equations

$$\begin{aligned} \zeta &= \left\{ \frac{3}{2} \int_x^1 \frac{(1-x^2)^{1/2}}{x}\, dx \right\}^{2/3} \\ &= \left\{ \frac{3}{2} \ln \frac{1+(1-x^2)^{1/2}}{x} - \frac{3}{2}(1-x^2)^{1/2} \right\}^{2/3}, \qquad 0 < x \leqq 1, \end{aligned} \tag{2.2}$$

$$\begin{aligned} \zeta &= -\left\{ \frac{3}{2} \int_1^x \frac{(x^2-1)^{1/2}}{x}\, dx \right\}^{2/3} \\ &= -\left\{ \frac{3}{2}(x^2-1)^{1/2} - \frac{3}{2} \sec^{-1} x \right\}^{2/3}, \qquad x \geqq 1, \end{aligned} \tag{2.3}$$

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and where

$$\varphi(\zeta) = \left(\frac{4\zeta}{1-x^2}\right)^{1/4}. \tag{2.4}$$

The coefficients $A_s(\zeta)$ and $B_s(\zeta)$ satisfy a set of recurrence relations, and are holomorphic functions in a region containing the real axis. This result is due to Olver, and can be found in [8] and [12, Chapter 11]. Precise bounds for the remainder terms in (2.1) have also been constructed by him; see [9] and [11]. To state this result, we first recall from [9] the modulus function $M(x)$ and the weight function $E(x)$ associated with the Airy functions:

$$\begin{aligned} E(x) &= \exp(\tfrac{2}{3}x^{3/2}) \qquad x > 0, \\ E(x) &= 1, \qquad x \leqq 0; \qquad E^{-1}(x) = 1/E(x), \end{aligned} \tag{2.5}$$

$$M(x) = \{E^2(x)Ai^2(x) + E^{-2}(x)Bi^2(x)\}^{1/2}, \tag{2.6}$$

$$\lambda = \max_{(-\infty,\infty)} \{\pi|x|^{1/2}M^2(x)\} = 1.430\ldots, \tag{2.7}$$

$$\mu = \max_{(-\infty,0)} \{\pi|x|^{1/2}M^2(x)\} = 1 \qquad ([10, \text{p. } 751]). \tag{2.8}$$

Olver's result then states that

$$\begin{aligned} J_\nu(\nu x) = \frac{1}{1+\delta_{2n+1}} \frac{\varphi(\zeta)}{\nu^{1/3}} \Bigg\{ & Ai(\nu^{2/3}\zeta) \sum_{s=0}^{n} \frac{A_s(\nu)}{\nu^{2s}} \\ & + \frac{Ai'(\nu^{2/3}\zeta)}{\nu^{4/3}} \sum_{s=0}^{n-1} \frac{B_s(\nu)}{\nu^{2s}} + \varepsilon_{2n+1}(\nu,\zeta) \Bigg\}, \end{aligned} \tag{2.9}$$

where

$$|\varepsilon_{2n+1}(\nu,\zeta)| \leq \frac{2M(\nu^{2/3}\zeta)}{E(\nu^{2/3}\zeta)} \exp\left\{\frac{2\lambda}{\nu} \mathcal{V}_{\zeta,\infty}(|\zeta|^{1/2}B_0)\right\} \frac{\mathcal{V}_{\zeta,\infty}(|\zeta|^{1/2}B_n)}{\nu^{2n+1}} \tag{2.10}$$

and

$$|\delta_{2n+1}| \leq 2e^{\nu_0/\nu}\nu^{-2n-1}\,\mathcal{V}_{-\infty,\infty}(|\zeta|^{1/2}B_n). \tag{2.11}$$

In (2.10) and (2.11) we have used $\mathcal{V}_{a,b}(f)$ to denote the total variation of a function $f(\zeta)$ on an interval (a,b). The following values are computed in [9, p. 9] and [11, p. 207]:

$$\mathcal{V}_{-\infty,\infty}\{|\zeta|^{1/2}B_0(\zeta)\} = 0.1051, \tag{2.12}$$

$$\nu_0 = 2\lambda\,\mathcal{V}_{-\infty,\infty}(|\zeta|^{1/2}B_0) = 0.30. \tag{2.13}$$

For our purpose, it suffices to take $n = 0$ in (2.9). Thus

$$J_\nu(\nu x) = \frac{1}{1+\delta_1} \frac{\varphi(\zeta)}{\nu^{1/3}} \left[Ai(\nu^{2/3}\zeta) + \varepsilon_1(\nu,\zeta)\right]. \tag{2.14}$$

In view of (2.12) and (2.13), (2.11) simplifies to

$$|\delta_1| \leq 2e^{0.30/\nu}\,\frac{0.1051}{\nu} \leq \frac{0.217}{\nu} \qquad \text{if } \nu \geq 10. \tag{2.15}$$

If ζ is negative, as it will be in our case, we also have from (2.10) and (2.8)

$$|\varepsilon_1(\nu,\zeta)| \leq \frac{0.2102}{\sqrt{\pi}(-\nu^{2/3}\zeta)^{1/4}\nu}\, e^{0.30/\nu}. \tag{2.16}$$

Squaring both sides of (2.14) gives

$$J_\nu^2(\nu x) = \frac{1}{(1+\delta_1)^2}\,\frac{\varphi^2(\zeta)}{\nu^{2/3}}\left[Ai^2(\nu^{2/3}\zeta) + \varepsilon^*(\nu,\zeta)\right], \tag{2.17}$$

where

$$\varepsilon^*(\nu,\zeta) = 2Ai(\nu^{2/3}\zeta)\varepsilon_1(\nu,\zeta) + \varepsilon_1^2(\nu,\zeta). \tag{2.18}$$

By (2.6) and (2.8),

$$|Ai(x)| \leq M(x) \leq \frac{1}{\sqrt{\pi}(-x)^{1/4}} \qquad \text{if } x < 0. \tag{2.19}$$

Hence it follows from (2.16) that

$$\begin{aligned} |\varepsilon^*(\nu,\zeta)| &\leqq \frac{0.134\, e^{0.30/\nu}}{\nu^{4/3}(-\zeta)^{1/2}}\left[1 + \frac{0.1051 e^{0.30/\nu}}{\nu}\right] \\ &\leqq \frac{0.1396}{\nu^{4/3}(-\zeta)^{1/2}} \qquad \text{if } \nu \geqq 10. \end{aligned} \tag{2.20}$$

Equation (2.17) can be further simplified to

$$J_\nu^2(\nu x) = \varphi^2(\zeta)\frac{Ai^2(\nu^{2/3}\zeta)}{\nu^{2/3}} + \tilde{\varepsilon}(\nu,\zeta) \tag{2.21}$$

with

$$\tilde{\varepsilon}(\nu,\zeta) = \frac{\varphi^2(\zeta)}{\nu^{2/3}}\left\{\frac{\varepsilon^*(\nu,\zeta) - \delta_1(2+\delta_1)Ai^2(\nu^{2/3}\zeta)}{(1+\delta_1)^2}\right\}. \tag{2.22}$$

Since $|1+\delta_1| \geq 1 - 0.0217$ if $\nu \geq 10$ by (2.15), a combination of (2.15), (2.19) and (2.20) gives

$$|\tilde{\varepsilon}(\nu,\zeta)| \leq \varphi^2(\zeta)\frac{0.2918}{\nu^2(-\zeta)^{1/2}} \qquad \text{if } \nu \geq 10. \tag{2.23}$$

A uniform asymptotic approximation of $J'_\nu(\nu x)$, corresponding to (2.9) with $n = 0$, is

$$\begin{aligned} J'_\nu(\nu x) = -\frac{1}{1+\delta_1}\frac{\psi(\zeta)}{\nu^{2/3}}\bigg[&\frac{Ai(\nu^{2/3}\zeta)}{\nu^{2/3}}\{C_0(\zeta) - \zeta B_0(\zeta)\} \\ &+ Ai'(\nu^{2/3}\zeta) + \eta_1(\nu,\zeta) + \chi(\zeta)\frac{\varepsilon_1(\nu,\zeta)}{\nu^{2/3}}\bigg], \end{aligned} \tag{2.24}$$

where $\varphi(\zeta)$ and $\varepsilon_1(\nu,\zeta)$ are as given in (2.4) and (2.16), respectively, $\psi(\zeta) = 2/\{x\varphi(\zeta)\}$, $\chi(\zeta) = \varphi'(\zeta)/\varphi(\zeta)$, $C_0(\zeta) = \chi(\zeta) + \zeta B_0(\zeta)$ and

(2.25) $$|\eta_1(\nu,\zeta)| \leq 2e^{\nu_0/\nu}\nu^{-1}\mathcal{V}_{\zeta,\infty}(|\zeta|^{1/2}B_0)E^{-1}(\nu^{2/3}\zeta)N(\nu^{2/3}\zeta);$$

see [11, p. 208]. In view of (2.5), (2.12) and (2.13), (2.25) can be simplified to

(2.26) $$|\eta_1(\nu,\zeta)| \leq \frac{0.2102}{\nu}e^{0.30/\nu}N(\nu^{2/3}\zeta)$$

for negative ζ. The modulus function $N(x)$ is defined by

(2.27) $$N(x) = \{E^2(x)Ai'^2(x) + E^{-2}(x)Bi'^2(x)\}^{1/2};$$

see [10, p. 750]. For $x \leq -1$, we also have from [10, p. 752],

(2.28) $$0 < |x|^{-1/4}N(x) < 0.60.$$

3. **The negative zeros of** $Ai(x)$ **and** $Ai'(x)$. Let a_n and a'_n denote the n^{th} negative zero of $Ai(x)$ and $Ai'(x)$, respectively. In [3], Hethcote has shown that

(3.1) $$a_n = -\left[\frac{3\pi}{8}(4n-1)\right]^{2/3}(1+\sigma_n),$$

where

(3.2) $$|\sigma_n| \leq 0.130\left[\frac{3\pi}{8}(4n-1.051)\right]^{-2}$$

for $n \geq 1$. For our purpose, we need a corresponding error estimate for a'_n. Our argument here parallels that of Hethcote. First we recall the following result from [3], which was derived from a method of Gatteschi [2].

LEMMA 1. *In the interval* $[n\pi - \psi - \rho, n\pi - \psi + \rho]$, *where* $\rho < \pi/2$, *suppose* $f(x) = \sin(x+\psi) + \varepsilon(x)$, $f(x)$ *is continuous and* $E = \max|\varepsilon(x)| < \sin\rho$. *Then there exists a zero d of* $f(x)$ *in the interval such that* $|d - (n\pi - \psi)| \leqq E/\cos\rho$.

Also we recall the asymptotic expansion [12, p. 392]

(3.3) $$Ai'(-x) = \pi^{-1/2}x^{1/4}\left\{\sin(\xi - \tfrac{1}{4}\pi)P(\xi) - \cos(\xi - \tfrac{1}{4}\pi)Q(\xi)\right\},$$

where $\xi = \frac{2}{3}x^{3/2}$,

(3.4) $$P(\xi) \sim 1 + \frac{13}{11}\cdot\frac{5\cdot 7\cdot 9\cdot 11}{(216)^2 2!\,\xi^2} - \frac{25}{23}\cdot\frac{9\cdot 11\cdot 13\cdot 15\cdot 17\cdot 19\cdot 21\cdot 23}{(216)^4 4!\,\xi^4} + \cdots$$

and

(3.5) $$Q(\xi) \sim -\frac{7}{5}\cdot\frac{3\cdot 5}{(216)\xi} + \frac{19}{17}\cdot\frac{7\cdot 9\cdot 11\cdot 13\cdot 15\cdot 17}{(216)^3 3!\,\xi^3} - \cdots.$$

It is known that if the expansions of $P(\xi)$ and $Q(\xi)$ are truncated at their n^{th} terms, then the error terms are bounded in absolute value by the first neglected terms, provided that $n \geq 1$ and 0, respectively. To apply the above lemma, we let $f(\xi) = \pi^{1/2}x^{-1/4}Ai'(-x)$ and $\psi = -\frac{1}{4}\pi$. Then

$$|\varepsilon(\xi)| \leq 0.0973\xi^{-1} + 0.0439\xi^{-2} + 0.0627\xi^{-4}.$$

With $\rho = 0.03$ and $\xi \geq 3.88$, we have $E = \max|\varepsilon(\xi)| \leq 0.02825$ and $\sin\rho > 0.029$. Since $E < \sin\rho$, by the above lemma there is a zero d_n of $f(\xi)$ in the interval $\left[n\pi + \frac{1}{4}\pi - 0.03, n\pi + \frac{1}{4}\pi + 0.03\right]$ such that

$$|d_n - (n\pi + \tfrac{1}{4}\pi)| \leq E/\cos\rho \leq 0.1097/(n\pi + \tfrac{1}{4}\pi - 0.03) \tag{3.6}$$

if $n \geq 1$. Here, use has been made of the fact that $|\varepsilon(\xi)| \leq 0.1096/\xi$ for $\xi \geq 3.88$. Note that d_1 lies in the interval [3.873, 3.932], and that $a'_1 = -1.01879$ and $a'_2 = -3.24820$. Thus $d_n = \frac{2}{3}\left(-a'_{n+1}\right)^{3/2}$ for $n = 1, 2, \ldots$, and the assumption that $\xi \geq 3.88$ is justified. From (3.6), we have

$$a'_{n+1} = -\left[\frac{3\pi}{8}(4n+1)\right]^{2/3}(1 + \tilde{\tau}_{n+1})^{2/3},$$

where

$$|\tilde{\tau}_{n+1}| \leq 0.2469\Big/\left[\frac{3\pi}{8}(4n + 0.9618)\right]^2.$$

If $n \geq 1$ then $|\tilde{\tau}_{n+1}| \leq 0.00723$. Applying the Mean-Value Theorem to $(1+x)^{2/3}$, we obtain

$$a'_n = -\left[\frac{3\pi}{8}(4n-3)\right]^{2/3}(1 + \tau_n), \tag{3.7}$$

where

$$|\tau_n| \leq 0.165\Big/\left[\frac{3\pi}{8}(4n - 3.0382)\right]^2 \tag{3.8}$$

for $n \geq 2$. The bound on τ_n is reasonable, since 0.165 is only slightly greater than the magnitude of the coefficient $-7/48$ of the next term in the asymptotic expansion of a'_n; see [12, p. 405].

The approximation (3.1) and (3.7) will now be used to estimate the numbers ρ_n, α_n and β_n defined by

$$\rho_n = \tfrac{1}{4}(a_n - a'_{n+1}), \qquad \alpha_n = a_n - \rho_n, \qquad \beta_n = a_n + \rho_n. \tag{3.9}$$

From (3.7) and the inequality [4]

$$a_n \leq -\left[\frac{3\pi}{8}(4n-1)\right]^{2/3},$$

we have

$$4\rho_n \leq \left[\frac{3\pi}{8}(4n)\right]^{2/3}\left\{\left(1+\frac{1}{4n}\right)^{2/3}(1+\tau_{n+1})-\left(1-\frac{1}{4n}\right)^{2/3}\right\}.$$

By using the Mean-Value Theorem, it can easily be shown that

$$\left(1+\frac{1}{4n}\right)^{2/3} \leq 1+\frac{1}{6n},$$

and

$$\left(1-\frac{1}{4n}\right)^{2/3} \geq 1-\frac{2}{3}\frac{1}{4n-1}$$

for $n \geq 1$. Consequently

$$4\rho_n \leqq \left(\frac{3\pi n}{2}\right)^{2/3}\left\{\frac{4}{3}\frac{1}{4n-1}+\left(1+\frac{1}{6n}\right)\tau_{n+1}\right\}.$$

Replacing τ_{n+1} by its upper bound (3.8), we obtain

$$4\rho_n \leqq \left(\frac{3\pi n}{2}\right)^{2/3}\left\{\frac{4}{3}\frac{1}{4n-1}+\frac{0.121}{(4n-1)^2}\right\}$$

for $n \geq 10$, from which it follows that

(3.10) $$0 < \rho_n \leq 0.241 n^{-1/3} \qquad \text{if } n \geq 10.$$

To estimate α_n, we note that

(3.11) $$\alpha_n = a_n - \rho_n = \tfrac{3}{4}a_n + \tfrac{1}{4}a'_{n+1}.$$

Let $B_n = [3\pi(4n-1)/8]^{2/3}$. Then substitution of (3.1) and (3.7) in (3.11) gives

$$-\alpha_n = B_n\left\{\frac{3}{4}(1+\sigma_n)+\frac{1}{4}\left(1+\frac{2}{4n-1}\right)^{2/3}(1+\tau_{n+1})\right\}.$$

By Taylor's theorem,

$$\left(1+\frac{2}{4n-1}\right)^{2/3} = 1+\frac{4}{3}\frac{1}{4n-1}+e_n$$

where $|e_n| \leq 4/\left\{9(4n-1)^2\right\}$ for $n \geq 1$. Thus

(3.12) $$-\alpha_n = B_n\left(1+\frac{1}{3}\frac{1}{4n-1}+\eta_n^*\right)$$

with

$$\eta_n^* = \frac{3}{4}\sigma_n+\frac{1}{4}e_n+\frac{1}{4}\left(1+\frac{4}{3}\frac{1}{4n-1}+e_n\right)\tau_{n+1}.$$

Using (3.2) and (3.8), it can be shown that if $n \geq 1$,

$$|\eta_n^*| \leq \frac{0.212}{(4n-1.051)^2} + \frac{0.040}{(4n-1.051)^3} + \frac{0.014}{(4n-1.051)^4}.$$

Consequently,

$$(3.13) \qquad |\eta_n^*| \leq \frac{0.213}{(4n-1.051)^2} \qquad \text{for } n \geq 10.$$

By Taylor's theorem again, we have

$$\frac{2}{3}(-\alpha_n)^{3/2} = \frac{2}{3}B_n^{3/2}\left(1 + \frac{1}{2}\frac{1}{4n-1} + \eta_n\right),$$

where

$$|\eta_n| \leqq \frac{0.3626}{(4n-1.051)^2}, \qquad \text{for } n \geq 10.$$

Since $\frac{2}{3}B_n^{3/2} = \frac{\pi}{4}(4n-1)$, it follows that

$$(3.14) \qquad \frac{2}{3}(-\alpha_n)^{3/2} = n\pi + \frac{\pi}{4} - \frac{3\pi}{8} + \theta_n$$

with $\theta_n = \frac{2}{3}B_n^{3/2}\eta_n$. Using the fact that $(4n-1) \leq 1.0014(4n-1.051)$, we obtain

$$(3.15) \qquad |\theta_n| \leq \frac{0.2852}{4n-1.051} \qquad \text{for } n \geq 10.$$

In exactly the same manner, one can show that

$$(3.16) \qquad \frac{2}{3}(-\beta_n)^{3/2} = n\pi - \frac{3\pi}{4} + \frac{3\pi}{8} + \phi_n$$

where

$$(3.17) \qquad |\phi_n| \leq \frac{0.375}{4n-1.051} \qquad \text{for } n \geq 10.$$

We conclude this section with the following result.

LEMMA 2. *For* $n \geqq 10$, $a'_{n+1} < \alpha_n < a_n < \beta_n < a'_n$.

Proof. From the graph of $Ai(-x)$ given in [1, p. 446], it is evident that $a'_{n+1} < a_n < a'_n$ for all $n \geqq 1$. Since $\rho_n = \frac{1}{4}(a_n - a'_{n+1}) > 0$, it is also clear that $a'_{n+1} < \alpha_n < a_n < \beta_n$. Hence we need show only that $\beta_n < a'_n$ for $n \geqq 10$. Now recall that $d_{n-1} = \frac{2}{3}(-a'_n)^{3/2}$. Consequently, it follows from (3.6) that if $n \geqq 2$,

$$(3.18) \qquad \frac{2}{3}\left(-a'_n\right)^{3/2} = n\pi - \frac{3\pi}{4} + \psi_n,$$

where

$$(3.19) \qquad |\psi_n| \leq \frac{0.1097}{n\pi - \frac{3\pi}{4} - 0.03} \leq \frac{0.140}{4n-3.01}.$$

Coupling (3.16) and (3.18) gives

$$\Delta_n = \tfrac{2}{3}(-\beta_n)^{3/2} - \tfrac{2}{3}\left(-a'_n\right)^{3/2} = \tfrac{3\pi}{8} + \phi_n - \psi_n.$$

From (3.17) and (3.19), we have $\phi_n \geq -0.0097$ and $\psi_n \leq 0.0038$ for $n \geq 10$. Hence, $\Delta_n \geq \frac{3\pi}{8} - 0.0097 - 0.0039 > 0$ and the lemma is proved.

4. **Estimates for** $Ai(\alpha_n)$, $Ai(\beta_n)$, $Ai'(\alpha_n)$ **and** $Ai'(\beta_n)$. From the asymptotic expansion of $Ai(-x)$, we have

$$\sqrt{\pi}\, x^{1/4} Ai(-x) = \cos(\xi - \tfrac{\pi}{4}) + \varepsilon(\xi), \tag{4.1}$$

where $\xi = \frac{2}{3}x^{3/2}$ and

$$|\varepsilon(\xi)| \leq \frac{5}{72}\xi^{-1} + \frac{385}{10368}\xi^{-2}; \tag{4.2}$$

see [12, pp. 392 and 394]. Let $\xi_n = \frac{2}{3}(-\beta_n)^{3/2}$. By (3.16)

$$\xi_n = n\pi - \frac{3\pi}{4} + \frac{3\pi}{8} + \phi_n, \tag{4.3}$$

where $|\phi_n| \leq 0.375/(4n - 1.051)$ if $n \geq 10$. The addition formula for the cosine function gives

$$\cos(\xi_n - \tfrac{\pi}{4}) = \cos\{(n-1)\pi + \tfrac{3\pi}{8} + \phi_n\} = \pm\cos(\tfrac{3\pi}{8} + \phi_n).$$

If $n \geq 10$ then $|\phi_n| \leq 0.0097$,

$$\xi_n \geqq 10\pi - \tfrac{3\pi}{8} - 0.0097 \geqq 30,$$

and from (4.2) it also follows that $|\varepsilon(\xi_n)| \leq 0.0024$. Furthermore, we have

$$1.1683 \leq \frac{3\pi}{8} - 0.0097 \leq \frac{3\pi}{8} + \phi_n \leq 1.1877, \tag{4.4}$$

and

$$0.3737 \leqq \cos(1.1877) \leqq \cos(\tfrac{3\pi}{8} + \phi_n) \leqq 1$$

for $n \geq 10$. The approximation in (4.1) then gives

$$\sqrt{\pi}\,(-\beta_n)^{1/4}|Ai(\beta_n)| \geqq |\cos(\xi_n - \tfrac{\pi}{4})| - |\varepsilon(\xi_n)| \geqq 0.3737 - 0.0024$$

or equivalently

$$|Ai(\beta_n)| \geq \frac{0.2094}{(-\beta_n)^{1/4}} \qquad \text{for } n \geq 10. \tag{4.5}$$

In exactly the same manner, it can be proved that

$$|Ai(\alpha_n)| \geqq \frac{0.2106}{(-\alpha_n)^{1/4}} \qquad \text{for } n \geqq 10.$$

Since $\alpha_n = \beta_n - 2\rho_n$, it follows that

$$|Ai(\alpha_n)| \geqq \frac{0.2084}{(-\beta_n)^{1/4}}\left\{1 + \frac{2\rho_n}{(-\beta_n)}\right\}^{-1/4}.$$

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For $n \geq 10$, we have $0 < \rho_n \leq 0.241 n^{-1/3} \leq 0.1119$ and $\beta_n < \beta_{10} = a_{10} + \rho_{10} < -12.8287 + 0.1119 < -12.716$. Consequently,

$$|Ai(\alpha_n)| \geq \frac{0.2096}{(-\beta_n)^{1/4}} \qquad \text{for } n \geq 10. \tag{4.6}$$

To derive similar estimates for $Ai'(\alpha_n)$ and $Ai'(\beta_n)$, we use, instead of (4.1), the asymptotic expansion (3.3), which gives in particular

$$\sqrt{\pi}\, x^{-1/4} Ai'(-x) = \sin(\xi - \tfrac{1}{4}\pi) + \varepsilon(\xi), \tag{4.7}$$

where $\xi = \frac{2}{3}x^{3/2}$ and

$$|\varepsilon(\xi)| \leq \frac{7}{72}\xi^{-1} + \frac{455}{10368}\xi^{-2} + \frac{40415375}{644972544}\xi^{-4}. \tag{4.8}$$

If $\xi \geq 30$, then $|\varepsilon(\xi)| \leq 0.0033$. We again let $\xi_n = \frac{2}{3}(-\beta_n)^{3/2}$. Then from (4.3), it follows that $\sin(\xi_n - \frac{\pi}{4}) = \pm\sin(\frac{3\pi}{8} + \phi_n)$. Furthermore, from (4.4) we have $0.92 \leq \sin(1.1683) \leq \sin(\frac{3\pi}{8} + \phi_n) \leq 1$ if $n \geq 10$. Since $\xi_n \geq 30$ if $n \geq 10$, (4.7) gives

$$\sqrt{\pi}\,(-\beta_n)^{-1/4}|Ai'(\beta_n)| \geqq |\sin(\xi_n - \tfrac{\pi}{4})| - |\varepsilon(\xi_n)| \geqq 0.92 - 0.0033,$$

or equivalently

$$|Ai'(\beta_n)| \geq 0.5171(-\beta_n)^{1/4} \qquad \text{if } n \geq 10. \tag{4.9}$$

A similar argument leads to

$$|Ai'(\alpha_n)| \geqq 0.5177\,(-\alpha_n)^{1/4} \qquad \text{if } n \geqq 10.$$

Since $-\alpha_n > -\beta_n > 0$ for all n, we also have

$$|Ai'(\alpha_n)| \geq 0.5177(-\beta_n)^{1/4} \qquad \text{whenever } n \geq 10. \tag{4.10}$$

5. **A uniform asymptotic approximation of $J_\nu''(\nu x)$.** A uniform asymptotic approximation of $J_\nu''(\nu x)$ can be derived from (2.14), (2.24) and the Bessel differential equation

$$x^2 J_\nu''(x) + xJ_\nu'(x) + (x^2 - \nu^2)J_\nu(x) = 0. \tag{5.1}$$

Since $C_0(\zeta) - \zeta B_0(\zeta) = \chi(\zeta)$ in (3.24), replacing x by νx in (5.1) and substituting (2.14) and (2.24) in the resulting equation gives

$$\begin{aligned} J_\nu''(\nu x) = \frac{1}{1+\delta_1}\frac{1}{\nu^{1/3}}\bigg\{ &\frac{1-x^2}{x^2}\varphi(\zeta)[Ai(\nu^{2/3}\zeta) + \varepsilon_1] \\ &+ \frac{\psi(\zeta)\chi(\zeta)}{\nu^2 x}[Ai(\nu^{2/3}\zeta) + \varepsilon_1] \\ &+ \frac{\psi(\zeta)}{\nu^{4/3}x}[Ai'(\nu^{2/3}\zeta) + \eta_1]\bigg\}. \end{aligned} \tag{5.2}$$

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With $H(\zeta) = \frac{1}{2}\varphi^2(\zeta)$ and $G(\zeta) = H(\zeta)\chi(\zeta)$, equation (5.2) becomes

$$J_\nu''(\nu x) = \frac{\theta(\zeta)}{\nu^{1/3}(1+\delta_1)}\left\{\left(\zeta + \frac{G(\zeta)}{\nu^2}\right)[Ai(\nu^{2/3}\zeta)+\varepsilon_1] + \frac{H(\zeta)}{\nu^{4/3}}[Ai'(\nu^{2/3}\zeta)+\eta_1]\right\}, \tag{5.3}$$

where $\theta(\zeta) = 4/\left\{x^2\varphi^3(\zeta)\right\}$.

It can be verified that $\varphi(\zeta)$ is a nonnegative and increasing function on $(-\infty, 0]$; see [5]. Hence $0 \leq \varphi(\zeta) \leq \varphi(0) = 2^{1/3}$ for $\zeta < 0$. Furthermore, it is known from [9, p. 10] that $|\varphi'(\zeta)/\varphi(\zeta)| \leq 0.160$ for $-\infty < \zeta < \infty$. Thus we have

$$|H(\zeta)| \leq 0.79391, \qquad |G(\zeta)| \leq 0.127 \tag{5.4}$$

for negative ζ.

Equation (5.3) can be simplified to

$$J_\nu''(\nu x) = \frac{\zeta\theta(\zeta)}{\nu^{1/3}(1+\delta_1)}\left[Ai(\nu^{2/3}\zeta) + \delta(\nu,\zeta)\right] \tag{5.5}$$

with

$$\delta(\nu,\zeta) = \frac{G(\zeta)}{\nu^2\zeta}Ai(\nu^{2/3}\zeta) + \left[1 + \frac{G(\zeta)}{\nu^2\zeta}\right]\varepsilon_1 + \frac{H(\zeta)}{\nu^{4/3}\zeta}[Ai'(\nu^{2/3}\zeta)+\eta_1]. \tag{5.6}$$

In (5.6), we first replace $Ai(x)$, ε_1 and η_1 by their upper bounds given in (2.19), (2.16) and (2.26), respectively. From (2.27), we can also replace $Ai'(x)$ by its associated modulus function $N(x)$. The result is

$$\begin{aligned} |\delta(\nu,\zeta)| \leqq{} & \frac{0.2102}{\sqrt{\pi}(-\nu^{2/3}\zeta)^{1/4}\nu}e^{0.30/\nu} \\ & + \frac{G(\zeta)}{\nu^2|\zeta|}\left[1 + \frac{0.2102}{\nu}e^{0.30/\nu}\right]\frac{1}{\sqrt{\pi}(-\nu^{2/3}\zeta)^{1/4}} \\ & + \frac{H(\zeta)}{\nu^{4/3}|\zeta|}\left[1 + \frac{0.2102}{\nu}e^{0.30/\nu}\right]N(\nu^{2/3}\zeta). \end{aligned}$$

Next, we replace G, H and N by their estimates given in (5.4) and (2.28), and obtain

$$\begin{aligned} |\delta(\nu,\zeta)| \leqq{} & \frac{1}{\nu^{2/3}(-\nu^{2/3}\zeta)^{1/4}}\left\{\frac{0.2102}{\sqrt{\pi}\,\nu^{1/3}}e^{0.30/\nu}\right. \\ & \left. + \left[1 + \frac{0.2102}{\nu}e^{0.30/\nu}\right]\left[\frac{0.072}{\nu^{2/3}(-\nu^{2/3}\zeta)} + \frac{0.477}{(-\nu^{2/3}\zeta)}\right]\right\}. \end{aligned} \tag{5.7}$$

If $\nu \geq 10$, then (5.7) reduces to

$$|\delta(\nu,\zeta)| \leq \frac{1}{\nu^{2/3}(-\nu^{2/3}\zeta)^{1/4}}\left[0.0568 + \frac{0.5032}{(-\nu^{2/3}\zeta)}\right]. \tag{5.8}$$

Recall that x and ζ in (5.5) are related in a one-to-one manner by equations (2.2) and (2.3). Let $x_{\nu,k} = j''_{\nu,k}/\nu$ and $\zeta_{\nu,k} = \zeta(x_{\nu,k})$. We now use (5.5) and the following result [3] to derive an asymptotic approximation for $\zeta_{\nu,k}$.

THEOREM. *In the interval* $[a-\rho, a+\rho]$, *suppose* $f(\tau)=g(\tau)+\varepsilon(\tau)$, *where* $f(\tau)$ *is continuous,* $g(\tau)$ *is differentiable,* $g(a)=0$, $m=\min|g'(\tau)|>0$, *and*

$$E=\max|\varepsilon(\tau)|<\min\{|g(a-\rho)|,|g(a+\rho)|\}. \tag{5.9}$$

Then there exists a zero c of $f(\tau)$ *in the interval such that* $|c-a| \leqq E/m$.

We apply this theorem to (5.5) with $\tau=\nu^{2/3}\zeta$ as the independent variable, $f(\nu^{2/3}\zeta)=(1+\delta_1)\nu^{1/3}J_\nu''(\nu x)/\{\zeta\theta(\zeta)\}$, $g(\nu^{2/3}\zeta)=Ai(\nu^{2/3}\zeta)$, $a=a_k$, a_k being the k^{th} negative zero of the Airy function, and the error $\varepsilon(\tau)$ given by $\varepsilon(\tau)=\delta(\nu,\zeta)$. For each k between 2 and 9, we shall choose a positive number ρ_k so that $a'_{k+1}<a_k-\rho_k<a_k<a_k+\rho_k<a'_k$ and (5.9) holds with a and ρ replaced by a_k and ρ_k. As in § 3, we let $\alpha_k=a_k-\rho_k$ and $\beta_k=a_k+\rho_k$. For convenience, we also introduce the notations $m_k=\min\{|Ai'(\tau)| : \alpha_k\le\tau\le\beta_k\}$ and $M_k=\min\{|Ai(\alpha_k)|,|Ai(\beta_k)|\}$. Since a'_{k+1} and a'_k are two consecutive zeros of $Ai'(\tau)$ and a_k is a critical point of $Ai'(\tau)$ in the interval $[a'_{k+1},a'_k]$, the minimum value m_k is attained at the endpoints α_k or β_k. Table 1 below lists the values of $-a_k$, $-a'_{k+1}$, ρ_k, m_k and M_k for $k=2,3,\ldots,9$; cf. [1, pp. 476–478]. On the interval $[\alpha_k,\beta_k]$, $-\nu^{2/3}\zeta=-\tau\ge-\beta_k\ge-\beta_2\ge 3.90$. Thus it follows from (5.8) that

$$|\varepsilon_k(\tau)|=|\delta(\nu,\zeta)|\le\frac{c_k}{\nu^{2/3}}, \tag{5.10}$$

where $c_k=0.1859/(-\beta_k)^{1/4}$. The values of c_k are also given in Table 1.

k	$-a_k$	$-a'_{k+1}$	ρ_k	m_k	M_k	c_k
2	4.08795	4.82010	0.18304	0.74713	0.14359	0.13224
3	5.52056	6.16331	0.16069	0.80309	0.13570	0.12218
4	6.78671	7.37218	0.14637	0.84450	0.13007	0.11581
5	7.94413	8.48849	0.13609	0.87771	0.12575	0.11121
6	9.02265	9.53545	0.12820	0.90563	0.12227	0.10765
7	10.04017	10.52766	0.12187	0.92981	0.11936	0.10475
8	11.00852	11.47506	0.11663	0.95120	0.11688	0.10233
9	11.93602	12.38479	0.11219	0.97043	0.11472	0.10025

TABLE 1

From (5.10) and Table 1, it is now evident that the conditions of the above theorem are all satisfied. Hence, there exists a zero $\tau_k=\nu^{2/3}\zeta_{\nu,k}$ in the interval $[\alpha_k,\beta_k]$ for each $k=2,\ldots,9$ such that

$$|\nu^{2/3}\zeta_{\nu,k}-a_k|\le\frac{c_k}{\nu^{2/3}m_k} \tag{5.11}$$

or equivalently

$$\nu^{2/3}\zeta_{\nu,k}=a_k+\nu^{2/3}\eta_k, \tag{5.12}$$

where

$$|\eta_k|\le\frac{d_k}{\nu^{4/3}} \tag{5.13}$$

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and $d_k = c_k/m_k$. The values of d_k, for $k = 2, \ldots, 9$, are listed below.

$$(5.14)\qquad \begin{array}{llll} d_2 = 0.17700, & d_3 = 0.15214, & d_4 = 0.13713, & d_5 = 0.12670, \\ d_6 = 0.11887, & d_7 = 0.11266, & d_8 = 0.10758, & d_9 = 0.10330. \end{array}$$

We now consider the case $k \geq 10$. Here we choose $\rho_k = \frac{1}{4}\left(a'_k - a'_{k+1}\right)$, and again let $\alpha_k = a_k - \rho_k$ and $\beta_k = a_k + \rho_k$; cf. (3.9). Since $a'_{k+1} < \alpha_k < a_k < \beta_k < a'_k$ for $k \geq 10$ (see § 3), it follows from (4.9) and (4.10) that in the interval $[\alpha_k, \beta_k]$, $m_k = \min|Ai'(\tau)| \geq 0.5171(-\beta_k)^{1/4} > 0$ if $k \geq 10$. To show that condition (5.9) holds, we note that $a_{10} = -12.82877675$ and $\rho_k \leq 0.241/k^{1/3}$ for $k \geq 10$. Thus for τ in $[\alpha_k, \beta_k]$, we have $\tau \leq \beta_k \leq \beta_{10} < -12.716$ if $k \geq 10$, and (5.8) gives

$$|\varepsilon_k(\tau)| = |\delta(\nu, \zeta)| \leqq \frac{0.0964}{\nu^{2/3}(-\beta_k)^{1/4}} \leqq \frac{0.0208}{(-\beta_k)^{1/4}} \qquad \text{for } \nu \geqq 10.$$

Also, from (4.5) and (4.6), we have

$$M_k = \min\{|Ai(\alpha_k)|, |Ai(\beta_k)|\} \geqq 0.2094/(-\beta_k)^{1/4}, \qquad k \geqq 10.$$

Consequently, condition (5.9) is satisfied. By the above theorem, if $\nu \geq 10$ and $k \geq 10$, there exists a zero $\tau_k = \nu^{2/3}\zeta_{\nu,k}$ in the interval $[\alpha_k, \beta_k]$ such that

$$(5.15)\qquad |\nu^{2/3}\zeta_{\nu,k} - a_k| \leq \frac{0.1865}{\nu^{2/3}(-\beta_k)^{1/2}},$$

or equivalently

$$(5.16)\qquad \nu^{2/3}\zeta_{\nu,k} = a_k + \nu^{2/3}\eta_k,$$

where $|\eta_k| \leq d_k/\nu^{4/3}$ and

$$(5.17)\qquad d_k = 0.1865/(-\beta_k)^{1/2} \qquad \text{for } k \geq 10.$$

From (5.14) and (5.17), it is evident that $\{d_k\}$ is a monotonically decreasing sequence.

6. **A Bound for $J_\nu(j''_{\nu,k})$.** In the asymptotic approximation (2.14), we replace x by $x_k = j''_{\nu,k}/\nu$ so that

$$(6.1)\qquad J_\nu(j''_{\nu,k}) = \frac{\varphi(\zeta_{\nu,k})}{(1+\delta_1)\nu^{1/3}}\left\{Ai(\nu^{2/3}\zeta_{\nu,k}) + \varepsilon_1(\nu, \zeta_{\nu,k})\right\},$$

where $\nu^{2/3}\zeta_{\nu,k}$ belongs to the interval $[\alpha_k, \beta_k]$ and satisfies (5.12) or (5.16), and

$$(6.2)\qquad |\varepsilon_1(\nu, \zeta_{\nu,k})| \leq \frac{0.2102}{\sqrt{\pi}(-\nu^{2/3}\zeta_{\nu,k})^{1/4}\nu} e^{0.30/\nu}.$$

Since $\zeta_{\nu,k}$ is negative and $\varphi(\zeta)$ is increasing in $(-\infty, 0]$, $0 \leq \varphi(\zeta_{\nu,k}) \leq \varphi(0) = 2^{1/3}$. Furthermore, since $Ai(a_k) = 0$, the Mean-Value Theorem gives

$$Ai(\nu^{2/3}\zeta_{\nu,k}) = Ai(a_k + \nu^{2/3}\eta_k) = Ai'(\xi_k)\nu^{2/3}\eta_k,$$

where $\xi_k \in [\alpha_k, \beta_k] \subset [a'_{k+1}, a'_k]$. From the Airy equation, it is easily seen that $Ai'(x)$ has only one critical point in $[a'_{k+1}, a'_k]$, which is located at a_k. Thus, $|Ai'(\xi_k)| \leq |Ai'(a_k)|$ and

$$|Ai(\nu^{2/3}\zeta_{\nu,k})| \leq |Ai'(a_k)|\nu^{2/3}\eta_k \leq \frac{1}{\nu^{2/3}}|Ai'(a_k)|d_k. \tag{6.3}$$

From (2.15), we also have $1 + \delta_1 \geq 0.9783$. A combination of these results yields

$$|J_\nu(j''_{\nu,k})| \leqq \frac{1.2879}{\nu}\left\{|Ai'(a_k)|d_k + \frac{0.2102}{\sqrt{\pi}(-\nu^{2/3}\zeta_{\nu,k})^{1/4}}\frac{e^{0.30/\nu}}{\nu^{1/3}}\right\}.$$

The values of $Ai'(a_k)$, $k = 2, \ldots, 9$, are given in [1, p. 478]. Since $-\nu^{2/3}\zeta_{\nu,k} \geq -\beta_k \geq -\beta_2 \geq 3.90$, simple computation gives

$$|J_\nu(j''_{\nu,k})| \leq \frac{e_k}{\nu} \qquad \text{for } \nu \geq 10, \tag{6.4}$$

where

$$\begin{aligned} &e_2 = 0.23506, \quad e_3 = 0.22151, \quad e_4 = 0.21285, \quad e_5 = 0.20657,\\ &e_6 = 0.20170, \quad e_7 = 0.19771, \quad e_8 = 0.19438, \quad e_9 = 0.19151. \end{aligned} \tag{6.5}$$

If $k \geq 10$, then by (6.3) and (5.17)

$$|Ai(\nu^{2/3}\zeta_{\nu,k})| \leqq \frac{0.0403}{\nu^{2/3}(-\beta_k)^{1/2}}|Ai'(a_k)|.$$

Since $\beta_k = a_k + \rho_k$, the last inequality can be written as

$$|Ai(\nu^{2/3}\zeta_{\nu,k})| \leqq \frac{0.0403}{\nu^{2/3}(-\beta_k)^{1/4}}\left(1 + \frac{\rho_k}{a_k}\right)^{-1/4}\frac{|Ai'(a_k)|}{|a_k|^{1/4}}.$$

Recall that $\rho_k \leq 0.241k^{-1/3} \leq 0.1119$ if $k \geq 10$, $a_{10} = -12.82878$ and $|Ai'(x)|/|x|^{1/4} \leq N(x)/|x|^{1/4} < 0.60$ if $x \leq -1$ (see (2.27) and (2.28)). Hence, $\left[1 + \rho_k/a_k\right]^{-1/4} \leq 1.0022$ and

$$|Ai(\nu^{2/3}\zeta_{\nu,k})| \leq \frac{0.0243}{\nu^{2/3}(-\beta_k)^{1/4}} \tag{6.6}$$

if $\nu \geq 10$ and $k \geq 10$. From (6.1), it follows that

$$|J_\nu(j''_{\nu,k})| \leqq \frac{1.2879}{\nu}\left\{\frac{0.0243}{(-\beta_k)^{1/4}} + \frac{0.2102}{\sqrt{\pi}(-\nu^{2/3}\zeta_{\nu,k})^{1/4}}\frac{e^{0.30/\nu}}{\nu^{1/3}}\right\}.$$

Since $-\nu^{2/3}\zeta_{\nu,k} \geq -\beta_k$, we obtain

$$|J_\nu(j''_{\nu,k})| \leq \frac{e_k}{\nu} \qquad \text{for } \nu \geq 10, \tag{6.7}$$

where

$$e_k = \frac{0.0811}{(-\beta_k)^{1/4}} \qquad \text{if } k \geq 10. \tag{6.8}$$

From (6.5), (6.8) and the fact that $-\beta_k \geq -\beta_{10} > -a_{10}$ for $k \geq 10$, it is evident that $e_{k+1} < e_k$ for all $k \geq 2$. This completes the proof of (1.2) with $\mu_k = e_k^2$.

7. **Asymptotic expansion of $I(\lambda)$.** It is well-known that

$$Ai(-x) = \frac{1}{2}\sqrt{\frac{x}{3}}\left[e^{i\pi/6}H^{(1)}_{1/3}(\xi) + e^{-i\pi/6}H^{(2)}_{1/3}(\xi)\right],$$

where $\xi = \frac{2}{3}x^{3/2}$ and $H^{(i)}_\nu(x)$, $i = 1, 2$, are the Hankel functions; see [1, p. 447]. Hence we may write

$$Ai^2(-x) = h_1(x) + h_2(x) + h_3(x) \tag{7.1}$$

with

$$\begin{aligned} h_1(x) &= \frac{x}{12}e^{i\pi/3}\left[H^{(1)}_{1/3}(\xi)\right]^2 \\ &= \frac{x}{12}e^{i\pi/3}\left\{[J^2_{1/3}(\xi) - Y^2_{1/3}(\xi)] + 2iJ_{1/3}(\xi)Y_{1/3}(\xi)\right\}, \\ h_2(x) &= \frac{x}{12}e^{-i\pi/3}\left[H^{(2)}_{1/3}(\xi)\right]^2 \\ &= \frac{x}{12}e^{-i\pi/3}\left\{[J^2_{1/3}(\xi) - Y^2_{1/3}(\xi)] - 2iJ_{1/3}(\xi)Y_{1/3}(\xi)\right\}, \end{aligned}$$

and

$$h_3(x) = \frac{x}{6}H^{(1)}_{1/3}(\xi)H^{(2)}_{1/3}(\xi) = \frac{x}{6}\left[J^2_{1/3}(\xi) + Y^2_{1/3}(\xi)\right].$$

The asymptotic expansion of $h_3(x)$ can be obtained from that of $J^2_\nu + Y^2_\nu$. More precisely, we have

$$h_3(x) \sim \frac{1}{2\pi}\sum_{s=0}^{\infty} 1\cdot 3\cdot 5\cdots(2s-1)\left(\frac{3}{2}\right)^{2s} A_s(\tfrac{1}{3})\,x^{-3s-1/2}, \tag{7.2}$$

where $A_0(\nu) = 1$ and

$$A_s(\nu) = \frac{(4\nu^2 - 1^2)(4\nu^2 - 3^2)\cdots\{4\nu^2 - (2s-1)^2\}}{s!\,8^s};$$

cf. [12, p. 342]. Furthermore, it is known that the remainder after n terms is of the same sign as, and numerically less than, the $(n+1)^{\text{th}}$ term. From the asymptotic expansions of the Hankel functions $H^{(1)}_\nu$ and $H^{(2)}_\nu$, we also have

$$h_1(x) \sim \frac{1}{4\pi}\exp\left\{i\left(\frac{4}{3}x^{3/2} - \frac{\pi}{2}\right)\right\}\sum_{s=0}^{\infty} i^s\left(\frac{3}{2}\right)^s \frac{C_s(\frac{1}{3})}{x^{(3s+1)/2}} \tag{7.3}$$

and

$$h_2(x) \sim \frac{1}{4\pi}\exp\left\{-i\left(\frac{4}{3}x^{3/2} - \frac{\pi}{2}\right)\right\}\sum_{s=0}^{\infty} (-i)^s\left(\frac{3}{2}\right)^s \frac{C_s(\frac{1}{3})}{x^{(3s+1)/2}} \tag{7.4}$$

where

$$C_s(\nu) = \sum_{\ell=0}^{s} A_\ell(\nu)A_{s-\ell}(\nu), \qquad \nu = \frac{1}{3}.$$

Bounds for the remainders associated with the expansions (7.3) and (7.4) can be constructed from those of the Hankel functions; see [12, pp. 266–269].

Inserting (7.1) in (1.4) gives

$$I(\lambda) = I_1(\lambda) + I_2(\lambda) + I_3(\lambda), \tag{7.5}$$

where

$$I_i(\lambda) = \int_0^\infty f(t) h_i(\lambda t)\, dt, \qquad i = 1, 2, 3. \tag{7.6}$$

Throughout this section we shall assume that $f(t)$ is an infinitely differentiable function on $(0, \infty)$ with an asymptotic expansion of the form

$$f(t) \sim \sum_{s=0}^{\infty} a_s t^{s+\alpha-1}, \qquad \text{as } t \longrightarrow 0^+, \tag{7.7}$$

where $0 < \alpha \leq 1$. We further assume that the asymptotic expansion of the derivatives of $f(t)$ can be obtained by termwise differentiation of (7.7), and that for each $j = 0, 1, 2, \ldots,$

$$f^{(j)}(t) = O(t^{-1-\varepsilon}), \qquad \text{as } t \longrightarrow \infty, \tag{7.8}$$

where ε is some fixed nonnegative number.

From (7.7) it follows that the Mellin transform of $f(t)$ defined by

$$M[f; z] = \int_0^\infty t^{z-1} f(t)\, dt, \qquad 1 - \alpha < \operatorname{Re} z < 1 + \varepsilon, \tag{7.9}$$

can be analytically continued to a meromorphic function in the half-plane $\operatorname{Re} z < 1 + \varepsilon$, with simple poles at $z = 1 - s - \alpha$ of residue a_s, $s = 0, 1, 2, \ldots$; see [14, p. 742] or [15, p. 425]. In this paper, the notation $M[f; z]$ is used to denote not only the integral in (7.9) but also its analytic continuation.

The Mellin transforms of $h_i(t)$ can be obtained from integral tables [3, p. 199, Eq. 23(1); p. 203, Eq. 32(1); p. 209, Eq. 45(1)], and we have

$$M[h_1; z] = -\frac{3^{s-2}}{4\pi^2} e^{i\pi s/2} \frac{1}{\Gamma(s)} \Gamma\left(\frac{s}{2} + \frac{1}{3}\right) \Gamma\left(\frac{s}{2} - \frac{1}{3}\right) \Gamma^2\left(\frac{s}{2}\right), \tag{7.10}$$

$$M[h_2; z] = -\frac{3^{s-2}}{4\pi^2} e^{-i\pi s/2} \frac{1}{\Gamma(s)} \Gamma\left(\frac{s}{2} + \frac{1}{3}\right) \Gamma\left(\frac{s}{2} - \frac{1}{3}\right) \Gamma^2\left(\frac{s}{2}\right), \tag{7.11}$$

$$M[h_3; z] = \frac{3^{s-2}}{\pi^2} \cos\left(\frac{\pi}{3}\right) \frac{\Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s}{2} + \frac{1}{3}\right) \Gamma\left(\frac{s}{2} - \frac{1}{3}\right)}{\Gamma\left(1 - \frac{s}{2}\right) \Gamma(s)} \frac{\pi}{\sin \pi s}, \tag{7.12}$$

where $s = \frac{2}{3}(z + 1)$.

We are now ready to apply the results in [14, 15]. For each $n \geq 1$, we set

$$f(t) = \sum_{s=0}^{n-1} a_s t^{s+\alpha-1} + f_n(t). \tag{7.13}$$

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By our assumption,

$$f_n^{(j)}(t) = O(t^{n+\alpha-j-1}), \qquad \text{as } t \longrightarrow 0^+,$$

for $j = 0, 1, 2, \ldots$. Similarly, we write (7.2) in the form

$$h_3(t) = \sum_{s=0}^{n-1} b_s t^{-s-1/2} + h_{3,n}(t) \tag{7.14}$$

with $b_{3s+1} = b_{3s+2} = 0$ and

$$b_{3s} = \frac{1 \cdot 3 \cdot 5 \cdots (2s-1)}{2\pi} \left(\frac{3}{2}\right)^{2s} A_s(\tfrac{1}{3}), \qquad s = 0, 1, 2, \ldots . \tag{7.15}$$

By an earlier remark, we also have

$$|h_{3,n}(t)| \leq |b_n|\, t^{-n-1/2} \qquad \text{for } t > 0, \text{ if } n = 0, 3, 6, \ldots . \tag{7.16}$$

If $\alpha \neq \frac{1}{2}$ then it follows from Theorem 1 in [14] that

$$I_3(\lambda) = \sum_{s=0}^{n-1} a_s M[h_3; s+\alpha]\lambda^{-s-\alpha} + \sum_{s=0}^{n-1} b_s M[f; 1-s-1/2]\lambda^{-s-1/2} + \delta_{3,n}(\lambda), \tag{7.17}$$

whereas if $\alpha = \frac{1}{2}$ then we obtain from Theorem 2 in [14]

$$I_3(\lambda) = \sum_{s=0}^{n-1} c_s \lambda^{-s-1/2} + (\ln \lambda) \sum_{s=0}^{n-1} a_s b_s \lambda^{-s-1/2} + \delta_{3,n}(\lambda), \tag{7.18}$$

where

$$c_s(\alpha) = a_s b_s^* + a_s^* b_s$$

$$a_s^* = \lim_{z \to s+1/2} \left\{ M[f; 1-z] + \frac{a_s}{z-s-1/2} \right\}$$

$$b_s^* = \lim_{z \to s+1/2} \left\{ M[h; z] + \frac{b_s}{z-s-1/2} \right\};$$

cf. [16, pp. 158–159]. In both cases the remainder is given by

$$\delta_{3,n}(\lambda) = \int_0^\infty f_n(t) h_{3,n}(\lambda t)\, dt. \tag{7.19}$$

Bounds for $\delta_{3,n}(\lambda)$ can also be found in [14] and [15]. In particular, if $\alpha > \frac{1}{2}$ then we have from (7.16)

$$|\delta_{3,n}(\lambda)| \leq \frac{|b_n|}{\lambda^{n+1/2}} \int_0^\infty |f_n(t)|\, t^{-n-1/2}\, dt, \qquad \text{if } n = 0, 3, 6, \ldots . \tag{7.20}$$

To the oscillatory integrals $I_1(\lambda)$ and $I_2(\lambda)$, we apply the result in [15, § 5], which gives

$$I_i(\lambda) = \sum_{s=0}^{n-1} a_s M[h_i; s+\alpha]\lambda^{-s-\alpha} + \delta_{i,n}(\lambda) \tag{7.21}$$

for $i = 1, 2$, where

$$\delta_{i,n}(\lambda) = \frac{(-1)^n}{\lambda^n} \int_0^\infty f_n^{(n)}(t) h_i^{(-n)}(\lambda t)\, dt, \tag{7.22}$$

and $h_i^{(-n)}(t)$ denotes an n^{th} iterated integral of $h_i(t)$. In the case of $h_1(t)$, we can write

$$h_1^{(-n)}(t) = \frac{(-1)^n}{(n-1)!} \int_t^{t+\infty e^{i\pi/3}} (w-t)^{n-1} h_1(w)\, dw.$$

On the path of integration, $w = t + \rho e^{i\pi/3}$ and ρ varies from 0 to ∞.

It is readily verified that

$$\operatorname{Im}(w^{3/2}) \geqq \left(\frac{\sqrt{3}}{2}\right)^{3/2} \rho^{3/2}.$$

In view of the well-known result [13, p. 219]

$$|H_{1/3}^{(1)}(\zeta)| \leqq \left| \sqrt{\frac{2}{\pi\zeta}}\, e^{i\zeta} \right|, \qquad 0 \leqq \arg\zeta \leqq \pi,$$

it follows that

$$|h_1(w)| \leqq \frac{1}{4\pi} t^{-1/2} \exp\left\{ -\frac{2^{1/2}}{3^{1/4}} \rho^{3/2} \right\}.$$

Consequently

$$|h_1^{(-n)}(t)| \leqq \frac{1}{(n-1)!} \frac{t^{-1/2}}{6\pi} \Gamma\left(\frac{2}{3}n\right) \left(\frac{\sqrt{3}}{2}\right)^{n/3}.$$

Similarly, we can write

$$h_2^{(-n)}(t) = \frac{(-1)^n}{(n-1)!} \int_t^{t+\infty e^{-i\pi/3}} (w-t)^{n-1} h_2(w)\, dw.$$

Using the estimate [13, p. 220]

$$|H_{1/3}^{(2)}(\zeta)| \leqq \left| \sqrt{\frac{2}{\pi\zeta}}\, e^{-i\zeta} \right|, \qquad -\pi \leqq \arg\zeta \leqq 0,$$

we have

$$|h_2^{(-n)}(t)| \leqq \frac{1}{(n-1)!} \frac{t^{-1/2}}{6\pi} \Gamma\left(\frac{2}{3}n\right) \left(\frac{\sqrt{3}}{2}\right)^{n/3}.$$

Thus, if $\frac{1}{2} < \alpha \leq 1$ then (7.22) gives

$$|\delta_{i,n}(\lambda)| \leq \frac{C_n}{\lambda^{n+1/2}} \int_0^\infty t^{-1/2} |f_n^{(n)}(t)|\, dt, \qquad i = 1, 2, \tag{7.23}$$

where

$$C_n = \frac{1}{(n-1)!\, 6\pi} \Gamma\left(\frac{2}{3}n\right) \left(\frac{\sqrt{3}}{2}\right)^{n/3}. \tag{7.24}$$

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8. **A special case.** We shall apply the results of the previous section to the integral

$$F_1(\nu) = \int_0^\infty \varphi^4(-\zeta) Ai^2(-\nu^{2/3}\zeta)\, d\zeta, \tag{8.1}$$

where $\varphi(\zeta)$ is given in (2.4). In the notations of § 7, we have $\lambda = \nu^{2/3}$,

$$f(t) = \varphi^4(-t) = 2^{4/3} - \frac{8}{5}t + \frac{12}{35}2^{2/3}t^2 - \cdots, \tag{8.2}$$

and $\alpha = 1$. The condition in (7.8) is readily verifiable. In fact, we have $f^{(j)}(t) = O(t^{-2-j})$ as $t \to +\infty$ for $j = 0, 1, 2, \ldots$. In (7.13) and (7.14), we shall take $n = 2$ and note that

$$a_0 = 2^{4/3}, \qquad a_1 = -\frac{8}{5}, \qquad b_0 = \frac{1}{2\pi}, \qquad b_1 = 0. \tag{8.3}$$

Using (7.10), (7.11) and (7.12), it is easily shown that

$$\begin{aligned} M[h_1; 1] + M[h_2; 1] + M[h_3; 1] &= -Ai'^2(0), \\ M[h_1; 2] + M[h_2; 2] + M[h_3; 2] &= -\frac{1}{3}Ai(0)Ai'(0). \end{aligned} \tag{8.4}$$

It is also easily verified that

$$M[f; 1/2] = \int_0^\infty \zeta^{-1/2}\varphi^4(-\zeta)\, d\zeta = 4\int_1^\infty \frac{1}{x\sqrt{x^2-1}}\, dx = 2\pi. \tag{8.5}$$

Hence a combination of (7.6), (7.17) and (7.21) gives

$$F_1(\nu) = \nu^{-1/3} - 2^{4/3}Ai'^2(0)\nu^{-2/3} + \frac{8}{15}Ai(0)Ai'(0)\nu^{-4/3} + \delta(\nu). \tag{8.6}$$

The remainder $\delta(\nu)$ is given by

$$\delta(\nu) = \delta_{1,2}(\lambda) + \delta_{2,2}(\lambda) + \delta_{3,2}(\lambda), \tag{8.7}$$

where $\lambda = \nu^{2/3}$, and $\delta_{3,2}$ and $\delta_{i,2}$, $i = 1, 2$, are as defined by (7.19) and (7.22) respectively.

By Taylor's theorem,

$$f(t) = 2^{4/3} - \frac{8}{5}t + f_2(t) \tag{8.8}$$

where

$$f_2(t) = \frac{1}{2}f''(\xi)t^2 = \frac{1}{2}(\varphi^4)''(-\xi)t^2, \qquad 0 < \xi < t. \tag{8.9}$$

It can be verified that $(\varphi^4)''(-\zeta)$ is a positive and decreasing function on $0 < \zeta < \infty$; see [5]. Hence, $0 < (\varphi^4)''(-\xi) \le (\varphi^4)''(0) = \frac{24}{35}2^{2/3} = 1.09$. Also, since $b_2 = 0$ in (7.14), we have $h_{3,2}(t) = h_{3,3}(t)$. In view of the remark following (7.2), $h_{3,2}(t)$ is negative in $0 < t < \infty$ and

$$|h_{3,2}(t)| \leqq \frac{5}{64\pi}t^{-3-1/2} \leqq \frac{5}{64\pi}t^{-2-1/2} \qquad \text{for } t \geqq 1.$$

Furthermore, since $h_3(t)$ is positive, we deduce from above that

$$\text{(8.10)}\quad \begin{aligned} \left|\int_0^{1/\lambda} f_2(t)h_{3,2}(\lambda t)\,dt\right| &\leqq \frac{1.09}{2}\int_0^{1/\lambda} t^2\left[\frac{1}{2\pi}-h_3(\lambda t)\right]dt \\ &\leqq \frac{0.545}{2\pi}\int_0^{1/\lambda} t^2\,dt \leqq \frac{0.029}{\lambda^3} \end{aligned}$$

and

$$\text{(8.11)}\quad \begin{aligned} \left|\int_{1/\lambda}^{\infty} f_2(t)h_{3,2}(\lambda t)\,dt\right| &\leqq \frac{5}{64\pi}\lambda^{-5/2}\int_{1/\lambda}^{\infty} f_2(t)\,t^{-5/2}\,dt \\ &\leqq \frac{5}{64\pi}\lambda^{-5/2}\int_0^{\infty} f_2(t)\,t^{-5/2}\,dt. \end{aligned}$$

From (8.9) we have

$$\int_0^1 f_2(t)\,t^{-5/2}\,dt \leqq \frac{1.09}{2}\int_0^1 t^{-1/2}\,dt = 1.09,$$

and from (8.5) we have

$$\text{(8.12)}\quad \int_1^{\infty} f(t)\,t^{-5/2}\,dt < \int_1^{\infty} f(t)\,t^{-1/2}\,dt < 2\pi.$$

Also, a straightforward calculation gives

$$\int_1^{\infty}\left[2^{4/3}-\frac{8}{5}t\right]t^{-5/2}\,dt = \frac{2}{3}2^{4/3}-\frac{16}{5} = -1.520.$$

Hence

$$0 < \int_0^{\infty} f_2(t)\,t^{-5/2}\,dt < 1.09 + (2\pi + 1.520) \leqq 8.894$$

and

$$\text{(8.13)}\quad \left|\int_{1/\lambda}^{\infty} f_2(t)h_{3,2}(\lambda t)\,dt\right| \leq \frac{0.222}{\lambda^{5/2}}.$$

Combining (7.19), (8.10) and (8.13), we obtain

$$\text{(8.14)}\quad |\delta_{3,2}(\lambda)| \leq \frac{0.029}{\lambda^3} + \frac{0.022}{\lambda^{5/2}}.$$

From (7.23), we have

$$\text{(8.15)}\quad |\delta_{i,2}(\lambda)| \leq \frac{0.044}{\lambda^{5/2}}\int_0^{\infty} t^{-1/2}|f_2^{(2)}(t)|\,dt.$$

Since $f_2^{(2)}(t) = f^{(2)}(t)$ and $0 \leq f^{(2)}(t) = (\varphi^4)''(-t) \leq \frac{24}{35}2^{2/3}$,

$$\text{(8.16)}\quad \int_0^1 t^{-1/2}|f_2^{(2)}(t)|\,dt \leq 2.178.$$

On the other hand, integration by parts gives

$$\text{(8.17)}\quad \begin{aligned} \int_1^{\infty} t^{-1/2}|f_2^{(2)}(t)|\,dt &= \int_1^{\infty} t^{-1/2}f^{(2)}(t)\,dt \\ &= \frac{3}{4}\int_1^{\infty} t^{-5/2}f(t)\,dt - f'(1) - \frac{1}{2}f(1). \end{aligned}$$

From (2.3) and (2.4), it is easily seen that $dx/d\zeta = x\varphi^2(\zeta)/2$. Straightforward differentiation yields

$$\frac{d}{d\zeta}\varphi^4(\zeta) = \frac{-4(x^2-1) - 4\zeta x^2\varphi^2(\zeta)}{(x^2-1)^2}.$$

Since $\varphi(-1) = 1.0821991971$ and $x(-1) = 1.9789626178$ (see [9, pp. 38 and 41]), $f(1) = \varphi^4(-1) = 1.371604273$ and $f'(1) = -(\varphi^4)'(-1) = -0.785580091$. Consequently, it follows from (8.17) and (8.12) that

$$\int_1^\infty t^{-1/2}|f_2^{(2)}(t)|\,dt \le \frac{3\pi}{2} + 0.0998 \le 4.813. \tag{8.18}$$

Coupling (8.16) and (8.18), we obtain

$$|\delta_{i,2}(\lambda)| \le \frac{0.31}{\lambda^{5/2}}, \qquad i = 1, 2. \tag{8.19}$$

A combination of (8.7), (8.14) and (8.19) gives

$$|\delta(\nu)| \le \frac{0.856}{\nu^{5/3}} \qquad \text{for } \nu \ge 10. \tag{8.20}$$

9. **Proof of (1.6).** We now turn to the integral in (1.5), and write

$$F(j''_{\nu,2}) = \int_{j''_{\nu,2}}^\infty \frac{J_\nu^2(t)}{t}\,dt. \tag{9.1}$$

In (9.1), we first make the change of variable $t = \nu x$ and replace $J_\nu(\nu x)$ by its asymptotic approximation (2.21). Next, we make ζ the variable of integration. Since $j''_{\nu,2} > j'_{\nu,1} > \nu$ (see[6, (2.4)] and [12, p. 246]), the point $x_{\nu,2} = j''_{\nu,2}/\nu$ is greater than 1 and its image $\zeta_{\nu,2}$ under the transformation (2.3) is negative. The final result is

$$F(j''_{\nu,2}) = \frac{1}{2\nu^{2/3}}\int_{\overline{\zeta}_\nu}^\infty \varphi^4(-\zeta)Ai^2(-\nu^{2/3}\zeta)\,d\zeta + \rho_1(\nu), \tag{9.2}$$

where $\overline{\zeta}_\nu = -\zeta_{\nu,2}$, $\varphi(\zeta)$ is the function defined by (2.4) and

$$\rho_1(\nu) = \frac{1}{2}\int_{\overline{\zeta}_\nu}^\infty \tilde{\varepsilon}(\nu,-\zeta)\varphi^2(-\zeta)\,d\zeta. \tag{9.3}$$

Since $\overline{\zeta}_\nu > 0$, it follows from (2.23) and (8.5) that

$$|\rho_1(\nu)| \le \frac{0.1458}{\nu^2}\int_0^\infty \varphi^4(-\zeta)\zeta^{-1/2}\,d\zeta \le \frac{0.917}{\nu^2}, \qquad \nu \ge 10. \tag{9.4}$$

For convenience, we set

$$F^*(\nu) = -\int_0^{\overline{\zeta}_\nu} \varphi^4(-\zeta)Ai^2(-\nu^{2/3}\zeta)\,d\zeta \tag{9.5}$$

so that we may write (9.2) as

$$F(j''_{\nu,2}) = \frac{1}{2\nu^{2/3}}\left[F_1(\nu) + F^*(\nu)\right] + \rho_1(\nu), \tag{9.6}$$

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where $F_1(\nu)$ is defined by (8.1).

In what follows we shall consider the integral in (9.5). From (5.12), we have

$$\bar{\zeta}_\nu = -\frac{a_2}{\nu^{2/3}} - \eta_2, \qquad \text{where } |\eta_2| \le \frac{0.40}{\nu^{4/3}}. \tag{9.7}$$

Hence we can write $F^*(\nu)$ in the form

$$F^*(\nu) = F_2(\nu) + \rho_2(\nu), \tag{9.8}$$

where

$$F_2(\nu) = -\frac{1}{\nu^{2/3}} \int_{a_2}^{0} \varphi^4(\nu^{-2/3}\tau) Ai^2(\tau)\, d\tau \tag{9.9}$$

and

$$\rho_2(\nu) = -\frac{1}{\nu^{2/3}} \int_{a_2+\nu^{2/3}\eta_2}^{a_2} \varphi^4(\nu^{-2/3}\tau) Ai^2(\tau)\, d\tau. \tag{9.10}$$

Since $0 \le \varphi(\zeta) \le \varphi(0) = 2^{1/3}$ and $|Ai(\zeta)| \le 0.53566$ for $-\infty < \zeta \le 0$ (see §5 and [1, pp. 446 and 478]), we have from (9.7)

$$|\rho_2(\nu)| \le \frac{0.290}{\nu^{4/3}} \qquad \text{for } \nu \ge 10. \tag{9.11}$$

To evaluate the integral in (9.9), we use the Taylor expansion

$$\varphi^4(\zeta) = 2^{4/3} + \frac{8}{5}\zeta + R_2(\zeta) \tag{9.12}$$

where

$$R_2(\zeta) = \frac{\zeta^2}{2!}(\varphi^4)''(\xi), \qquad \zeta < \xi < 0. \tag{9.13}$$

Since $(\varphi^4)''(\xi)$ is an increasing function in $(-\infty, 0]$ (see[5]), $0 < (\varphi^4)''(\xi) \le (\varphi^4)''(0) = \frac{24}{35} 2^{2/3} = 1.09$ for $-\infty < \zeta < 0$. Thus, it follows that

$$|R_2(\zeta)| \le 0.55\zeta^2, \qquad -\infty < \zeta < 0. \tag{9.14}$$

Using the fact that $Ai(z)$ satisfies the differential equation $w'' - zw = 0$, we have by integration by parts

$$\int Ai^2(z)\, dz = zAi^2(z) - Ai'^2(z) \equiv M_0(z),$$

$$\int zAi^2(z)\, dz = \frac{1}{3}[zM_0(z) + Ai(z)Ai'(z)] \equiv M_1(z),$$

$$\int z^2 Ai^2(z)\, dz = \frac{1}{5}[3zM_1(z) + zAi(z)Ai'(z) - Ai^2(z)] \equiv M_2(z),$$

from which it follows that

$$M_0(0) = -Ai'^2(0), \qquad M_0(a_2) = -Ai'^2(a_2),$$
$$M_1(0) = \frac{1}{3}Ai(0)Ai'(0), \qquad M_1(a_2) = -\frac{1}{3}a_2Ai'^2(a_2),$$
$$M_2(0) = -\frac{1}{5}Ai^2(0), \qquad M_2(a_2) = -\frac{1}{5}a_2^2Ai'^2(a_2).$$

Consequently we obtain

$$\begin{aligned} F_2(\nu) = &\frac{2^{4/3}}{\nu^{2/3}}[Ai'^2(0) - Ai'^2(a_2)] \\ &- \frac{8}{15\nu^{4/3}}[Ai(0)Ai'(0) + a_2Ai'^2(a_2)] + \rho_3(\nu), \end{aligned} \tag{9.15}$$

where

$$|\rho_3(\nu)| \leq \frac{0.11}{\nu^2}\left[a_2^2Ai'^2(a_2) - Ai^2(0)\right]. \tag{9.16}$$

Numerical computation gives $a_2^2Ai'^2(a_2) - Ai^2(0) = 10.6526$; see[1, pp. 476 and 478]. Hence

$$|\rho_3(\nu)| \leq \frac{1.172}{\nu^2}. \tag{9.17}$$

Coupling (9.6) and (9.8), we have

$$F(j''_{\nu,2}) = \frac{1}{2\nu^{2/3}}[F_1(\nu) + F_2(\nu) + \rho_2(\nu)] + \rho_1(\nu). \tag{9.18}$$

Inserting (8.6) and (9.15) in (9.18) gives

$$F(j''_{\nu,2}) = \frac{1}{2\nu} - \frac{2^{1/3}Ai'^2(a_2)}{\nu^{4/3}} - \frac{4}{15}a_2Ai'^2(a_2)\frac{1}{\nu^2} + \rho(\nu), \tag{9.19}$$

where $\rho(\nu) = \rho_1(\nu) + [\delta(\nu) + \rho_2(\nu) + \rho_3(\nu)]/2\nu^{2/3}$. From (8.20), (9.4), (9.11) and (9.17), it follows that

$$|\rho(\nu)| \leqq \frac{1.376}{\nu^2}.$$

The approximation formula (1.6) is obtained from (9.19) with

$$\frac{\varepsilon(\nu)}{\nu^2} = -\frac{4}{15}a_2Ai'^2(a_2)\frac{1}{\nu^2} + \rho(\nu).$$

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