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RESEARCH ARTICLE

An extension theorem for weak solutions of the 3d incompressible Euler equations and applications to singular flows

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Abstract

We prove an extension theorem for local solutions of the 3d incompressible Euler equations. More precisely, we show that if a smooth vector field satisfies the Euler equations in a spacetime region $\Omega \times (0,T)$, one can choose an admissible weak solution on $\mathbb{R}^3 \times (0,T)$ of class C^β for any $\beta < 1/3$ such that both fields coincide on $\Omega \times (0,T)$. Moreover, one controls the spatial support of the global solution. Our proof makes use of a new extension theorem for local subsolutions of the incompressible Euler equations and a $C^{1/3}$ convex integration scheme implemented in the context of weak solutions with compact support in space. We present two nontrivial applications of these ideas. First, we construct infinitely many admissible weak solutions of class C^β_{loc} with the same vortex sheet initial data, which coincide with it at each time t outside a turbulent region of width $O(t)$. Second, given any smooth solution v of the Euler equation on $\mathbb{T}^3 \times (0,T)$ and any open set $U \subset \mathbb{T}^3$, we construct admissible weak solutions which coincide with v outside U and are uniformly close to it everywhere at time 0, yet blow up dramatically on a subset of $U \times (0,T)$ of full Hausdorff dimension. These solutions are of class C^β outside their singular set.

Contents

1. Introduction

Convex integration methods, introduced by Nash [40] in the context of the C^1 isometric embedding problem and subsequently refined by Gromov in his work on flexible geometric PDEs and by Müller and Šverák [39] in their theory of differential inclusions, have experimented an extraordinary development in connection with the study of weak solutions of the incompressible Euler equations. This system reads:

$$\partial_t v + \operatorname{div}(v \otimes v) + \nabla p = 0\,, \qquad \operatorname{div} v = 0\,,$$

where the time-dependent vector field v is the velocity of the fluid and the scalar function p is the hydrodynamic pressure. One typically considers the Euler equations either on the whole space $\mathbb{R}^3$, or on the torus $\mathbb{T}^3 := (\mathbb{R}/\mathbb{Z})^3$, or on a bounded domain $\Omega \subset \mathbb{R}^3$ with smooth boundary (where additional complications may arise).

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The motivation to consider weak solutions in this setting is twofold. First, the 3d Euler equations are expected to dynamically produce singularities from smooth initial conditions [34, 51]. Second, weak solutions are necessary to describe some of the phenomena that appear in turbulence, such as the energy dissipation in nonsmooth Euler flows famously conjectured by Onsager in 1949 [42]. Roughly speaking, Onsager's conjecture asserts that weak solutions that are Hölder continuous in space with exponent greater than $1/3$ must conserve energy, while for any smaller exponent there should be weak solutions that do not.

The rigidity part of Onsager's conjecture was proved by Constantin, E and Titi [18] after a partial result of Eyink [29]. The endpoint case was addressed in [13]. Concerning the flexible part of the conjecture, following the construction of L^2 solutions with compact support in space and time due to Scheffer [44] and Shnirelman [46], a systematic approach was introduced in the seminal work of De Lellis and Székelyhidi, who introduced L^∞-convex integration and C^0-Nash iteration schemes in this setting [23, 24]. After a series of significant intermediate results [21, 6, 36], the flexible part of Onsager's conjecture was finally established by Isett [36], and further refined by Buckmaster, De Lellis, Székelyhidi, and Vicol [7] to construct solutions for which the kinetic energy is strictly decreasing. In addition to the classical Hölder-based approach, the so-called intermittent L^p-based flavor of convex integration, introduced by Buckmaster and Vicol [10] to prove the nonuniqueness of weak solutions of the 3d Navier–Stokes equations, has also attracted much attention, as it can effectively capture new aspects of Kolmogorov's theory of turbulence. For detailed expositions of these and other results on various models in fluid mechanics, we refer the reader to the surveys [11, 22] and the papers [1, 8, 9, 12, 15, 14, 16, 30, 31, 41, 47].

A key property of the solutions that one constructs using convex integration techniques is their *flexibility*. This refers to the fact that, at a certain regularity level, the equations are no longer predictive: there exist infinitely many solutions, in stark contrast to what happens in the case of smooth solutions. Three possible formulations of this property are as follows; as discussed in [41, Remark 1.3], once one of them has been established within a certain functional framework, it is usually straightforward to pass to another formulation using techniques that are now standard.

Restricting to the case of $\mathbb{T}^3$ for concreteness, let us denote by $\mathcal{V} \subset L^2(\mathbb{T}^3)$ some suitable function space, which in our case will be some Hölder space $C^\beta(\mathbb{T}^3)$. Three standard ways of stating the flexibility of weak solutions in this regularity class are as follows:

1. *Solutions of compact time support:* Given any positive constants E, T, there exists a weak solution $v \in C([-T,T],\mathcal{V})$ whose time support is contained in $(-T,T)$ and such that $\|v(0)\|_{L^2(\mathbb{T}^3)} > E$.
2. *Solutions with fixed energy profile:* Given any smooth positive function $e : [0,T] \to (0,\infty)$, there exists a weak solution $v \in C([0,T],\mathcal{V})$ such that $\|v(t)\|_{L^2(\mathbb{T}^3)} = e(t)$ for all $t \in [0,T]$.
3. *Arbitrary initial and final states:* Given any divergence-free vector fields $v_{\mathrm{start}}, v_{\mathrm{end}} \in \mathcal{V}$ with the same mean, any $T > 0$ and any $\epsilon > 0$, there exists a weak solution $v \in C([0,T],\mathcal{V})$ such that

$$\|v(0) - v_{\mathrm{start}}\|_{L^2(\mathbb{T}^3)} + \|v(T) - v_{\mathrm{end}}\|_{L^2(\mathbb{T}^3)} < \epsilon\,. \tag{1.1}$$

(If $v_{\mathrm{start}}, v_{\mathrm{end}}$ are smooth, one can take $\epsilon = 0$ by gluing in time.)

1.1. Main result

Our objective in this paper is to prove an extension theorem for local solutions of the 3d incompressible Euler equations. Roughly speaking, we prove that if a smooth vector field satisfies the Euler equations in a spacetime region $\Omega \times (0,T)$ (so it is a "local" solution of Euler), one can choose a weak solution on $\mathbb{R}^3 \times (0,\infty)$ of class C^β for any $\beta < 1/3$ (which is the sharp Hölder regularity) such that both fields coincide on $\Omega \times (0,T)$. Moreover, one controls the spatial support of the "global solution" which extends the local one.

This property is very different from the approximation theorems that one can prove for smooth solutions of various classes of linear PDEs [26, 27, 25, 28], and also from the fact (often known as

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h-principle) that weak solutions of certain regularity can approximate, in Sobolev spaces of negative index, any given subsolution of the Euler equations.

Before presenting this result, let us recall the definition of weak solution (which, as we will be dealing with continuous functions exclusively, is just the distributional one). More precisely, given some $T \in (0, \infty]$ and some open set $\Omega \subseteq \mathbb{R}^3$ with smooth boundary, we will say that a vector field $v \in C(\Omega \times [0, T), \mathbb{R}^3)$ is a *weak solution* of the Euler equations on $\Omega \times (0, T)$ if

$$\int_0^T \int_\Omega (\partial_t \varphi \cdot v + \nabla\varphi : (v \otimes v)) dx dt = 0$$

for all divergence-free $\varphi \in C_c^\infty(\Omega \times (0, T), \mathbb{R}^3)$, and $\operatorname{div} v = 0$ in the sense of distributions.

The main result of this paper can then be stated as follows:

Theorem 1.1. *Fix some $T > 0$ and a bounded open set $\Omega \subset \mathbb{R}^3$ with smooth boundary and with a finite number of connected components. Assume that $v_0 \in C^\infty(\overline{\Omega} \times [0, T], \mathbb{R}^3)$ is a solution of the Euler equations on the spacetime region $\Omega \times (0, T)$. Then, for any $0 < \beta < 1/3$, there exists an admissible weak solution $v \in C^\beta(\mathbb{R}^3 \times [0, T])$ of the Euler equations such that $v|_{\overline{\Omega}\times[0,T]} = v_0$ if and only if*

$$\int_\Sigma v_0 \cdot \nu = \int_\Sigma [(a \cdot x)\partial_t v_0 + (a \cdot v_0)v_0 + p_0 a] \cdot \nu = 0 \tag{1.2}$$

for all $a \in \mathbb{R}^3$, all $t \in [0, T]$ and all connected components Σ of $\partial\Omega$. These conditions are automatically satisfied if $\partial\Omega$ is connected. Furthermore, there exists $e_0 > 0$ such that we may prescribe any energy profile $e \in C^\infty([0, T], [e_0, +\infty))$, that is, $\|v(t)\|_{L^2(\mathbb{R}^3)} = e(t)$. In addition, given any open set $\Omega' \supset \overline{\Omega}$, one can in fact assume that the spatial support of v is contained in this region.

Remark 1.2. In fact, one can obtain a global weak solution $v \in C^\beta(\mathbb{R}^3 \times [0, +\infty))$ such that $v|_{\mathbb{R}^3\times[0,T]}$ satisfies the claimed properties. Its temporal support may be assumed to be contained in $[0, T']$ for any $T' > T$. We cannot then prescribe the energy profile for $t > T$, but we can still choose v so that it remains admissible, that is,

$$\int_{\mathbb{R}^3} |v(x, t)|^2 dx \le \int_{\mathbb{R}^3} |v(x, 0)|^2 dx \qquad \forall t \in [0, +\infty).$$

Remark 1.3. It can be proved [17] that the pressure function $p := -\Delta^{-1} \operatorname{div}\operatorname{div}(v \otimes v)$ associated to this weak solution is in $L_t^\infty C_x^{2\beta} \cap C_{xt}^{2\beta-\delta}$ for any $\delta > 0$.

Before moving on to discuss some applications, let us provide some intuition about the compatibility conditions (1.2). When $\partial\Omega$ is connected, it is easy to see that any smooth Euler flow v_0 on Ω satisfies this condition. Indeed, these two conditions are respectively obtained by integrating over the domain Ω the incompressibility condition $\operatorname{div} v_0 = 0$ and the projected Euler equation $a\cdot(\partial_t v_0 + \operatorname{div}(v_0 \otimes v_0) + \nabla p_0) = 0$. If v_0 is the restriction to Ω of a global Euler flow, one can refine the argument to show that these conditions must hold on each connected component Σ of the boundary $\partial\Omega$, and not every field satisfying the Euler equations on Ω will satisfy them. Details are given in Lemma 2.11.

1.2. Applications

We shall next present two applications of the above extension result to the analysis of weak solutions of the 3d Euler equations. These applications do not follow directly from our main theorem, but they use it in an essential way.

Specifically, for these applications we consider subsolutions that are not smooth up to the endpoints of the interval $(0, T)$, which implies a lack of uniform-in-time bounds. Thus the scheme does not work as is because the available bounds are not uniform, but we will show in Section 9 that one can tweak the construction in many interesting situations.

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The first application we consider concerns the case of the standard vortex sheet u_0, which we can define as the periodic extension to $\mathbb{T}^3$ of:

$$u_0(x) := \begin{cases} +e_1 & \text{if } x_3 \in \left[0, \frac{1}{4}\right] \cup \left[\frac{3}{4}, 1\right], \\ -e_1 & \text{if } x_3 \in \left(\frac{1}{4}, \frac{3}{4}\right). \end{cases}$$

It follows from the classical local existence results and from the weak-strong uniqueness property [5, 52] that wild initial data must be somewhat irregular. However, until the publication of [49] it was not known how irregular they must be. In that paper it was proved that the vortex sheet u_0 is a wild initial data but the constructed solutions are only in L^∞. Results for nonflat vortex sheets have been recently established in [38].

One can use a suitable modification of our main theorem to extend this result to solutions of class C^β_{loc}:

Theorem 1.4. *Let $0 < \beta < 1/3$ and let $T > 0$. There exist infinitely many admissible weak solutions of the Euler equations $v \in C^\beta_{\text{loc}}(\mathbb{T}^3 \times (0, T))$ with initial datum u_0. For all $t \in (0, T)$, $v(x, t)$ coincides with $u_0(x)$ outside a "turbulent" zone of size $O(t)$.*

The second application we will present concerns the existence of a wealth of reasonably well behaved solutions that blow up on a set of maximal Hausdorff dimension. To make this precise, let us say that a point (x_0, t_0) in spacetime is in the *singular set* of v, which we will denote by $\mathcal{S}^\infty_v$, if $v \notin L^\infty((t_0 - \delta, t_0 + \delta) \times B)$ for any ball $B \ni x_0$ and any $\delta > 0$. More generally, the *q-singular set* of v, $\mathcal{S}^q_v$, consists of the spacetime points (x_0, t_0) such that $v \notin L^\infty((t_0 - \delta, t_0 + \delta), L^q(B))$ for any ball B and any $\delta > 0$ as above. Clearly $\mathcal{S}^q_v \subset \mathcal{S}^{q'}_v$ if $q < q'$ and $\mathcal{S}^q_v$ is a closed set.

We are now ready to state the result. Basically, the theorem says that, given any smooth solution v_0 on $\Omega \times (0, T)$ and any open set $U \subset \Omega$, there is an admissible weak solution v which coincides with v_0 outside U and which is uniformly close to v_0 at time 0, yet blows up dramatically on a subset of $U \times (0, T)$ of full dimension. Interestingly, smooth stationary Euler flows with compact support [32, 19] are very useful as building blocks in the construction of these solutions.

Theorem 1.5. *Consider some $0 < \beta < 1/3$ and some $q > 2$. Let $T > 0$ and let Ω be $\mathbb{T}^3$ or an open subset of $\mathbb{R}^3$. Fix some open set U whose closure is contained in Ω. Let v_0 be a smooth solution of the Euler equations in $\Omega \times (0, T)$. For any $\varepsilon > 0$ there exists a weak solution $v \in L^2(\Omega \times (0, T))$ of the Euler equations whose q-singular set $\mathcal{S}^q_v$ is contained in $U \times (0, T]$ and has Hausdorff dimension 4. Furthermore, v coincides with v_0 on $(\Omega \backslash U) \times [0, T]$ and satisfies*

$$\|v(\cdot, 0) - v_0(\cdot, 0)\|_{C^0(\Omega)} < \varepsilon.$$

Moreover, $v \in C^\beta_{loc}((\Omega \times [0, T]) \backslash \mathcal{S}^q_v)$ and the energy profile $\int_\Omega |v|^2\, dx$ can be chosen to be nonincreasing.

1.3. Strategy of the proof

We prove Theorem 1.1 in two stages: first we extend the field to $\mathbb{R}^3 \times [0, T]$ as a smooth subsolution (see Definition 2.1 in the main text), and then we use a Nash iteration to perturb it into a weak solution. These stages are interrelated in that tools and ideas that we develop to manipulate subsolutions also play a fundamental role in our convex integration scheme.

Concerning the extension of a local smooth solution of the Euler equations as a subsolution, the key result we prove is the following. In view of future applications of this result, which will appear elsewhere, we are stating these results for the Euler equations in any spatial dimension $n \geq 2$.

Theorem 1.6. *Let $\Omega_0 \subset \mathbb{R}^n$, $n \geq 2$ be a bounded open set with smooth boundary and finitely many connected components and let $I \subset \mathbb{R}$ be a closed and bounded interval. Let $(v_0, p_0, \mathring{R}_0) \in C^\infty(\overline{\Omega}_0 \times I)$ be a subsolution in $\Omega_0 \times I$. Let Ω be a neighborhood of $\overline{\Omega}_0$. There exists a subsolution $(v, p, \mathring{R}) \in C^\infty(\mathbb{R}^n \times I)$*

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that extends $(v_0, p_0, \mathring{R}_0)$ *and such that* $\operatorname{supp}(v, p, \mathring{R})(\cdot, t) \subset \Omega$ *for all* $t \in I$ *if and only if for each connected component* Σ *of* $\partial\Omega_0$ *and all times* $t \in I$ *we have*

$$\int_\Sigma v_0 \cdot \nu = \int_\Sigma \left[(a \cdot x)\partial_t v_0 + (a \cdot v_0)v_0 + p_0 a - a^t \mathring{R}_0\right] \cdot \nu = 0 \qquad \forall a \in \mathbb{R}^n.$$

These conditions are automatically satisfied if $\partial\Omega_0$ *is connected.*

Regarding the convex integration scheme, we start off with the strategy from [7], which we implement in the context of solutions with compact support. The main issue we have to address is that, as we want the resulting solution to coincide with v_0 in $\Omega \times [0, T]$, we must ensure that the scheme does not modify the subsolution in that region.

The result of our construction is:

Theorem 1.7. *Fix some* $T > 0$ *and let* $\Omega \subset \mathbb{R}^3$ *be a bounded open set with smooth boundary and with a finite number of connected components. Let* $(v_0, p_0, \mathring{R}_0) \in C^\infty(\mathbb{R}^3 \times [0, T])$ *be a subsolution of the Euler equations. Suppose that* $\operatorname{supp} \mathring{R}_0 \subset \overline{\Omega} \times [0, T]$. *Let* $0 < \beta < 1/3$ *and let* $e \in C^\infty([0, T], (0, \infty))$ *be an energy profile such that*

$$e(t) > \int_\Omega |v_0(x, t)|^2 dx + 6\|\mathring{R}_0\|_{L^\infty} |\Omega| \tag{1.3}$$

for all $0 \le t \le T$. *Then, there exists a weak solution of the Euler equations,* $v \in C^\beta_c(\mathbb{R}^3 \times [0, \infty))$, *such that* $v = v_0$ *in* $(\mathbb{R}^3 \backslash \Omega) \times [0, T]$ *and*

$$\int_\Omega |v(x, t)|^2 dx = e(t)$$

for all $t \in [0, T]$.

1.4. Organization of the paper

In Section 2 we develop a set of tools to handle the construction, extension, and gluing of subsolutions that will be used throughout the paper; in particular we prove Theorem 1.6. In Section 3 we present the iterative process used to prove Theorem 1.7, which is carried out in a number of stages. The technical details of each stage of the construction are discussed in detail in Sections 4 to 7. The very short Section 8 shows how to pass from Theorems 1.6 and 1.7 to Theorem 1.1 and Remark 1.2. The modification of this scheme to account for the lack of uniform-in-time bounds is carried out in Section 9. The applications concerning vortex sheets and blowup, cf. Theorem 1.4 and Theorem 1.5, are discussed in Sections 10 and 11, respectively. The paper concludes with two appendices, one about Hölder norms and another with some auxiliary estimates.

2. Extension of subsolutions

The goal of this section is to prove the extension theorem of smooth subsolutions stated in Theorem 1.6. This is a key ingredient to prove our main theorem on the extension of weak solutions of the Euler equations. In Subsection 2.1 we sketch the strategy to prove Theorem 1.6. Some instrumental tools from Hodge theory are presented in Subsection 2.2, and in Subsection 2.3 we show how to construct compactly supported solutions to the key divergence equation. Finally, the proof of Theorem 1.6 is completed in Subsection 2.4.

In addition to constructing the desired solutions, we must estimate their derivatives. We refer to Appendix A for the definition of the Hölder norms used. Specifically, we warn the reader that when dealing with time-dependent functions, we consider the supremum in time of the corresponding norms in

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space. Nevertheless, obtaining bounds on the derivatives of the solutions to certain differential equations is not enough for our construction. As we will see, we also need to control their C^0 norm. This can be achieved if we work with Besov spaces, which are defined in Appendix A.

Throughout this section, we denote the space of $n \times n$ symmetric matrices as $\mathcal{S}^n$ and the space of $n \times n$ skew-symmetric matrices as $\mathcal{A}^n$. Unless otherwise stated, the dimension is $n \geq 2$. We define the divergence of a matrix $M \in C^\infty(\mathbb{R}^n \times \mathbb{R}, \mathbb{R}^{n\times n})$ as the vector field whose coordinates are given by

$$(\operatorname{div} M)_i := \sum_{j=1}^n \partial_j M_{ij},$$

where the derivatives are taken only with respect to the spatial variables. More generally, partial derivatives with Latin subscripts denote partial derivatives in the spatial coordinates, whereas temporal partial derivatives are always denoted by ∂_t.

We will repeatedly use Einstein's summation convention: when an index appears twice in an expression, it is implicitly summed over its range. Indices that appear only once in an expression are free indices and are not summed over.

Let us now recall the definition of subsolution of the Euler equations:

Definition 2.1. Let $V \subset \mathbb{R}^n \times \mathbb{R}$ be an open set. We will say that a triplet $(v, p, \mathring{R}) \in C^\infty(V, \mathbb{R}^n \times \mathbb{R} \times \mathcal{S}^n)$ is a *subsolution* of the Euler equations if

$$\begin{cases} \partial_t v + v \cdot \nabla v + \nabla p = \operatorname{div} \mathring{R}, \\ \operatorname{div} v = 0. \end{cases} \tag{2.1}$$

The symmetric matrix $\mathring{R}$ is known as the *Reynolds stress* and it measures the deviation from being a solution of the Euler equations. It is customary to also impose that

$$\operatorname{tr} \mathring{R} = 0. \tag{2.2}$$

All along this article, $\circ$ above a symmetric matrix will indicate that it is trace-free.

Finally, let us fix some notation that will be used all along this section. We introduce the following norms in the space of $n \times n$ matrices:

$$|M| := \max_{\zeta \in \mathbb{S}^{n-1}} |M\zeta|, \qquad |M|_2 := \left(\sum_{i,j=1}^n M_{ij}^2 \right)^{1/2}. \tag{2.3}$$

Unless otherwise stated, we will always use the operator norm $|\cdot|$. However, in some parts of the article we will exploit the elementary property that $|\cdot|_2^2$ depends smoothly on the matrix entries. Note that $|M|_2^2 = \operatorname{tr}(M^t M)$, which is invariant under orthogonal transformations. Hence, in the case of a symmetric matrix $S \in \mathcal{S}^n$, we have

$$|S| \leq |S|_2, \tag{2.4}$$

which can be easily deduced by using an orthonormal basis of eigenvectors.

2.1. General strategy

Our techniques for extending subsolutions and performing convex integration in the nonperiodic setting rely on obtaining compactly supported solutions to the (matrix) divergence equation when the source term is compactly supported. Let us illustrate the key ideas behind our method with the following toy problem:

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Problem 2.2. *Given $\rho \in C_c^\infty(\mathbb{R}^3)$ such that $\int \rho = 0$, find $v \in C_c^\infty(\mathbb{R}^3, \mathbb{R}^3)$ such that* $\operatorname{div} v = \rho$.

It is easy to see that $v_0 = \nabla \Delta^{-1}\rho$ solves our equation, but in general it is not compactly supported. To fix this, let B be a ball containing the support of ρ, so that v_0 is divergence-free outside B. In addition, it follows from the divergence theorem that

$$0 = \int_B \rho = \int_B \operatorname{div} v_0 = \int_{\partial B} v_0 \cdot \nu.$$

This ensures that in $\mathbb{R}^3 \backslash B$ the divergence-free field v_0 can be written as $v_0 = \operatorname{curl} w_0$ for some smooth field w_0. We extend w_0 to a smooth field $w \in C^\infty(\mathbb{R}^3, \mathbb{R}^3)$ and we define

$$v := v_0 - \operatorname{curl} w.$$

Since w extends w_0, we see that v vanishes outside B. Furthermore, $\operatorname{div} v = \rho$ because $\operatorname{div}\operatorname{curl} \equiv 0$. Therefore, v is the sought field, which is clearly not unique.

Our approach to solving the divergence equation in the matrix case is the same: the potential-theoretic solution of the equation is not compactly supported. However, far from the support of the source our matrix will be the image of certain differential operator $\mathscr{L}$ applied to a smooth potential, which is in the kernel of the divergence. We will extend the potential to the whole space and then subtract it from the potential-theoretic solution, obtaining a compactly supported solution.

Just like in the vector case, we will have to impose certain integrability conditions on the source term for this to be possible. As we will see, these conditions are related to the classical conservation laws of linear and angular momentum in the Euler equations.

A totally different method to construct compactly supported solutions to the divergence equation in the (symmetric) matrix case was developed by Isett and Oh in [36]. Their theorem is stated in a very different setting and adapting it to what we need would require certain work. On the other hand, it will be relatively easy to deduce our result as a consequence of our analysis of the operator $\mathscr{L}$ introduced below, which is necessary for our result on the extension of subsolutions. Hence, we have preferred to take this path, which we believe is simpler (partly because it has a nice interpretation in terms of the elementary operations of vector calculus).

2.2. Basic tools

The tools that we will need come from the Hodge decomposition theorem for manifolds with boundary. A good reference for this topic is [45]. Nevertheless, we do not need the full generality of these results, as we will work in bounded domains of $\mathbb{R}^n$. Let us summarize the notation and main definitions that we will need.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. We denote by Λ^k the vector space of skew-symmetric k-forms over $\mathbb{R}^n$, for $0 \leq k \leq n$. In this setting, differential k-forms are maps $\omega \in C^\infty(\overline{\Omega}, \Lambda^k)$. They form vector spaces in which we have two differential operators: the exterior derivative $d : C^\infty(\overline{\Omega}, \Lambda^k) \to C^\infty(\overline{\Omega}, \Lambda^{k+1})$ and the codifferential $\delta : C^\infty(\overline{\Omega}, \Lambda^{k+1}) \to C^\infty(\overline{\Omega}, \Lambda^k)$. The Euclidean product induces a natural scalar product $(\cdot,\cdot)$ in $C^\infty(\overline{\Omega}, \Lambda^k)$. The tangential part of a differential form is $\mathbf{t}\omega := j^*\omega$, where $j : \partial\Omega \hookrightarrow \overline{\Omega}$ is the natural inclusion, and j^* is the pushforward. We define the Dirichlet harmonic k-forms as:

$$\mathcal{H}_D^k(\overline{\Omega}) := \{\omega \in C^\infty(\overline{\Omega}, \Lambda^k) : d\omega = 0, \delta\omega = 0, \mathbf{t}\omega = 0\}.$$

Finally, to obtain quantitative estimates we will need to work in Hölder spaces. We refer to Appendix A for the definition of these norms. We also recommend to take a look at the appendix to check our convention of Hölder norms when the field is time-dependent.

With this notation, the first basic lemma that we shall use is:

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Lemma 2.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary and let $\rho \in C^\infty(\overline{\Omega}, \Lambda^k)$. The boundary value problem*

$$(P)\begin{cases}\delta\omega = \rho,\\ d\omega = 0,\\ \mathbf{t}\omega = 0,\\ (\omega,\lambda) = 0 \quad \forall \lambda \in \mathcal{H}_D^{k+1}(\overline{\Omega})\end{cases}$$

is solvable if and only if

$$\delta\rho = 0 \qquad \text{and} \qquad \int_{C_{n-k}} \star\rho = 0 \quad \forall\ (n-k)\text{-cycle } C_{n-k}.$$

In that case, the solution is unique and we have

$$\|\omega\|_{C^{N+1+\alpha}(\Omega)} \leq C\,\|\rho\|_{C^{N+\alpha}(\Omega)}$$

for any $N \geq 0$, $\alpha \in (0,1)$ and certain constants $C \equiv C(N,\alpha,\Omega)$.

For a proof, see [20, Theorem 7.2] and [45, Corollary 3.2.4]. In [45, Theorem 3.2.5] we can find the general case of a Riemannian manifold with boundary, but it does not include estimates for Hölder norms, only for Sobolev norms. We recall that, as usual, $\star$ is the Hodge star operator acting on differential forms and an $(n-k)$-cycle is an $(n-k)$-chain (in the sense of algebraic topology) whose boundary is zero.

We are mainly interested in the problem $\delta\omega = \rho$, but we have to add the other conditions to select a single solution. This allows us to obtain a time-dependent version of Lemma 2.3:

Lemma 2.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary and let $I \subset \mathbb{R}$ be a closed and bounded interval. Let $\rho \in C^\infty(\overline{\Omega}\times I, \Lambda^k)$. There exists a differential form $\omega \in C^\infty(\overline{\Omega}\times I, \Lambda^{k+1})$ solving the boundary value problem (P) at each $t \in I$ if and only if*

$$\delta\rho = 0 \qquad \text{and} \qquad \int_{C_{n-k}} \star\rho = 0 \quad \forall\ (n-k)\text{-cycle } C_{n-k}, \forall t \in I.$$

In that case, the solution is unique and we have

$$\|\omega\|_{N+1+\alpha} \leq C\,\|\rho\|_{N+\alpha}$$

for any $N \geq 0$, $\alpha \in (0,1)$ and certain constants $C \equiv C(N,\alpha,\Omega)$.

Proof. Given a time-dependent differential form, we denote by a subscript the differential form at a given time. By Lemma 2.3, the necessity of the conditions is clear. To prove that they are also sufficient, let us suppose that ρ_t satisfies the conditions of Lemma 2.3 at all times $t \in I$. Hence, applying Lemma 2.3 at each time, we see that there exists a time-dependent $(k+1)$-form ω solving (P) at each $t \in I$. The question is whether ω depends smoothly on t.

Since $\partial_t\rho$ also satisfies the hypotheses of Lemma 2.3 at all times $t \in I$, there exists a $(k+1)$-form $\widetilde{\omega}$ solving (P) with data $\partial_t\rho$. For a fixed $t_0 \in I$ and $h \neq 0$ small we see that $h^{-1}(\omega_{t_0+h} - \omega_{t_0}) - \widetilde{\omega}_{t_0}$ is the unique solution of (P) with data $h^{-1}(\rho_{t_0+h} - \rho_{t_0}) - (\partial_t\rho)_{t_0}$. Therefore,

$$\left\|\frac{\omega_{t_0+h} - \omega_{t_0}}{h} - \widetilde{\omega}_{t_0}\right\|_{k+1+\alpha} \leq C\left\|\frac{\rho_{t_0+h} - \rho_{t_0}}{h} - (\partial_t\rho)_{t_0}\right\|_{k+\alpha} \longrightarrow 0 \quad \text{as } h \longrightarrow 0.$$

We deduce that $\widetilde{\omega}$ is the partial derivative with respect to time of ω. Iterating this argument, we conclude that ω depends smoothly on time. The claimed estimates are easily obtained by taking the supremum on $t \in I$ in the estimates of Lemma 2.3. □

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Remark 2.5. If $\delta\rho = 0$, the integral of $\star\rho$ on an $(n-k)$-cycle depends only on the homology class of the cycle. Indeed, if C and C' are two $(n-k)$-cycles in $\overline{\Omega}$ that are the boundary of an $(n-k+1)$-chain, by Stokes' theorem we have

$$\int_{C'} \star\rho - \int_C \star\rho = \int_{\partial\mathcal{N}} \star\rho = \int_{\mathcal{N}} d\star\rho = (-1)^k \int_{\mathcal{N}} \star\delta\rho = 0.$$

The machinery of differential geometry is quite powerful, but we are interested in the simpler setting of bounded domains $\Omega \subset \mathbb{R}^n$. Taking advantage of the canonical basis of Euclidean space, we may forget about differential forms and work with simpler objects. Indeed, there is a natural correspondence between 1-forms and vector fields and between 2-forms and skew-symmetric matrices $\mathcal{A}^n$:

$$\begin{aligned} C^\infty\left(\overline{\Omega}, \Lambda^1\right) &\to C^\infty\left(\overline{\Omega}, \mathbb{R}^n\right), & \sum_{i,j=1}^n a_i\, dx_i &\mapsto \sum_{i=1}^n a_i e_i, \\ C^\infty\left(\overline{\Omega}, \Lambda^2\right) &\to C^\infty\left(\overline{\Omega}, \mathcal{A}^n\right), & \frac{1}{2}\sum_{i,j=1}^n a_{ij}\, dx_i \wedge dx_j &\mapsto \sum_{i,j=1}^n a_{ij} e_i \otimes e_j. \end{aligned}$$

Here we have used that $dx_i \wedge dx_j = -dx_j \wedge dx_i$. Using the canonical base of 1-forms, the action of the codifferential can be summarized as

$$\delta(f\, dx_i) = \partial_i f, \delta(f\, dx_i \wedge dx_j) = -\partial_j f\, dx_i,$$

where f is any smooth function and $1 \le i < j \le n$. One can then check that the following diagram commutes:

$$\begin{array}{ccccc} C^\infty\left(\overline{\Omega}, \Lambda^2\right) & \xrightarrow{\delta} & C^\infty\left(\overline{\Omega}, \Lambda^1\right) & \xrightarrow{\delta} & C^\infty\left(\overline{\Omega}, \Lambda^0\right) \\ \updownarrow & & \updownarrow & & \| \\ C^\infty\left(\overline{\Omega}, \mathcal{A}^n\right) & \xrightarrow{div} & C^\infty\left(\overline{\Omega}, \mathbb{R}^n\right) & \xrightarrow{div} & C^\infty\left(\overline{\Omega}\right) \end{array}$$

This allows us to write everything in terms of matrices and to simplify the notation. Using this correspondence and Remark 2.5, we may formulate a particular case of Lemma 2.4 as follows:

Lemma 2.6. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary and let $I \subset \mathbb{R}$ be a closed and bounded interval. Let $v \in C^\infty(\overline{\Omega} \times I, \mathbb{R}^n)$. The following are equivalent:*

1. *there exists $A \in C^\infty(\overline{\Omega} \times I, \mathcal{A}^n)$ such that* $\operatorname{div} A = v$,
2. $\operatorname{div} v = 0$ *and $\int_\Sigma v \cdot \nu = 0$ for any connected component Σ of $\partial\Omega$ and any fixed $t \in I$,*
3. *$\int_{C_{n-1}} v \cdot \nu = 0$ for any $(n-1)$-cycle C_{n-1} and any $t \in I$.*

In that case, $A \in C^\infty(\overline{\Omega} \times I, \mathcal{A}^n)$ may be chosen so that

$$\|A\|_{N+1+\alpha} \le C\, \|v\|_{N+\alpha}$$

for any $N \ge 0$, $\alpha \in (0,1)$ and certain constants $C \equiv C(N, \alpha, \Omega)$.

The proof is straightforward taking into account that the codifferential δ becomes the operator div and the integral on cycles becomes the flux of the corresponding vector field across a closed surface. By Remark 2.5, the integral only depends on the homology class, so we can choose to integrate on the connected component of $\partial\Omega$ belonging to each class.

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2.3. The divergence equation

After collecting some basic tools from Hodge theory in the previous subsection, we will now show how to obtain compactly supported solutions to the divergence equation. We begin by introducing some potential-theoretic solutions, which we will later modify in order to fix the support. Let us consider the following differential operator that maps smooth vector fields (with bounded derivatives) to $C^\infty(\mathbb{R}^n, \mathcal{S}^n)$:

$$(\mathcal{R}f)_{ij} := \Delta^{-1}(\partial_i f_j + \partial_j f_i) - \delta_{ij}\Delta^{-1}\operatorname{div} f. \tag{2.5}$$

Here Δ^{-1} refers to the potential-theoretic solution of the Poisson equation, that is, the (spatial) convolution of the source term with the fundamental solution of the Laplace equation in $\mathbb{R}^n$. We remind the reader that partial derivatives with Latin subscripts denote partial derivatives in the spatial coordinates, whereas temporal partial derivatives are always denoted by ∂_t.

A direct calculation shows that $\operatorname{div}\mathcal{R}f = f$. We notice that $\mathcal{R}$ is not trace-free. This is not an issue in our proofs, because our constructions with potentials do not preserve being trace-free, so we will usually absorb the trace into the pressure at the end.

Let us now derive a very useful identity. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with smooth boundary. Let $v \in C^\infty(\overline{\Omega}, \mathbb{R}^n)$ and $S \in C^\infty(\overline{\Omega}, \mathcal{S}^n)$. Integrating by parts, we have

$$\int_\Omega v \cdot \operatorname{div} S + \int_\Omega (\nabla_{\mathrm{sym}} v) : S = \int_{\partial\Omega} v^t S\,\nu, \tag{2.6}$$

where ν is the unitary normal vector associated to the exterior orientation and the operator ∇_{sym} is given by:

$$\nabla_{\mathrm{sym}} : C^\infty(\Omega, \mathbb{R}^n) \to C^\infty(\Omega, \mathcal{S}^n), v \mapsto \frac{1}{2}(\nabla v + \nabla v^t).$$

Its kernel are the so-called Killing vector fields. It is a finite-dimensional vector space that plays an important role in Riemannian geometry. It is well known (see [43, page 52]) that in $\mathbb{R}^n$ a basis of this vector space is given by

$$\mathcal{B} := \{e_1, \dots, e_n, \xi_{12}, \dots, \xi_{(n-1)n}, \} \tag{2.7}$$

where

$$\xi_{ij} := x_i e_j - x_j e_i, \qquad 1 \le i < j \le n. \tag{2.8}$$

Next, we introduce two vector spaces and a differential operator that will be very important in our construction:

Definition 2.7. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $I \subset \mathbb{R}$ be a closed and bounded interval. We define two vector spaces:

$$\mathcal{P}(\overline{\Omega} \times I) := \left\{A \in C^\infty(\overline{\Omega} \times I, \mathbb{R}^{n^4}) : A^{ik}_{jl} = -A^{ki}_{jl},\ A^{ik}_{jl} = -A^{ik}_{lj}\right\},$$

$$\mathcal{G}(\overline{\Omega} \times I) := \left\{ S \in C^\infty(\overline{\Omega} \times I, \mathcal{S}^n) \ :\ \operatorname{div} S = 0,\ \int_\Sigma \xi^t S\,\nu = 0 \begin{array}{c} \forall \xi \in \ker \nabla_{\mathrm{sym}}, \\ \forall \Sigma \text{ comp. of } \partial\Omega, \\ \forall t \in I. \end{array} \right\}$$

and we consider the differential operator

$$\mathcal{L} : \mathcal{P}(\overline{\Omega} \times I) \to C^\infty(\overline{\Omega} \times I, \mathbb{R}^{n^2}),\ \ [\mathcal{L}(A)]_{ij} = \frac{1}{2}\sum_{k,l} \partial_{kl}\left(A^{ik}_{jl} + A^{jk}_{il}\right).$$

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This operator already appeared in the context of convex integration in the original article by De Lellis and Székelyhidi [23], who noticed that the image of the operator $\mathscr{L}$ is contained in the space of divergence-free matrices.

For our purposes, this operator can be regarded as a matrix analog of the curl operator in arbitrary dimension. In order to perform the construction sketched in Subsection 2.1, the next step is to understand how to invert this operator (under the appropriate boundary conditions). The following lemma is the key to our approach to solve the divergence equation:

Lemma 2.8. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary and let $I \subset \mathbb{R}$ be a closed and bounded interval. Then $\mathcal{G}(\overline{\Omega} \times I)$ is the image of the differential operator $\mathscr{L}$. Furthermore, given $S \in \mathcal{G}(\overline{\Omega} \times I)$, there exists $A \in \mathcal{P}(\overline{\Omega} \times I)$ such that $S = \mathscr{L}(A)$ and for any $\alpha \in (0,1)$ we have*

$$\|A\|_{N+2+\alpha} \leq C \, \|S\|_{N+\alpha}$$

for all $N \geq 0$ and certain constants $C \equiv C(N, \alpha, \Omega)$.

Proof. First, we prove that the image of $\mathscr{L}$ is contained in $\mathcal{G}(\overline{\Omega} \times I)$. Fix an arbitrary $A \in \mathcal{P}(\overline{\Omega} \times I)$. It is clear from the definition that $\mathscr{L}(A)$ is symmetric. The fact that it is divergence-free follows from the skew-symmetric properties of A:

$$\begin{aligned}
[\operatorname{div} \mathscr{L}(A)]_i &= \frac{1}{2} \sum_{j,k,l} \partial_{jkl} \left(A^{ik}_{jl} + A^{jk}_{il} \right) = \\
&= \sum_k \frac{1}{2} \partial_k \left(\sum_{j,l} \partial_{jl} A^{ik}_{jl} \right) + \sum_l \frac{1}{2} \partial_l \left(\sum_{j,k} \partial_{jk} A^{jk}_{il} \right) = 0.
\end{aligned}$$

Next, we fix an arbitrary Killing vector field ξ and a connected component Σ of $\partial\Omega$. We choose a smooth cut-off function φ that vanishes in a neighborhood of Σ and it is identically 1 in a neighborhood of the other connected components of $\partial\Omega$. By the choice of φ we have

$$\int_{\partial\Omega} \xi^t \mathscr{L}(\varphi A) \, \nu = \int_{\partial\Omega} \xi^t \mathscr{L}(A) \, \nu - \int_{\Sigma} \xi^t \mathscr{L}(A) \, \nu.$$

By our previous discussion, $\mathscr{L}(A)$ and $\mathscr{L}(\varphi A)$ are symmetric and divergence-free. In addition, ξ is a Killing vector, so $\nabla_{\mathrm{sym}} w = 0$. Thus, from (2.6) we deduce that the term on the left-hand side of the previous equation vanishes and so does the first term on the right-hand side. Therefore, we see that $\int_\Sigma \xi^t \mathscr{L}(A) \, n = 0$ and, since A, ξ, and Σ are arbitrary, we conclude that the image of $\mathscr{L}$ is contained in $\mathcal{G}(\overline{\Omega} \times I)$.

Now we will prove the other inclusion and the stated estimate. We fix an arbitrary $S \in \mathcal{G}(\overline{\Omega} \times I)$. By definition, when choosing the canonical basis of $\mathbb{R}^n$ as Killing vectors, we obtain

$$\int_\Sigma S_{ij} \nu_j = 0 \qquad \forall \Sigma \text{ connected component of } \partial\Omega$$

for any $i = 1, \ldots, m$. If we fix i, we may apply Lemma 2.6 to conclude that there exists $B^i \equiv B^i_{jl} \in C^\infty(\overline{\Omega} \times I, \mathcal{A}^n)$ such that $\partial_l B^i_{jl} = S_{ij}$.

Next we fix a Killing field ξ of the form $\xi_i = R_{ik} x_k$, where $R \in \mathcal{A}^n$. For $j = 1, \ldots m$ we compute

$$\partial_l (\xi_i B^i_{jl}) = \xi_i \partial_l B^i_{jl} + (\partial_l \xi_i) B^i_{jl} = \xi_i S_{ij} + R_{il} B^i_{jl}.$$

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Note that, since $S \in \mathcal{G}(\overline{\Omega} \times I)$,

$$\int_\Sigma \xi_i S_{ij} \nu_j = 0 \qquad \forall \Sigma \text{ connected component of } \partial\Omega, \quad \forall t \in I.$$

Regarding the left-hand side term, we define the forms

$$\omega := \sum_{i,j,l} \partial_l(\xi_i B^i_{\ jl})\, dx_j,$$
$$\eta := \sum_{j<l,i} \xi_i B^i_{\ jl}\, dx_j \wedge dx_l.$$

Since $B^i_{\ jl}$ is skew-symmetric in the lower indices, we see that $\delta\eta = \omega$. Using the properties of the Hodge star operator and the codifferential, we have $\star\delta\eta = d \star \eta$. These forms allow us to rewrite the integral on an $(n-1)$-cycle at any $t \in I$ as:

$$\int_{C_{n-1}} \partial_l(\xi_i B^i_{\ jl})\, \nu_j = \int_{C_{n-1}} \omega(n)\, \tilde{\mu} = \int_{C_{n-1}} \star\, \omega = \int_{C_{n-1}} d(\star\, \eta) = 0,$$

where $\tilde{\mu}$ is the measure induced by the standard measure in $\mathbb{R}^n$. We have used Stokes' theorem and the fact that $(n-1)$-cycles have no boundary. We conclude that for any $(n-1)$-cycle

$$\int_{C_{n-1}} R_{il} B^i_{\ jl} \nu_j = 0.$$

Choosing $R = e_{i_0} \otimes e_{l_0} - e_{l_0} \otimes e_{i_0}$, that is, choosing ξ as $\xi_{l_0 i_0}$, we see that for any $i, l = 1, \dots m$ we have:

$$\int_{C_{n-1}} (B^i_{\ jl} - B^l_{\ ji})\, \nu_j = 0 \qquad \text{for any } (n-1)\text{-cycle } C_{n-1} \text{ and all } t \in I.$$

Applying again Lemma 2.6, we obtain A^{ik}_{jl} skew-symmetric in j, k such that

$$\partial_k A^{ik}_{jl} = B^i_{\ jl} - B^l_{\ ji}.$$

Therefore,

$$\begin{aligned}\frac{1}{2}\partial_{kl}(A^{ik}_{jl} + A^{jk}_{il}) &= \frac{1}{2}\partial_l \left[\partial_k A^{ik}_{jl} + \partial_k A^{jk}_{il}\right] = \frac{1}{2}\partial_l \left[(B^i_{\ jl} - B^l_{\ ji}) + (B^j_{\ il} - B^l_{\ ij})\right] \\ &= \frac{1}{2}(\partial_l B^i_{\ jl} + \partial_l B^j_{\ il}) = S_{ij},\end{aligned}$$

where we have used that B is skew-symmetric in the lower indices and the symmetry of S: $\partial_l B^i_{\ jl} = S_{ij} = S_{ji} = \partial_l B^j_{\ il}$. In summary, we have found $A \in C^\infty\left(\overline{\Omega} \times I, \mathbb{R}^{n^4}\right)$ such that:

(i) $A^{ik}_{jl} = -A^{ij}_{kl}$,
(ii) $\partial_k A^{ik}_{jl} = -\partial_k A^{lk}_{ji}$,
(iii) $\frac{1}{2}\partial_{kl}\left(A^{ik}_{jl} + A^{jk}_{il}\right) = S_{ij}$.

We define

$$\widetilde{A}^{ik}_{jl} := \frac{1}{2}\left(A^{il}_{jk} - A^{kl}_{ji}\right).$$

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It is clear that $\widetilde{A}^{ik}_{jl}$ is skew-symmetric in i, k. In addition, it is skew-symmetric in j, l by (i). Furthermore,

$$\partial_l \widetilde{A}^{ik}_{jl} = \frac{1}{2}\left(\partial_l A^{il}_{jk} - \partial_l A^{kl}_{ji}\right) \overset{\text{(ii)}}{=} \partial_l A^{il}_{jk}.$$

Hence, using (iii) we conclude

$$\frac{1}{2}\partial_{kl}\left(\widetilde{A}^{ik}_{jl} + \widetilde{A}^{jk}_{il}\right) = \frac{1}{2}\partial_{kl}\left(A^{ik}_{jl} + A^{jk}_{il}\right) = S_{ij}.$$

Therefore, $\widetilde{A} \in \mathcal{P}(\overline{\Omega} \times I)$ and $\mathscr{L}(\widetilde{A}) = S$, as we wanted. The estimates for $\widetilde{A}$ follow from applying twice the estimates from Lemma 2.6. □

Finally, we are ready to prove the main result of this subsection, which establishes the existence of compactly supported solutions to the divergence equation. In a different setting, a related class of compactly supported solutions to the divergence equation were constructed by Isett and Oh [36, Theorem 10.1]. Our approach is based on the operator $\mathscr{L}$, which will be essential for the extension of subsolutions in Lemma 2.15. We observe that the compatibility conditions (2.9) in Lemma 2.9 are precisely the conditions (202) in [36].

We recall that the Besov norms that we use are defined in Appendix A. We need to work with these norms because they are necessary to derive estimates for the C^α norm of the resulting matrix. This will be essential in the proof of Theorem 1.7.

Lemma 2.9. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary and let $I \subset \mathbb{R}$ be a closed and bounded interval. Let $f \in C^\infty(\mathbb{R}^n \times I, \mathbb{R}^n)$ such that* $\operatorname{supp} f(\cdot, t) \subset \Omega$ *for all $t \in I$. Then, there exists $S \in C^\infty(\mathbb{R}^n \times I, \mathcal{S}^n)$ such that* $\operatorname{div} S = f$ *and* $\operatorname{supp} S(\cdot, t) \subset \Omega$ *for all $t \in I$ if and only if*

$$\int_\Omega f \cdot \xi = 0 \qquad \forall \xi \in \ker \nabla_{sym},\ \forall t \in I. \tag{2.9}$$

In that case, we may choose S so that for all $N \geq 0$ and any $\alpha \in (0, 1)$ we have

$$\|S\|_{N+\alpha} \leq C\,\|f\|_{B^{N-1+\alpha}_{\infty,\infty}}$$

for certain constants $C = C(\Omega, N, \alpha)$.

Proof. First of all, we show that the integrability condition (2.9) is necessary. Let us suppose that such an S exists. We fix a ball $B \supset \overline{\Omega}$ and use the identity (2.6) to obtain

$$0 = \int_{\partial B} \xi^t S \nu = \int_B \xi \cdot \operatorname{div} S = \int_\Omega \xi \cdot f \qquad \forall \xi \in \ker \nabla_{\text{sym}},\ \forall t \in I.$$

Let us show that condition (2.9) is also sufficient. The field $S_0 \in C^\infty(\mathbb{R}^n \times I, \mathcal{S}^n)$ given by $S_0 := \mathscr{R} f$ solves the equation $\operatorname{div} S_0 = f$, where $\mathscr{R}$ was defined in (2.5). It is easy to check that it satisfies the estimates

$$\|S_0\|_{N+\alpha} \leq C\,\|f\|_{B^{N-1+\alpha}_{\infty,\infty}} \tag{2.10}$$

for certain constants $C = C(N, \alpha)$ because $\mathscr{R}$ is an operator of order -1. However, it is not compactly supported, in general. We must modify it far from the support of f.

We begin by studying the boundary conditions. Let Σ_i be a connected component of $\partial\Omega$ and let U_i be the domain bounded by it. We claim that

$$\int_{U_i} \xi \cdot f = 0 \qquad \forall \xi \in \ker \nabla_{\text{sym}},\ \forall t \in I. \tag{2.11}$$

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Indeed, since Ω is a bounded domain, U_i must be either the complement of the unbounded connected component of $\mathbb{R}^3\backslash\overline{\Omega}$ or one of the bounded connected components of $\mathbb{R}^3\backslash\overline{\Omega}$ (if there are any). In the first case, (2.11) follows from the integrability condition (2.9) because $f(\cdot,t)\subset\Omega\subset U_i$. In the second case, (2.11) is trivial because $f(\cdot,t)$ vanishes on $U_i\subset\mathbb{R}^3\backslash\overline{\Omega}$. Thus, applying the identity (2.6) to each U_i we obtain

$$\int_{\Sigma_i}\xi^t S_0\nu = 0 \qquad \forall\xi\in\ker\nabla_{\mathrm{sym}},\ \forall t\in I. \tag{2.12}$$

Next, note that for sufficiently small $r>0$ the boundary of the open set

$$G := \{x\in\Omega : \mathrm{dist}(x,\partial\Omega)<r\}$$

has twice as many connected components as $\partial\Omega$. Furthermore, the boundary of each connected component G_i of G consists of exactly two hypersurfaces, which we denote as Σ_i and Σ_i', and we have $\Sigma_i\subset\partial\Omega$. By further reducing $r>0$, we may assume that $f(\cdot,t)$ vanishes on G at all times $t\in I$. Then, it follows from (2.12) and the identity (2.6) that

$$\int_{\Sigma_i'}\xi^t S\nu = -\int_{\Sigma_i}\xi^t S\nu + \int_{\partial G_i}\xi^t S\nu = -\int_{\Sigma_i}\xi^t S\nu + \int_{G_i}\xi\cdot f = 0$$

for any $\xi\in\ker\nabla_{\mathrm{sym}}$. Next, we fix a ball $B\supset\overline{\Omega}$ and we consider the domain

$$U := (B\backslash\Omega)\cup G.$$

We see that

$$\partial U = \partial B\cup\bigcup_i\Sigma_i'.$$

Again, it follows from (2.6) and the integrability condition (2.9) that

$$\int_{\partial B}\xi^t S_0\nu = \int_B \xi\cdot f = \int_\Omega \xi\cdot f = 0 \qquad \forall\xi\in\ker\nabla_{\mathrm{sym}},\ \forall t\in I.$$

We conclude that S_0 is divergence-free on U and in each connected component Σ of ∂U we have

$$\int_{\Sigma}\xi^t S_0\nu = 0 \qquad \forall\xi\in\ker\nabla_{\mathrm{sym}},\ \forall t\in I,$$

that is, $S_0\in\mathcal{G}(\overline{U}\times I)$. By Lemma 2.8 there exists $A_0\in\mathcal{P}(\overline{U}\times I)$ such that $S_0(x,t)=\mathscr{L}(A_0)(x,t)$ for all $x\in\overline{G}$ and $t\in I$. Furthermore, for any $N\geq 0$ and $\alpha\in(0,1)$ we have

$$\|A_0\|_{N+2+\alpha}\leq C(U,N,\alpha)\,\|S_0\|_{N+\alpha}\leq C(U,N,\alpha)\,\|f\|_{B^{N-1+\alpha}_{\infty,\infty}}.$$

The constants depend on U, which depends not only on the geometry of Ω but also on the minimum distance between the support of $f(\cdot,t)$ and $\partial\Omega$ through the parameter r. However, since U tends to $B\backslash\overline{\Omega}$ as $r\to 0$, the constants remain uniformly bounded, so they ultimately depend only on Ω. For this, smoothness of $\partial\Omega$ is essential, as it allows us to choose parametrizations of $\overline{U}$ converging to parametrizations of $\overline{B}\backslash\Omega$ in any Hölder norm as $r\to 0$.

Applying Theorem B.3 and antisymmetrizing, we see that there exists a map $A\in C^\infty(\mathbb{R}^n\times I,\mathbb{R}^{n^4})$ such that $A^{ik}_{jl}=-A^{ki}_{jl}$, $A^{ik}_{jl}=-A^{ik}_{lj}$ that extends A_0 outside $\overline{U}\times I$. Furthermore, for any $N\geq 0$ and $\alpha\in(0,1)$, we have

$$\|A\|_{N+2+\alpha}\leq C(U,N)\,\|A_0\|_{N+2+\alpha}\leq C(U,\Omega,N,\alpha)\,\|f\|_{B^{N-1+\alpha}_{\infty,\infty}}. \tag{2.13}$$

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Again, since U tends to $B\backslash\overline{\Omega}$ as $r \to 0$ in a suitable manner, the constants ultimately depend only on Ω, N, and α, since they will be uniformly bounded on $r \in (0,1)$.

Finally, for $x \in B$ and $t \in I$ we define

$$S := S_0 - \mathscr{L}(A).$$

Since the image of $\mathscr{L}$ is contained in the kernel of the divergence, we see that $\operatorname{div} S = f$. By construction A extends A_0, so $\mathscr{L}(A) = \mathscr{L}(A_0) = S_0$ on $\overline{U} \times I$. Therefore, $S(\cdot,t)$ vanishes in U, so we may extend it by 0 to $\mathbb{R}^n \times I$.

In conclusion, we have constructed $S \in C^\infty(\mathbb{R}^n \times I, \mathcal{S}^n)$ such that $\operatorname{div} S = f$ and $\operatorname{supp} S(\cdot,t) \subset \Omega$ for all $t \in I$. Furthermore, the desired estimate follows from (2.10) and (2.13) because $\mathscr{L}$ is a second-order differential operator. □

2.4. Subsolutions and proof of Theorem 1.6

In this subsection we use Lemma 2.8 and Lemma 2.9 to glue and extend subsolutions, which will yield the proof of Theorem 1.6. It should be apparent by now that controlling the L^2-product with the Killing fields is very important in these constructions. It is not difficult to construct $f \in C_c^\infty(\mathbb{R}^n,\mathbb{R}^n)$ with the desired L^2-product with the Killing fields. However, when working with subsolutions we will also need that f be divergence-free. In addition, in our constructions we will work in domains of a certain form. Our approach is based on the following:

Lemma 2.10. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary and let $I \subset \mathbb{R}$ be a closed and bounded interval. Let $r > 0$ and let $L_{ij} \in C^\infty(I)$ for $1 \le i < j \le n$. There exists a divergence-free field $w \in C^\infty(\mathbb{R}^n \times I, \mathbb{R}^n)$ such that the support of $w(\cdot,t)$ is contained in $\{x \in \mathbb{R}^n : 0 < \operatorname{dist}(x,\Omega) < r\}$ and*

$$\int a \cdot w \, dx = 0, \qquad \int \xi_{ij} \cdot w \, dx = L_{ij}(t)$$

for all $t \in I$ and $1 \le i < j \le n$, where ξ_{ij} is given by (2.8). Furthermore, for any $N \ge 0$ we have

$$\|w\|_N \le C(N,n)\,|\Omega|^{-1}\, r^{-(N+1)} \max_{ij,\, t\in I} \left|L_{ij}(t)\right|,$$
$$\|\partial_t w\|_N \le C(N,n)\,|\Omega|^{-1}\, r^{-(N+1)} \max_{ij,\, t\in I} \left|L'_{ij}(t)\right|.$$

Proof. We will construct our field as $w = \operatorname{div} A$ for some $A \in C_c^\infty(\mathbb{R}^n \times I, \mathcal{A}^n)$ that we will choose later. Since A is compactly supported, it follows from the divergence theorem that $\int a \cdot w = 0$ for any $a \in \mathbb{R}^n$. Furthermore, for any $1 \le i < j \le n$ we have

$$\int \xi_{ij} \cdot w = \int (\xi_{ij})_k \partial_l A_{kl} = -\int \partial_l (\xi_{ij})_k A_{kl} = -\int (A_{ji} - A_{ij}) = 2\int A_{ij} \tag{2.14}$$

because $\partial_l (\xi_{ij})_k = \delta_{il}\delta_{jk} - \delta_{jl}\delta_{ik}$. Here we have denoted by $(\xi_{ij})_k$ the k-th component of the vector ξ_{ij} and we have used Einstein's summation convention when summing over k and l. By Lemma B.1 we may choose a nonnegative cutoff function $\varphi \in C_c^\infty(\Omega + B(0,r))$ that is identically 1 in a neighborhood of Ω and such that

$$\|\varphi\|_N \le C(N,n)\, r^{-N}.$$

We define

$$A(x,t) := \sum_{1\le i<j\le n} L_{ij}(t)\,(e_i \otimes e_j - e_j \otimes e_i) \left(2\int \varphi\right)^{-1} \varphi(x).$$

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Since φ is constant in a neighborhood of Ω and its support is contained in $\Omega + B(0, r)$, we see that the support of $w(\cdot, t)$ is contained in $\{x \in \mathbb{R}^n : 0 < \operatorname{dist}(x, \Omega) < r\}$. By construction

$$2A_{ij}(x,t) = L_{ij}(t) \left(\int \varphi \right)^{-1} \varphi(x),$$

so it follows from Equation (2.14) that $\int \xi_{ij} \cdot w = L_{ij}$. Finally, the claimed estimates follow at once from the bounds for φ and the fact that $\int \varphi \geq |\Omega|$. □

Now we have all the ingredients that we need to glue subsolutions in space. The following lemma is the key tool in this section. It will be used not only in the proof of Theorem 1.6, but also in the convex integration scheme. We use skew-symmetric matrices instead of potential vectors because the lemma is stated in any dimension $n \geq 2$.

Lemma 2.11. *Let $T > 0$ and let $\Omega_1 \Subset \Omega_2 \subset \mathbb{R}^n$ be bounded domains with smooth boundary. Let $(v_i, p_i, \mathring{R}_i) \in C^\infty(\Omega_2 \times [0,T])$ be subsolutions for $i = 1, 2$. Let $r > 0$ be sufficiently small. There exists a subsolution $(v, p, \mathring{R}) \in C^\infty(\Omega_2 \times [0,T])$ such that*

$$(v, p, \mathring{R})(x,t) = \begin{cases} (v_1, p_1, \mathring{R}_1)(x,t) & x \in \overline{\Omega}_1, \\ (v_2, p_2, \mathring{R}_2)(x,t) & \operatorname{dist}(x, \Omega_1) \geq r \end{cases} \tag{2.15}$$

if and only if for each connected component Σ of $\partial\Omega_1$, and all times $t \in [0,T]$, we have

$$\int_\Sigma v_1 \cdot \nu = \int_\Sigma v_2 \cdot \nu, \tag{2.16}$$

$$\int_\Sigma \left[(a \cdot x)\partial_t v_1 + (a \cdot v_1)v_1 + p_1 a - a^t \mathring{R}_1 \right] \cdot \nu = \int_\Sigma \left[(a \cdot x)\partial_t v_2 + (a \cdot v_2)v_2 + p_2 a - a^t \mathring{R}_2 \right] \cdot \nu \qquad \forall a \in \mathbb{R}^n. \tag{2.17}$$

Suppose that, in addition, we have $v_1 = \operatorname{div} A_1$ and $v_2 = \operatorname{div} A_2$ for some potentials $A_i \in C^\infty(\Omega_2 \times I, \mathcal{A}^n)$. Then, there exists $A \in C^\infty(\Omega_2 \times I, \mathcal{A}^n)$ such that $v = \operatorname{div} A$ and $A(x,t) = A_2(x,t)$ if $\operatorname{dist}(x, \Omega_1) \geq r$.

Remark 2.12. The compatibility conditions (2.16) and (2.17) are automatically satisfied if $\partial\Omega_1$ is connected or $\Omega_2 = \mathbb{R}^n$. This will be explained in the proof of the lemma.

Remark 2.13. The subsolution $(v_1, p_1, \mathring{R}_1)$ need not be defined in all of Ω_2 and the subsolution $(v_2, p_2, \mathring{R}_2)$ need not be defined in Ω_1. We have assumed this to simplify slightly the statement of the lemma.

Proof. First of all, note that a subsolution $(v_0, p_0, \mathring{R}_0)$ in a bounded domain G with smooth boundary satisfies

$$0 = \int_G \operatorname{div} v_0 = \int_{\partial G} v_0 \cdot \nu, \tag{2.18}$$

$$0 = \int_G a \cdot \left[\partial_t v_0 + \operatorname{div}\left(v_0 \otimes v_0 + p_0 \operatorname{Id} - \mathring{R}_0 \right) \right] = \int_{\partial G} \left[(a \cdot x)\partial_t v_0 + (a \cdot v_0)v_0 + p_0 a - a^t \mathring{R}_0 \right] \cdot \nu \qquad \forall a \in \mathbb{R}^n, \tag{2.19}$$

where we have used the divergence theorem, identity (2.6) and the fact that $\operatorname{div}[(a \cdot x)\partial_t v_0] = a \cdot \partial_t v_0$ because $\partial_t v_0$ is divergence-free.

From these equations we readily deduce that the compatibility conditions (2.16) and (2.17) are automatically satisfied if $\partial\Omega_1$ is connected, as both integrals vanish for each field. In the case $\Omega_2 = \mathbb{R}^n$,

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we apply Equations (2.18) and (2.19) to the domain bounded by each connected component of $\partial\Omega_1$. We conclude that both integrals vanish for each field in each connected component of $\partial\Omega_1$.

Next, we check that the conditions are necessary; we study (2.16) because the expressions are shorter, but the argument for (2.17) is exactly the same. First, if $\partial\Omega_1$ is connected, it readily follows from (2.18) that (2.16) must be satisfied. Hence, we focus on bounded domains Ω_1 whose boundary is not connected. In that case, $\mathbb{R}^n\backslash\Omega_1$ must have at least one bounded connected component. Given a bounded connected component of $\mathbb{R}^n\backslash\Omega_1$, we define G to be its intersection with Ω_2. Then, ∂G is composed of a connected component Σ of $\partial\Omega_1$ and (possibly) some connected components $\Sigma'_1, \dots, \Sigma'_m$ of $\partial\Omega_2$. Since v equals v_1 on Σ and v_2 on the other connected components of ∂G, it follows from (2.18) that:

$$0 = \int_{\partial G} v\cdot \nu = \int_{\Sigma} v_1 \cdot \nu + \sum_{i=1}^{n} \int_{\Sigma'_i} v_2 \cdot \nu.$$

On the other hand, applying (2.18) to v_2 on G, we have

$$-\int_{\Sigma} v_2 \cdot \nu = \sum_{i=1}^{n} \int_{\Sigma'_i} v_2 \cdot \nu,$$

which, combined with the previous equation, yields

$$\int_{\Sigma} v_1 \cdot \nu = \int_{\Sigma} v_2 \cdot \nu.$$

Since this applies to any bounded connected component of $\mathbb{R}^n\backslash\Omega_1$, we can combine it with (2.18) with $G = \Omega_1$ to obtain an analogous identity for the remaining connected component of $\partial\Omega_1$, that is, the boundary of the unbounded connected component of $\mathbb{R}^n\backslash\Omega_1$. We conclude (2.16).

Let us now prove that the compatibility conditions (2.16) and (2.17) are also sufficient. Let $r > 0$ be small enough so that $\{x \in \Omega_2 : \operatorname{dist}(x, \Omega_1) = r\}$ is diffeomorphic to $\partial\Omega_1$. We define $U := \{x \in \Omega_2 : 0 < \operatorname{dist}(x, \Omega_1) < r\}$. Then, the condition (2.16) ensures that there exists $A_{12} \in C^\infty(\overline{U}\times[0,T], \mathcal{A}^n)$ such that $v_2 - v_1 = \operatorname{div} A_{12}$ in $U \times [0,T]$. Indeed, let U_i be a connected component of U and let Σ_i and Σ'_i be the connected components of ∂U_i, where $\Sigma_i \subset \partial\Omega_1$. Using (2.16) and the fact that $v_2 - v_1$ is divergence-free:

$$\int_{\Sigma'_i} (v_2 - v_1)\cdot \nu = \int_{\partial U} (v_2 - v_1)\cdot \nu - \int_{\Sigma_i} (v_2 - v_1)\cdot \nu = 0.$$

Hence, the flux of $v_2 - v_1$ through each connected component of U vanishes, so by Lemma 2.6 there exists $A_{12} \in C^\infty(\overline{U}\times[0,T], \mathcal{A}^n)$ such that $v_2 - v_1 = \operatorname{div} A_{12}$.

Next, using Lemma B.1 we choose a cutoff function $\varphi \in C_c^\infty(\Omega_1 + B(0,r))$ that equals 1 in a neighborhood of Ω_1. We define

$$\begin{aligned} v &:= \varphi v_1 + (1-\varphi)v_2 + w_c + w_L \equiv \varphi v_1 + (1-\varphi)v_2 + w, \\ \widetilde{p} &:= \varphi p_1 + (1-\varphi)p_2, \end{aligned}$$

where $w_c := A_{12}\cdot\nabla\varphi$ so that $\varphi v_1 + (1-\varphi)v_2 + w_c$ is divergence-free. The additional correction w_L is a divergence-free field supported within U that will be defined later. Its purpose is to cancel the angular momentum so that the gluing can be performed in the interior of U. After a tedious computation we obtain

$$\partial_t v + \operatorname{div}(v\otimes v) + \nabla\widetilde{p} = \operatorname{div}\left(\varphi \mathring{R}_1 + (1-\varphi)\mathring{R}_2 + S_1\right) + \partial_t w_L + M\cdot\nabla\varphi, \tag{2.20}$$

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where

$$S_1 := -\varphi(1-\varphi)(v_1-v_2)\otimes(v_1-v_2) + w\otimes\left(v-\frac{1}{2}w\right) + \left(v-\frac{1}{2}w\right)\otimes w, \tag{2.21}$$

$$M := \partial_t A_{12} + v_1\otimes v_1 - v_2\otimes v_2 + (p_1-p_2)\,\mathrm{Id} - \mathring{R}_1 + \mathring{R}_2. \tag{2.22}$$

Let $\widetilde{\rho} := M\cdot\nabla\varphi$ and $\rho = \widetilde{\rho} + \partial_t w_L$. Our goal is to find $S_2 \in C^\infty(\Omega_2\times[0,T],\mathcal{S}^n)$ supported on U for all $t\in[0,T]$ and such that $\operatorname{div} S_2 = \rho$. Thus, we may set $R = \varphi\mathring{R}_1 + (1-\varphi)\mathring{R}_2 + S_1 + S_2$ and absorb the trace into the pressure, obtaining $\mathring{R}$ and the final pressure p. To do so, first we must check that ρ satisfies the compatibility conditions (2.9).

Note that $\widetilde{\rho} = \operatorname{div}(\varphi M)$ because $\operatorname{div} M = 0$, since $(v_i, p_i, \mathring{R}_i)$ are subsolutions. Hence, by the divergence theorem for any $a\in\mathbb{R}^n$ we have

$$\int_U a\cdot\widetilde{\rho} = \int_{\partial U} a^t(\varphi M)\nu = \int_{\partial\Omega_1} a^t M\nu. \tag{2.23}$$

Note that

$$\begin{aligned}\int_{\partial\Omega_1} a^t(\partial_t A_{12})\nu &= -\int_{\partial\Omega_1}\nu^t(\partial_t A_{12})\nabla(a\cdot x) = \int_{\partial\Omega_1}(a\cdot x)\operatorname{div}(\partial_t A_{12})\cdot\nu\\ &= \int_{\partial\Omega_1}(a\cdot x)(\partial_t v_1 - \partial_t v_2)\cdot\nu.\end{aligned}\tag{2.24}$$

Therefore, combining Equations (2.17), (2.23) and (2.24) we conclude that $\int_U a\cdot\widetilde{\rho} = 0$ for all $a\in\mathbb{R}^n$. Since $\partial_t w_L$ is divergence-free and its support is contained in U, the same holds for $\partial_t w_L$, so $\int_U a\cdot\rho = 0$ for all $a\in\mathbb{R}^n$.

Next, we study the product with nonconstant Killing fields. For each pair $1\le i<j\le n$ we define

$$l_{ij}(t) := \int_U \xi_{ij}\cdot\widetilde{\rho}(x,t)\,dx,$$

where ξ_{ij} are the elements of the basis of Killing fields defined in (2.8). It will be useful later on to write the coefficients as:

$$l_{ij} = \int_U \xi_{ji}\cdot\operatorname{div}(\varphi M) = \int_{\partial\Omega_1}\xi_{ij}^t M\nu, \tag{2.25}$$

where we have used the fact that Killing fields are divergence-free as well as the values of φ on $\partial\Omega_1$ and $\partial\Omega_2$. We define

$$L_{ij}(t) := -\int_0^t l_{ij}(s)\,ds. \tag{2.26}$$

We then define the correction w_L to be the divergence-free field obtained by applying Lemma 2.10 to the domain Ω_1 with coefficients $L_{ij}\in C^\infty([0,T])$. Thus, we have

$$\int \xi_{ij}\cdot\partial_t w_L = \frac{d}{dt}\int\xi_{ij}\cdot w_L = L'_{ij} = -l_{ij}.$$

We conclude that $\int_U \xi\cdot\rho = 0$ for any Killing field ξ, as we wanted. Therefore, by Lemma 2.9 there exists $S_2\in C^\infty(\Omega_2\times[0,T],\mathcal{S}^n)$ supported on U for all $t\in[0,T]$ and such that $\operatorname{div} S_2 = \rho$. We define the final pressure and the Reynolds stress as

$$p := \widetilde{p} - \frac{1}{n}\operatorname{tr}(S_1 + S_2) = \varphi p_1 + (1-\varphi)p_2 - \frac{1}{n}\operatorname{tr}(S_1+S_2)\,,$$
$$\mathring{R} := \varphi \mathring{R}_1 + (1-\varphi)\mathring{R}_2 + S_1 + S_2 - \frac{1}{n}\operatorname{tr}(S_1+S_2)\,\mathrm{Id}\,.$$

It follows from Equation (2.20) that the resulting triplet $(v, p, \mathring{R})$ is a subsolution and it satisfies (2.15) because S_1 and S_2 are supported in U for all $t \in [0, T]$.

Finally, let us consider that the velocity fields are given by $v_i = \operatorname{div} A_i$. In that case, we may simply take $A_{12} = A_2 - A_1$ instead of constructing a suitable potential using Lemma 2.6. We see that

$$v - w_L = \varphi v_1 + (1-\varphi)v_2 + (A_2 - A_1)\cdot\nabla\varphi = \operatorname{div}(\varphi A_1 + (1-\varphi)A_2).$$

Inspecting Lemma 2.10 leads us to define

$$A := \varphi A_1 + (1-\varphi)A_2 + \sum_{1\le i<j\le n} L_{ij}(t)\,(e_i\otimes e_j - e_j\otimes e_i)\left(2\int\varphi\right)^{-1}\varphi(x). \tag{2.27}$$

Hence, we have $v = \operatorname{div} A$ and we see that A equals A_2 in $\{\operatorname{dist}(x,\Omega_1) \ge r\}$ because φ vanishes in a neighborhood of this set. □

In the convex integration scheme we will need estimates of the glued subsolution. For the sake of clarity, we keep them separate in a different lemma:

Lemma 2.14. *Let $\alpha \in (0,1)$. In the conditions of Lemma 2.11 and using the notation of its proof, the new subsolution satisfies:*

$$\|v - (\varphi v_1 + (1-\varphi)v_2)\|_N \lesssim T r^{-(N+1)}\,\|M\|_{0;U} + \sum_{k=0}^{N} r^{-(k+1)}\,\|A_{12}\|_{N-k;U}\,, \tag{2.28}$$

$$\|\partial_t(v - \varphi v_1 - (1-\varphi)v_2)\|_N \lesssim r^{-(N+1)}\,\|M\|_{0;U} + \sum_{k=0}^{N} r^{-(k+1)}\,\|\partial_t A_{12}\|_{N-k;U}\,, \tag{2.29}$$

$$\begin{aligned}\|\mathring{R} - \varphi\mathring{R}_1 - (1-\varphi)\mathring{R}_2\|_0 \lesssim{}& r^{-\alpha}\,\|M\|_{0;U} + \|v_1 - v_2\|_{0;U}^2 \\ &+ (\|v_1\|_{0;U} + \|v_2\|_{0;U} + \|w\|_{0;U})\,\|w\|_{0;U}\,,\end{aligned} \tag{2.30}$$

In addition, if $v_1 = \operatorname{div} A_1$ and $v_2 = \operatorname{div} A_2$, the potential A satisfies

$$\|A - (\varphi A_1 + (1-\varphi)A_2)\|_N \lesssim T r^{-N}\,\|M\|_{0;U}\,, \tag{2.31}$$

$$\|\partial_t(A - \varphi A_1 - (1-\varphi)A_2)\|_N \lesssim r^{-N}\,\|M\|_{0;U}\,. \tag{2.32}$$

The implicit constants in these inequalities depend on Ω_1, N, and α.

Proof. We begin by estimating $w_c = A_{12}\cdot\nabla\varphi$. Since φ satisfies $\|\varphi\|_N \lesssim r^{-N}$ and it is independent of time, it is clear that

$$\|w_c\|_N \lesssim \sum_{k=0}^{N} r^{-(k+1)}\,\|A_{12}\|_{N-k;U}\,, \qquad \|\partial_t w_c\|_N \lesssim \sum_{k=0}^{N} r^{-(k+1)}\,\|\partial_t A_{12}\|_{N-k;U}$$

because the support of $\nabla\varphi$ is contained in U. Regarding w_L, it follows from (2.25) that $|l_{ij}| \lesssim \|M\|_{0;U}$, so $|L_{ij}| \lesssim T\,\|M\|_{0;U}$. Hence, by Lemma 2.10 we have the bounds

$$\|w_L\|_N \lesssim Tr^{-(N+1)}\,\|M\|_{0;U}\,, \qquad \|\partial_t w_L\|_N \lesssim r^{-(N+1)}\,\|M\|_{0;U}\,.$$

The claimed estimates for $v - (\varphi v_1 + (1-\varphi)v_2) = w_c + w_L$ follow at once. Let us now focus on the Reynolds stress. Using the assumption $\|w\|_0 \le \|v_1\|_0 + \|v_2\|_0$, we deduce from the definition (2.21) that

$$\|S_1\|_0 \lesssim \|v_1 - v_2\|_{0;U}^2 + (\|v_1\|_{0;U} + \|v_2\|_0)\,\|w\|_{0;U}\,.$$

Concerning S_2, let us first estimate ρ:

$$\|\rho\|_0 \le \|M\cdot\nabla\varphi\|_{0;U} + \|\partial_t w_L\|_0 \lesssim r^{-1}\,\|M\|_{0;U}\,.$$

Since the support of $\rho(\cdot,t)$ is contained in $\{x \in \mathbb{R}^n : 0 < \operatorname{dist}(x,\Omega_1) < r\}$, we may apply Lemma B.4, obtaining

$$\|\rho\|_{B^{-1+\alpha}_{\infty,\infty}} \lesssim r^{1-\alpha}\,\|\rho\|_0 \lesssim r^{-\alpha}\,\|M\|_{0;U}\,.$$

Hence, it follows from the estimates in Lemma 2.9 that

$$\|S_2\|_0 \lesssim \|\rho\|_{B^{-1+\alpha}_{\infty,\infty}} \lesssim r^{-\alpha}\,\|M\|_{0;U}\,.$$

Since

$$\mathring{R} - (\varphi\mathring{R}_1 + (1-\varphi)\mathring{R}_2) = S_1 + S_2 - \frac{1}{n}\operatorname{tr}(S_1+S_2)\operatorname{Id},$$

the claimed bound follows.

Finally, let us estimate A in the case that the velocities are given by $v_i = \operatorname{div} A_2$. By (2.27) we have

$$A - (\varphi A_1 - (1-\varphi)A_2) = \sum_{1\le i<j\le n} L_{ij}(t)\,(e_i\otimes e_j - e_j\otimes e_i)\left(2\int\varphi\right)^{-1}\varphi(x).$$

The claimed bounds then follow from the estimates derived in Lemma 2.10. □

Lemma 2.11 is almost what we want, but it can be made a bit sharper. In particular, in Theorem 1.6 we do not want to assume that the subsolution $(v_0, p_0, \mathring{R}_0)$ is defined in a neighborhood of $\overline{\Omega}_0$. Fortunately, it turns out that all subsolutions can be extended, at least a little bit. Our operator $\mathscr{L}$ is essential for this:

Lemma 2.15. *Let $\Omega_0 \subset \mathbb{R}^n$ be a bounded domain with smooth boundary and let $I \subset \mathbb{R}$ be a closed and bounded interval. Let $(v_0, p_0, \mathring{R}_0) \in C^\infty(\overline{\Omega}_0 \times I)$ be a subsolution in $\Omega_0 \times I$. Let Ω be a sufficiently small open neighborhood of $\overline{\Omega}_0$. Then, there exists a subsolution $(v, p, \mathring{R}) \in C^\infty(\Omega\times I)$ that extends $(v_0, p_0, \mathring{R}_0)$.*

Remark 2.16. The open neighborhood Ω need not be very small. It only needs to be bounded and such that each connected component of $\mathbb{R}^n\backslash\overline{\Omega}_0$ has nonempty intersection with $\mathbb{R}^n\backslash\overline{\Omega}$.

Proof. We begin by constructing the velocity field v. We choose $\rho \in C_c^\infty(\mathbb{R}^n\times I, \mathbb{R})$ such that $\operatorname{supp}\rho(\cdot,t)$ is contained in $\mathbb{R}^n\backslash\overline{\Omega}$ for all $t \in I$ and such that

$$\int_G \rho(x,t)\,dx = \int_{\partial G} v_0\cdot\nu \qquad \forall t \in I$$

for each bounded connected component G of $\mathbb{R}^n\backslash\overline{\Omega}_0$, whose boundary we have oriented with the outer normal with respect to G. This can be done if Ω is a sufficiently small neighborhood of $\overline{\Omega}_0$ so that the intersection of G with $\mathbb{R}^n\backslash\overline{\Omega}$ is nonempty.

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Next, let $\widetilde{v} := \nabla\Delta^{-1}\rho$ so that $\widetilde{v} \in C^\infty(\mathbb{R}^n \times I, \mathbb{R}^n)$ and $\operatorname{div}\widetilde{v} = \rho$ for all $t \in I$. By the divergence theorem we have

$$\int_{\partial G} \widetilde{v} \cdot \nu = \int_G \rho = \int_{\partial G} v_0 \cdot \nu$$

in each bounded connected component G of $\mathbb{R}^n \backslash \overline{\Omega}_0$ and for all $t \in I_0$. In addition, $\widetilde{v} - v_0$ is divergence-free in $\Omega_0 \times I$ because ρ vanishes in this set by construction. In particular, by the divergence theorem we have $\int_{\partial\Omega_0} (\widetilde{v} - v_0) \cdot \nu = 0$, from which we conclude

$$\int_\Sigma (\widetilde{v} - v_0) \cdot \nu = 0 \qquad \forall t \in I_0$$

for all connected components Σ of $\partial\Omega_0$. Therefore, by Lemma 2.6 there exists $A \in C^\infty(\overline{\Omega}_0 \times I, \mathcal{A}^n)$ such that $\operatorname{div} A = \widetilde{v} - v_0$ in $\Omega_0 \times I$. We choose a smooth extension $\widetilde{A} \in C^\infty(\mathbb{R}^n \times I, \mathbb{R}^{n\times n})$ and then we take the skew-symmetric part, so that $\widetilde{A} \in C^\infty(\mathbb{R}^n \times \mathbb{R}, \mathcal{A}^n)$. We define:

$$v := \widetilde{v} - \operatorname{div}\widetilde{A} \in C^\infty(\mathbb{R}^n \times I, \mathbb{R}^n).$$

Since the support of $\rho(\cdot, t)$ is contained in $\mathbb{R}^n \backslash \overline{\Omega}$ and the second term is divergence-free, v is divergence-free in $\Omega \times I$. In addition, our choice of $\widetilde{A}$ ensures that the restriction of v to $\overline{\Omega}_0 \times I$ is v_0, as we wanted.

Next, we will extend $S_0 := v_0 \otimes v_0 + p_0 \operatorname{Id} - \mathring{R}_0$ in a similar manner. First, we choose $f \in C_c^\infty(\mathbb{R}^n \times I, \mathbb{R}^n)$ such that $\operatorname{supp} f(\cdot, t)$ is contained in $\mathbb{R}^n \backslash \overline{\Omega}$ for all $t \in I$ and such that

$$\int_{G_i} \xi \cdot f = \int_{G_i} \xi \cdot \partial_t v + \int_{\partial G_i} \xi^t U_0 \nu \qquad \forall \xi \in \ker \nabla_{\mathrm{sym}},\ \forall t \in I \tag{2.33}$$

for all bounded connected components G_i of $\mathbb{R}^n \backslash \overline{\Omega}_0$, whose boundary we have oriented with the outer normal with respect to G_i. To find such an f, we fix a nonnegative radial function $\psi \in C_c^\infty(B(0,1))$ and a ball $\overline{B}(x_i, r_i) \subset G_i$. Due to symmetry, for any two elements $w_j \neq w_k$ of the basis $\mathcal{B}$ defined in (2.7) we have

$$\int \psi\left(r_i^{-1}(x - x_i)\right) w_j(x) \cdot w_k(x)\, dx = 0.$$

Therefore, it suffices to choose

$$f(x) := \sum_i \sum_{j=1}^{n(n+1)/2} c_{ij}\, \psi\left(r_i^{-1}(x - x_i)\right) w_j(x)$$

for the appropriate coefficients c_{ij}. Then, noticing that $\partial_t v$ is a smooth vector field with bounded derivatives, we define $\widetilde{S} := \mathscr{R}(-\partial_t v + f)$ so that $\widetilde{S} \in C^\infty(\mathbb{R}^n \times I, \mathcal{S}^n)$, and

$$\operatorname{div}\widetilde{S} = -\partial_t v + f. \tag{2.34}$$

Since v extends v_0 and $(v_0, p_0, \mathring{R}_0)$ is a subsolution in $\Omega_0 \times I$, we have

$$\operatorname{div}(\widetilde{S} - S_0)(x, t) = 0 \qquad \forall (x, t) \in \Omega_0 \times I \tag{2.35}$$

because f vanishes in that set. In addition, using (2.6) it follows from (2.34) and (2.33) that

$$\int_{\partial G} \xi^t (\widetilde{S} - S_0)\nu = 0 \qquad \forall \xi \in \ker \nabla_{\mathrm{sym}}\ \forall t \in I$$

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for all bounded connected components G of $\mathbb{R}^m\backslash\overline{\Omega}_0$. Due to (2.6) and (2.35), this integral also vanishes for the remaining connected component of $\partial\Omega_0$, that is, the connected component that separates Ω_0 and the unbounded connected component of $\mathbb{R}^n\backslash\Omega_0$.

We conclude that $\widetilde{S} - S_0$ is in $\mathcal{G}(\overline{\Omega}_0 \times I)$. Therefore, by Lemma 2.8 there exists $\widetilde{E} \in \mathcal{P}(\overline{\Omega}_0 \times I)$ such that $\mathscr{L}(\widetilde{E}) = \widetilde{S} - S_0$ in $\Omega_0 \times I$. We choose a smooth extension $E \in C^\infty(\mathbb{R}^n \times I, \mathbb{R}^{n^4})$ and then we make the appropriate antisymmetrization. We define

$$S := \widetilde{S} - \mathscr{L}(E) \in C^\infty(\mathbb{R}^n \times I, \mathcal{S}^n)$$

By construction of E, we see that S extends S_0. Additionally, we have

$$\operatorname{div} S = \operatorname{div} \widetilde{S} = -\partial_t v + f$$

because the image of $\mathscr{L}$ is contained in the kernel of the divergence. We define

$$\begin{aligned}\mathring{R} &:= v \otimes v - S - \frac{1}{n}\operatorname{tr}(v \otimes v - S)\,\mathrm{Id},\\ p &:= -\frac{1}{n}\operatorname{tr}(v \otimes v - S).\end{aligned}$$

Since $f(\cdot, t)$ vanishes in Ω for all $t \in I$, we conclude that $(v, p, \mathring{R})$ is a subsolution in $\Omega \times I$ that extends $(v_0, p_0, \mathring{R}_0)$. □

Combining Lemma 2.11 and Lemma 2.15 we can finally prove Theorem 1.6:

Proof of Theorem 1.6. Working with each connected component of Ω_0, we may assume that both Ω_0 and Ω are connected (i.e., domains). Then, reducing Ω if necessary, by Lemma 2.15 we may assume that $(v_0, p_0, \mathring{R}_0)$ is a subsolution in $\Omega \times I$. The result then follows by applying Lemma 2.11 with the domains Ω_0, Ω and subsolutions $(v_0, p_0, \mathring{R}_0)$ and $(0, 0, 0)$. □

3. Proof of Theorem 1.7

The construction of a weak solution to the Euler equations stated in Theorem 1.7 consists in an iterative argument which is presented in Subsection 3.1, cf. Proposition 3.2. This proposition together with Lemma 3.3 allow us to prove the theorem in Subsection 3.2. We want to remark that most of this article, that is, Sections 4 to 7, is devoted to prove Proposition 3.2, which is the key result for our convex integration scheme.

Section 3.1 and Section 3.2 follow the general outline of Sections 2.1 and 2.2 in [7] but with two important differences: here the initial subsolution will be nontrivial, that is, different from $(0, 0, 0)$, and the perturbations will be supported in a subset of Ω instead of the whole space. Regarding the first issue, we use Lemma 3.3 to help start the iterative process. Concerning the second point, we introduce suitable sets related to the distance to $\partial\Omega$ and the size of $\mathring{R}_0$. The perturbations will be localized to these sets, which is summarized in an additional inductive hypothesis, (3.13).

3.1. The iterative process

Let us assume all along this subsection that the subsolution $(v_0, p_0)(\cdot, t)$ is compactly supported for each time $t \in [0, T]$. We will construct the desired weak solution of the Euler equations as the limit of a sequence of subsolutions, that is, at a given step $q \geq 0$ we have $(v_q, p_q, \mathring{R}_q) \in C^\infty(\mathbb{R}^3 \times [0, T])$ solving the Euler-Reynolds system:

$$\begin{cases}\partial_t v_q + \operatorname{div}(v_q \otimes v_q) + \nabla p_q = \operatorname{div} \mathring{R}_q,\\ \operatorname{div} v_q = 0,\end{cases} \tag{3.1}$$

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to which we add the constraint that

$$\operatorname{tr} \mathring{R}_q = 0. \tag{3.2}$$

The matrix $\mathring{R}_q$ measures the deviation from being a solution of the Euler equations. The goal of the process is to make $\mathring{R}_q$ vanish at the limit $q \to +\infty$, so that the limit field is a weak solution of the Euler equations.

Assume we are given the initial subsolution $(v_0, p_0, \mathring{R}_0) \in C^\infty(\mathbb{R}^3 \times [0,T])$. Let us then show how to construct the rest of the terms iteratively. To construct the subsolution at step q from the one in step $q-1$, we will add an oscillatory perturbation with frequency λ_q. Meanwhile, the size of the Reynolds stress will be measured by an amplitude δ_q. These parameters are given by

$$\lambda_q = 2\pi \lceil a^{b^q} \rceil, \tag{3.3}$$

$$\delta_q = \lambda_q^{-2\beta}, \tag{3.4}$$

where $\lceil x \rceil$ denotes the ceiling, that is, the smallest integer $n \geq x$. The parameters $a, b > 1$ are very large and very close to 1, respectively. They will be chosen depending on the exponent $0 < \beta < 1/3$ that appears in Theorem 1.7, on Ω and on the initial subsolution. We introduce another parameter $\alpha > 0$ that will be very small. The necessary size of all of the parameters will be discovered in the proof.

Throughout the process we will also try to achieve a given energy profile $e \in C^\infty([0,T])$, which must satisfy the inequality (1.3). We will also assume

$$\sup_{t \in [0,T]} \left| \frac{d}{dt} e(t) \right| \leq 1. \tag{3.5}$$

We will see that this can be assumed without losing generality.

Unlike the construction on the torus in [7], it is essential that we only perturb the field in the region where the Reynolds stress is nonzero. Hence, we have to pay special attention to the support of the fields.

Since the map $(v_0, p_0, \mathring{R}_0)(\cdot, t)$ is assumed to be compactly supported at each time $t \in [0,T]$, we shall see that with a suitable rescaling we may assume that its support and Ω are contained in $(0,1)^3$. This is useful because sometimes it will be convenient to consider that we are working with periodic boundary conditions (that is, in $\mathbb{T}^3$) to reuse the results in [7]. On the other hand, $\mathring{R}_0(\cdot, t)$ is supported in a potentially smaller domain $\overline{\Omega}$. In our construction we must ensure that we do not perturb the subsolution outside of this set.

It will be convenient to do an additional rescaling in our problem. In the rescaled problem the initial subsolution will depend on a, but we assume that nevertheless there exists a sequence $\{y_N\}_{N=0}^\infty$ independent of the parameters such that

$$\|v_0\|_N + \|\partial_t v_0\|_N \leq y_N, \tag{3.6}$$

$$\|p_0\|_N \leq y_N, \tag{3.7}$$

$$\|\mathring{R}_0\|_N + \|\partial_t \mathring{R}_0\|_N \leq y_N. \tag{3.8}$$

Since the initial Reynolds stress $\mathring{R}_0$ and its derivatives vanish at $\partial\Omega \times [0,T]$, for any $k \in \mathbb{N}$ there exists a constant C_k such that for any $x \in \Omega$ we have

$$|\mathring{R}_0(x,t)| \leq C_k \operatorname{dist}(x, \partial\Omega)^k. \tag{3.9}$$

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The constants C_k are independent of a by (3.8). We define

$$d_q := \left(\frac{\delta_{q+2}\lambda_{q+1}^{-6\alpha}}{4C_{10}}\right)^{1/10}. \tag{3.10}$$

Hence, we have

$$|\mathring{R}_0(x,t)| \leq \frac{1}{4}\delta_{q+2}\lambda_{q+1}^{-6\alpha} \qquad \forall x \in \Omega,\ \operatorname{dist}(x,\partial\Omega) \leq d_q. \tag{3.11}$$

At step q the perturbation will be localized in a central region

$$A_q := \{x \in \Omega : \operatorname{dist}(x,\partial\Omega) \geq d_q\} \tag{3.12}$$

so that $(v_q, p_q, \mathring{R}_q)$ equals the initial subsolution in $(\mathbb{R}^3\backslash A_q) \times [0,T]$. Note that $d_q \to 0$ as $q \to \infty$ because so does δ_{q+2}. Therefore, in the limit the perturbation covers all of the region where $\mathring{R}_0$ is nonzero. However, the velocity is not modified outside this set.

As we have mentioned, the error introduced in the gluing step of [7] is spread throughout the whole space. To avoid this, we introduce an additional gluing *in space*, which we will explain in more detail in the following sections.

The complete list of inductive estimates is the following:

$$(v_q, p_q, \mathring{R}_q) = (v_0, p_0, \mathring{R}_0) \qquad \text{outside } A_q \times [0,T], \tag{3.13}$$

$$\|\mathring{R}_q\|_0 \leq \delta_{q+1}\lambda_q^{-6\alpha}, \tag{3.14}$$

$$\left\|v_q\right\|_1 \leq M\delta_q^{1/2}\lambda_q, \tag{3.15}$$

$$\left\|v_q\right\|_0 \leq 1 - \delta_q^{1/2}, \tag{3.16}$$

$$\delta_{q+1}\lambda_q^{-\alpha} \leq e(t) - \int_\Omega \left|v_q\right|^2 dx \leq \delta_{q+1}, \tag{3.17}$$

where M is a geometric constant that depends on Ω and is fixed throughout the iterative process.

Remark 3.1. If Ω has several connected components Ω^j, we may fix an energy profile e^j in each of them. In that case, (3.17) would have to be replaced by

$$\delta_{q+1}\lambda_q^{-\alpha} \leq e^j(t) - \int_{\Omega^j} \left|v_q\right|^2 dx \leq \delta_{q+1}.$$

Since the construction does not differ much, for simplicity we will assume that Ω is connected.

The following proposition is the key result to prove Theorem 1.7, because it establishes the existence of the iterative scheme in the convex integration process.

Proposition 3.2. *Let $T > 0$ and let $\Omega \subset (0,1)^3 \subset \mathbb{R}^3$ be an open set with smooth boundary and with a finite number of connected components. Let $(v_0, p_0, \mathring{R}_0) \in C^\infty(\mathbb{R}^3 \times [0,T])$ be a subsolution whose support is contained in $(0,1)^3 \times [0,T]$ and such that $\operatorname{supp} \mathring{R}_0 \subset \overline{\Omega} \times [0,T]$. Furthermore, assume that (3.6)–(3.8) are satisfied for some sequence of positive numbers $\{y_N\}_{N=0}^\infty$. There exists a constant M depending only on Ω with the following property: Assume $0 < \beta < 1/3$ and*

$$1 < b < \min\left\{\frac{1-\beta}{2\beta},\ \frac{11}{10}\right\}. \tag{3.18}$$

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Then there exists an α_0 *depending on* β *and* b *such that for any* $0 < \alpha < \alpha_0$ *there is an* a_0 *depending on* β, b, α, Ω, *and* $\{y_N\}_{N=0}^{\infty}$ *such that for any* $a \geq a_0$ *the following holds: Given a strictly positive energy profile satisfying (3.5) and a subsolution* $(v_q, p_q, \mathring{R}_q)$ *satisfying (3.13)–(3.17), there exists a subsolution* $(v_{q+1}, p_{q+1}, \mathring{R}_{q+1})$ *satisfying the same equations (3.13)–(3.17) with* q *replaced by* $q+1$. *Furthermore, we have the estimate*

$$\left\|v_{q+1} - v_q\right\|_0 + \frac{1}{\lambda_{q+1}} \left\|v_{q+1} - v_q\right\|_1 \leq M\delta_{q+1}^{1/2}. \tag{3.19}$$

We wish to iterate this result to construct a sequence of subsolutions whose limit will be the desired weak solution. However, in order to start the process, the first term in the sequence must satisfy the inductive hypotheses (3.13)–(3.17). Since we do not assume any bounds on $(v_0, p_0, \mathring{R}_0)$, these hypotheses will not be satisfied by the initial subsolution, in general. Although a time dilation would almost solve the problem, we need the following lemma to fully prepare the initial subsolution:

Lemma 3.3. *Let* $T > 0$ *and let* $\Omega \subset (0,1)^3 \subset \mathbb{R}^3$ *be an open set with smooth boundary and with a finite number of connected components. Let* $(v_0, p_0, \mathring{R}_0) \in C^\infty(\mathbb{R}^3 \times [0,T])$ *be a subsolution whose support is contained in* $(0,1)^3 \times [0,T]$ *and such that* $\operatorname{supp} \mathring{R}_0 \subset \overline{\Omega} \times [0,T]$. *Let* $\lambda > 0$ *be a sufficiently large constant. There exists a subsolution* $(v, p, \mathring{R}) \in C^\infty(\mathbb{R}^3 \times [0,T])$ *such that for any* $N \geq 0$ *we have*

$$\|v\|_N \lesssim \lambda^N, \|\mathring{R}\|_0 \leq \lambda^{-1/2}$$

where the implicit constants are independent of λ. *In addition, the energy satisfies*

$$\int_\Omega |v_0|^2 dx < \int_\Omega |v|^2 dx < \int_\Omega |v_0|^2 dx + 6\|\mathring{R}_0\|_0 \, |\Omega| \,. \tag{3.20}$$

Furthermore, $(v, p, \mathring{R})$ *equals the initial subsolution outside the set*

$$A_* := \left\{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \lambda^{-1/12}\right\} \times [0,T].$$

While necessary, this result is nothing new and one could easily obtain it by combining [24] and Lemma 2.9, or using the ideas in [36]. For completeness, we sketch its proof at the end of Section 7, considering a simplified version of the preceding construction.

3.2. Proof of Theorem 1.7

We first prove the theorem under the assumption that the subsolution $(v_0, p_0, \mathring{R}_0)(\cdot, t)$ is compactly supported for each time $t \in [0,T]$. This assumption will be relaxed in next subsection to a condition on the support of $\mathring{R}_0$. We fix $0 < \beta < 1/3$, we choose b satisfying (3.18) and α smaller than the threshold given by Proposition 3.2. Next, we use the scale invariance of the Euler equations and subsolutions

$$v_0(x,t) \mapsto v_0(\rho x, \rho t), \quad p_0(x,t) \mapsto p_0(\rho x, \rho t), \quad \mathring{R}_0(x,t) \mapsto \mathring{R}_0(\rho x, \rho t)$$

to assume that $\Omega \subset (0,1)^3$ and $\operatorname{supp}(v_0, p_0, \mathring{R}_0) \subset (0,1)^3 \times [0,T]$. The desired energy profile must also be modified: $e(t) \mapsto \rho^{-3} e(t)$. Note that this preserves (1.3). For convenience, we denote the rescaled subsolution like the original. It then suffices to construct the desired solution in this case and then reverse the change of variables.

Next, we use Lemma 3.3 with $\lambda = \lambda_1^{12\alpha}$ to obtain a subsolution $(v_1, p_1, \mathring{R}_1)$ that equals the initial subsolution outside the set

$$\{x \in \Omega : d(x, \partial\Omega) > \lambda_1^{-\alpha}\} \times [0,T]$$

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and satisfies the estimates

$$\|v_1\|_N \leq c_N \lambda_1^{12N\alpha}, \|\mathring{R}_1\|_0 \leq \lambda_1^{-6\alpha},$$

where the constants c_N are independent of λ_1 but they will depend on Ω and the initial subsolution. In addition, by (1.3) we have

$$e(t) > \int_\Omega |v_0|^2 dx + 6\|\mathring{R}_0\|_0 |\Omega| > \int_\Omega |v_1|^2 dx.$$

Next, we use another scale invariance of the Euler equations (and the definition of subsolution):

$$v(x,t) \mapsto \Gamma v(x, \Gamma t), \quad p(x,t) \mapsto \Gamma^2 p(x, \Gamma t), \quad \mathring{R}(x,t) \mapsto \Gamma^2 \mathring{R}(x, \Gamma t).$$

We choose

$$\Gamma := \delta_2^{1/2} \max\left\{1, \operatorname{supp}_t \left(e(t) - \int_\Omega |v_1(x,t)|^2 dx\right)\right\}^{-1/2}$$

and we begin to work in this rescaled setting, which we will indicate with a superscript r. We are thus working in the interval $[0, \widetilde{T}]$, where $\widetilde{T} := \Gamma^{-1} T$, and we try to prescribe the energy profile $\tilde{e} := \Gamma^2 e(t)$. By construction of Γ we have

$$\sup_t \left(\tilde{e}(t) - \int_\Omega \left|v_1^r(x,t)\right|^2 dx\right) \leq \delta_2$$

and

$$\inf_t \left(\tilde{e}(t) - \int_\Omega \left|v_1^r(x,t)\right|^2 dx\right) = \Gamma^2 \inf_t \left(e(t) - \int_\Omega |v_1(x,t)|^2 dx\right).$$

It follows from the definition of Γ that if a is sufficiently large we have

$$\lambda_1^\alpha \left(\frac{\Gamma^2}{\delta_2}\right) \inf_t \left(e(t) - \int_\Omega |v_0(x,t)|^2 dx\right) \geq 1.$$

so (3.17) holds. On the other hand,

$$\sup_t |\tilde{e}'(t)| \leq \Gamma^{3/2} \sup_t |e'(t)| \leq 1$$

because Γ becomes arbitrarily small by increasing a.

Next, we observe that $(v_0^r, p_0^r, \mathring{R}_0^r)$ still satisfies (3.6)–(3.8) with the same sequence $\{y_N\}_{N=0}^\infty$. Regarding $(v_1^r, p_1^r, \mathring{R}_1^r)$, it follows from the definition of the rescaling that

$$\|\mathring{R}_1^r\|_0 \leq \delta_2 \lambda_1^{-6\alpha}.$$

On the other hand, since the constants c_N are independent of λ_1, for sufficiently large a we have

$$\begin{aligned} \left\|v_1^r\right\|_0 &\leq \delta_2^{1/2} \|v_1\|_0 \leq \delta_2^{1/2} c_0 \leq 1 - \delta_1^{1/2}, \\ \left\|v_1^r\right\|_1 &\leq \delta_2^{1/2} \|v_1\|_1 \leq \delta_2^{1/2} c_1 \lambda_1^{12\alpha} \leq M \delta_1^{1/2} \lambda_1. \end{aligned}$$

Finally, $(v_1^r, p_1^r, \mathring{R}_1^r) = (v_0^r, p_0^r, \mathring{R}_0^r)$ outside

$$\{x \in \Omega : d(x, \partial\Omega) > \lambda_1^{-\alpha}\} \times [0, \widetilde{T}].$$

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Let us consider the sets A_q defined in (3.12). We see that $(v_1^r, p_1^r, \mathring{R}_1^r) = (v_0^r, p_0^r, \mathring{R}_0^r)$ outside $A_1 \times [0, \widetilde{T}]$ for sufficiently small α and sufficiently large a. Remember that δ_q and λ_q depend on a through the expressions (3.4) and (3.3), respectively.

From now on we assume that we are working in this rescaled problem and we omit the superscript r. Once we obtain the desired weak solution in this setting, to obtain the solution to the original problem it suffices to undo the scaling. By the previous discussion, the energy profile satisfies the inductive hypotheses (3.5) and the subsolution $(v_1, p_1, \mathring{R}_1)$ satisfies (3.13)–(3.17). In addition, the initial subsolution $(v_0, p_0, \mathring{R}_0)$ satisfies (3.6)–(3.8) for some sequence $\{y_N\}_{N=0}^\infty$ that does not depend on a. Therefore, we may apply Proposition 3.2 iteratively, obtaining a sequence of compactly supported smooth subsolutions $\{(v_q, p_q, \mathring{R}_q)\}_{q=1}^\infty$.

It follows from (3.19) that v_q converges uniformly to some continuous map v. On the other hand, note that the pressure p_q is the only compactly supported solution of

$$\Delta p_q = \operatorname{div}\operatorname{div}(-v_q \otimes v_q + \mathring{R}_q).$$

Therefore, p_q also converges to some pressure $p \in L^r(\mathbb{R}^3)$ for any $1 \leq r < \infty$. Since $\mathring{R}_q$ converges uniformly to 0, we conclude that the pair (v, p) is a weak solution of the Euler equations.

Furthermore, using (3.19) we obtain

$$\begin{aligned}\sum_{q=1}^\infty \left\|v_{q+1} - v_q\right\|_{\beta'} &\leq \sum_{q=1}^\infty C(\beta', \beta) \left\|v_{q+1} - v_q\right\|_0^{1-\beta'} \left\|v_{q+1} - v_q\right\|_\beta^{\beta'} \\ &\leq C(\beta', \beta) \sum_{q=1}^\infty (M\delta_{q+1}^{1/2})^{1-\beta'} \left(M\delta_{q+1}^{1/2}\lambda_q\right)^{\beta'} \\ &\leq M\, C(\beta', \beta) \sum_{q=1}^\infty \lambda_q^{\beta'-\beta},\end{aligned}$$

so $\{v_q\}_{q=1}^\infty$ is uniformly bounded in $C_t^0 C_x^{\beta'}$ for all $\beta' < \beta$. To recover the time regularity, see [7].

Next, note that $(v_q, p_q, \mathring{R}_q) = (v_0, p_0, 0)$ in $(\mathbb{R}^3 \backslash \Omega) \times [0, T]$ for all q by (3.13) and the definition of A_q. Hence, we have $(v, p) = (v_0, p_0)$ in $(\mathbb{R}^3 \backslash \Omega) \times [0, T]$.

Finally, it follows from (3.17) and the fact that $\delta_{q+1} \to 0$ as $q \to \infty$ that $\int_\Omega |v(x,t)|^2 dx = e(t)$, as we wanted. This completes the proof of the theorem for the case that the initial subsolution is compactly supported for all time.

3.3. Dropping the compact support condition

Once we have proved Theorem 1.7 for the case that $(v_0, p_0, \mathring{R}_0)(\cdot, t)$ is compactly supported, it is easy to relax this condition and show that it suffices that $\mathring{R}_0(\cdot, t)$ is compactly supported.

Indeed, let us choose a bounded domain with smooth boundary $U \Supset \Omega$. As we mentioned in Lemma 2.11 and Remark 2.12, any subsolution defined in all $\mathbb{R}^3$ automatically satisfies the conditions in Theorem 1.6. Hence, there exists a subsolution $(\widetilde{v}_0, \widetilde{p}_0, \widetilde{\mathring{R}}_0) \in C^\infty(\mathbb{R}^3 \times [0, T])$ that extends $(v_0, p_0, \mathring{R}_0)$ outside $\overline{U}$ and that is compactly supported in each time slice. Since it is an extension, we see that the Reynolds stress vanishes in $\overline{U} \backslash \Omega$.

We can now apply Theorem 1.7 in the case that the subsolution is compactly supported for each time slice, thus obtaining a weak solution (v, p) with the appropriate regularity and such that $(v, p) = (v_0, p_0)$ in $(\overline{U} \backslash \Omega) \times [0, T]$. Since $\Omega \Subset U$, we may glue this region back into (v_0, p_0), obtaining a weak solution that equals (v_0, p_0) outside $\Omega \times [0, T]$.

Finally, since Ω and $\mathbb{R}^3 \backslash \overline{U}$ are disjoint, when we apply Theorem 1.7 we may fix the energy profile in each region independently. Therefore, we can prescribe the energy profile in Ω of the final weak solution to be any smooth function satisfying condition (1.3).

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4. Proof of Proposition 3.2

The different steps in the proof of Proposition 3.2 are presented in Subsection 4.1. These steps are elaborated in the next subsections, and we complete the proof of the proposition in Subsection 4.6. Roughly speaking, our proof adapts the arguments of [7] to the nonperiodic setting. To do so, we introduce an additional step in the iteration: a gluing in space that ensures that the error does not spread out to the whole space when we glue in time.

4.1. Stages of the proof

1. Preparing the subsolution. We mollify our subsolution $(v_q, p_q, \mathring{R}_q)$ to avoid the *loss of derivatives problem*, obtaining a new subsolution $(v_\ell, p_\ell, \mathring{R}_\ell)$. It is convenient to glue it *in space* to the original subsolution far from the turbulent zone.
2. Gluing in space. We pick a collection of times $\{t_i\} \subset [0, T]$ and we consider the solutions (v_i, p_i) of the Euler equations with initial data $\widetilde{v}_\ell(t_i)$. We glue them in space to $(\widetilde{v}_\ell, \widetilde{p}_\ell, \widetilde{\mathring{R}}_\ell)$, obtaining new subsolutions $(\widetilde{v}_i, \widetilde{p}_i, \widetilde{\mathring{R}}_i)$. The error $\widetilde{\mathring{R}}_i$ is small and these subsolutions equal $(v_0, p_0, \mathring{R}_0)$ near $\partial\Omega$.
3. Gluing in time. We glue together the subsolutions $(\widetilde{v}_i, \widetilde{p}_i, \widetilde{\mathring{R}}_i)$, obtaining a new subsolution $(\overline{v}_q, \overline{p}_q, \overline{\mathring{R}}_q)$ in which most of the error is concentrated in temporally disjoint regions. The error remains localized within Ω owing to the fact that the differences between the subsolutions $(\widetilde{v}_i, \widetilde{p}_i, \widetilde{\mathring{R}}_i)$ vanish near $\partial\Omega$.
4. Perturbation. We add a highly oscillatory perturbation to reduce the error. In fact, we add many corrections, each of them reducing the error in one of the temporally disjoint regions. These perturbations do not interact with each other, which yields the optimal estimates for the new subsolution $(v_{q+1}, p_{q+1}, \mathring{R}_{q+1})$.

Throughout the iterative process we will use the notation $x \lesssim y$ to denote $x \leq Cy$ for a sufficiently large constant $C > 0$ that is independent of a, b, and q. However, the constant is allowed to depend on α, β, Ω, and $\{y_N\}_{N=0}^{\infty}$ and it may change from line to line.

4.2. Preparing the subsolution

The first step consists in mollifying the field in order to avoid the *loss of derivatives* problem, which is typical of convex integration. The problem is the following: to control a Hölder norm of $(v_{q+1}, p_{q+1}, \mathring{R}_{q+1})$ we need estimates of higher-order Hölder norms of $(v_q, p_q, \mathring{R}_q)$. As the iterative process goes on, we need to estimate higher and higher Hölder norms of the initial terms of the sequence to control just the first few Hölder norms of the subsolution. However, if we mollify the subsolution $(v_q, p_q, \mathring{R}_q)$, we can control all the derivatives in terms of the first few Hölder norms and the mollification parameter.

Note that this process changes the subsolution in the whole space, yet mollification is only strictly necessary in $A_q \times [0, T]$, as $(v_q, p_q, \mathring{R}_q)$ equals $(v_0, p_0, \mathring{R}_0)$ outside of this set. Furthermore, it will be convenient for later estimates that the resulting subsolution equals the initial subsolution far from the turbulent zone, as this is a property that we want to impose unto $(v_{q+1}, p_{q+1}, \mathring{R}_{q+1})$. Hence, our approach consists in gluing the mollified subsolution to the initial subsolution, which is not quite demanding because both subsolutions are very close far from the turbulent zone.

We begin by fixing a mollification kernel *in space* $\psi \in C_c^\infty(\mathbb{R}^3)$ and we introduce the mollification parameter

$$\ell := \frac{\delta_{q+1}^{1/2}}{\delta_q^{1/2} \lambda_q^{1+3\alpha}}. \tag{4.1}$$

Since $(\delta_{q+1}/\delta_q)^{1/2} = \lambda_q^{-\beta(b-1)}$ and our assumption (3.18) implies that $\beta(b-1) < 1/2$, we see that we may choose α sufficiently small and a sufficiently large so as to have

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$$\lambda_q^{-3/2} \leq \ell \leq \lambda_q^{-1}. \tag{4.2}$$

Note that our definition of ℓ differs from the one in [7] by a factor of $\lambda_q^{-3\alpha/2}$. The coefficients of α used in [7] are fine-tuned to their proof. Since we do some things differently, it is not surprising that we have to change some of these coefficients. In general the factor $\lambda_q^{-3\alpha/2}$ is quite harmless, as only simple relationships like (4.2) are used throughout most of the paper, and these are the same here and in [7]. The actual value of ℓ is only used at the very end, when fine relationships between the parameters are needed to estimate $\mathring{R}_{q+1}$. We will study these situations when they arise, but in any case we will see that the extra factor is essentially irrelevant. Indeed, α is assumed to be so small that in those inequalities the term containing α is negligible. Our definition of ℓ leads to a different coefficient multiplying α, but this only changes how small α has to be chosen, so it is not important.

After this brief digression, we define

$$\begin{aligned} v_\ell &:= v_q * \psi_\ell, \\ p_\ell &:= p_q * \psi_\ell + \left|v_q\right|^2 * \psi_\ell - |v_\ell|^2, \\ \mathring{R}_\ell &:= \mathring{R}_q * \psi_\ell - (v_q \mathring{\otimes} v_q) * \psi_\ell + v_\ell \mathring{\otimes} v_\ell, \end{aligned}$$

where the convolution with ψ_ℓ is in space only and $f \mathring{\otimes} g$ denotes the traceless part of the tensor $f \otimes g$. It is easy to check that the triplet $(v_\ell, p_\ell, \mathring{R}_\ell)$ is a subsolution and by [7, Proposition 2.2] we have the following estimates:

$$\begin{aligned} \left\|v_\ell - v_q\right\|_0 &\lesssim \delta_{q+1}^{1/2} \lambda_q^{-\alpha}, \\ \|v_\ell\|_{N+1} &\lesssim \delta_q^{1/2} \lambda_q \, \ell^{-N} \forall N \geq 0, \\ \|\mathring{R}_\ell\|_{N+\alpha} &\lesssim \delta_{q+1} \ell^{-N+3\alpha} \quad \forall N \geq 0, \\ \left|\int_\Omega \left|v_q\right|^2 - |v_\ell|^2 \, dx\right| &\lesssim \delta_{q+1} \ell^{\alpha}. \end{aligned}$$

Note that we have an extra factor $\ell^{2\alpha}$ in the estimate for the Reynolds stress in comparison to [7]. This comes from the extra factor $\lambda_q^{-3\alpha}$ in $\|\mathring{R}_q\|_0$ and from our definition of ℓ. They cause an extra factor $\lambda_q^{-3\alpha}$ to appear in the estimate, which yields an extra factor $\ell^{2\alpha}$ by (4.2).

Obtaining an extra factor $\ell^{2\alpha}$ (actually ℓ^{α} would suffice) is our reason for modifying the inductive estimate of the Reynolds stress and the definition of ℓ. We will use this extra factor to compensate a suboptimal estimate that we will be forced to use in Section 5.

Once we have mollified the subsolution, we will glue it to the initial subsolution far from the turbulent zone. It will be convenient to divide $A_{q+1} \backslash A_q$ into several pieces because we will have to do several constructions in this region. We define

$$\sigma = \frac{1}{5}(d_q - d_{q+1}), \tag{4.3}$$

where d_q was defined in (3.10). Using the elementary inequalities

$$2\pi \leq \frac{\lambda_q}{a^{b^q}} \leq 4\pi \tag{4.4}$$

we deduce $\delta_{q+2} \gtrsim \lambda_q^{-2\beta b^2}$. By (3.18) we have $b^2 < 5/4$, so for sufficiently small α we have

$$d_q \gtrsim \lambda_q^{-1/11}.$$

Therefore,

$$\sigma^{-1} \lesssim \lambda_q^{1/11}. \tag{4.5}$$

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In particular, $\sigma \gg \ell$. For $1 \leq j \leq 5$ we define

$$B_j := \mathbb{R}^3 \backslash [A_q + B(0, j)]$$

By hypothesis, $(v_q, p_q, \mathring{R}_q)$ equals $(v_0, p_0, \mathring{R}_0)$ outside $A_q \times [0, T]$. Hence, it follows from (3.6)–(3.8) that

$$\|v_\ell - v_0\|_{N;B_1} + \|\partial_t v_\ell - \partial_t v_0\|_{N;B_1} \lesssim \ell^2, \tag{4.6}$$

$$\|p_\ell - p_0\|_{N;B_1} \lesssim \ell^2, \tag{4.7}$$

$$\|\mathring{R}_\ell - \mathring{R}_0\|_{N;B_1} \lesssim \ell^2. \tag{4.8}$$

Thus, both subsolutions are very close in this region, which makes gluing them much easier. Taking into account (4.6), it follows from Lemma 2.6 that there exists a potential $A \in C^\infty(B_1 \times [0, T], \mathcal{A}^3)$ such that $\operatorname{div} A = v_\ell - v_0$ in $B_1 \times [0, T]$ and

$$\|A\|_{N+1+\alpha;B_1} + \|\partial_t A\|_{N+1+\alpha;B_1} \lesssim \ell^2 \qquad \forall N \geq 0.$$

Therefore, the matrices S_1 and M that appear in Lemma 2.11 satisfy the estimates $\|S_1\|_{N+\alpha} + \|M\|_{N+\alpha} \lesssim \ell^2$. We introduce this estimates into Lemma 2.11 to perform a gluing in the region $B_1 \backslash B_2$. Since $\sigma \gg \ell$ we may essentially ignore any factor coming from the derivatives of the cutoff by absorbing it into the ℓ factor. Carrying out the rest of the construction of Lemma 2.11, we conclude that there exists a smooth subsolution such that

$$(\widetilde{v}_\ell, \widetilde{p}_\ell, \mathring{\widetilde{R}}_\ell)(x,t) = \begin{cases} (v_\ell, p_\ell, \mathring{R}_\ell)(x,t) & \text{if } x \in A_q + B(0, \sigma), \\ (v_0, p_0, \mathring{R}_0)(x,t) \text{if } x \in B_2 \end{cases} \tag{4.9}$$

and satisfies the estimates

$$\left\|\widetilde{v}_\ell - v_q\right\|_0 \lesssim \delta_{q+1}^{1/2} \lambda_q^{-\alpha}, \tag{4.10}$$

$$\|\widetilde{v}_\ell\|_{N+1} \lesssim \delta_q^{1/2} \lambda_q \, \ell^{-N} \forall N \geq 0, \tag{4.11}$$

$$\|\mathring{\widetilde{R}}_\ell\|_{N+\alpha} \lesssim \delta_{q+1} \ell^{-N+3\alpha} \quad \forall N \geq 0, \tag{4.12}$$

$$\left| \int_\Omega \left|\widetilde{v}_q\right|^2 - |v_\ell|^2 \, dx \right| \lesssim \delta_{q+1} \ell^\alpha. \tag{4.13}$$

That is, we have obtained a new subsolution satisfying the same estimates as $(v_\ell, p_\ell, \mathring{R}_\ell)$ but with the additional property of being equal to the initial subsolution far from the turbulent zone.

Although many constructions in [7] work in $\mathbb{R}^3$ with very little or no modification, it is more convenient to work with periodic fields so that we may use results from [7] directly. Note that the subsolution that we have just obtained is supported in $(0, 1)^3 \times [0, T]$ because it equals the initial subsolution far form the turbulent zone. This allows us to consider its periodic extension to $\mathbb{R}^3/\mathbb{Z}^3$, which we denote the same. From now on, we consider that we are working in this setting.

4.3. Overview of the gluing in space

The key idea introduced by Isett in [35] is to glue *in time* exact solutions of the Euler equations to obtain a new subsolution $(\overline{v}_q, \overline{p}_q, \mathring{\overline{R}}_q)$ such that $\overline{v}_q$ is close to v_q but $\mathring{\overline{R}}_q$ is supported in a series of disjoint temporal regions of the appropriate length. This allows the use of Mikado flows, leading to the optimal regularity C^β for any $\beta < 1/3$ (as in Onsager's conjecture).

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More specifically, we define the length

$$\tau_q = \frac{\ell^{2\alpha}}{\delta_q^{1/2}\lambda_q} \tag{4.14}$$

and we consider the smooth solutions of the Euler equations

$$\begin{cases} \partial_t v_i + \operatorname{div}(v_i \otimes v_i) + \nabla p_i = 0, \\ \operatorname{div} v_i = 0, \\ v_i(\cdot, t_i) = \widetilde{v}_\ell(\cdot, t_i), \end{cases} \tag{4.15}$$

where $t_i := i\tau_q$. We will see that they are defined in the time interval $[t_i - \tau_q, t_i + \tau_q]$. The pressure is recovered as the unique solution to the equation $-\Delta p_i = \operatorname{tr}(\nabla v_i \nabla v_i)$ with the normalization

$$\int_{\mathbb{T}^3} p_i(x,t)\,dx = \int_{\mathbb{T}^3} \widetilde{p}_\ell(x,t)\,dx. \tag{4.16}$$

We will see that these solutions remain sufficiently close to $\widetilde{v}_\ell$ in their respective intervals. Following Isett's ideas, we would like to glue them in time to obtain a subsolution in the whole interval $[0, T]$. The velocity field would remain sufficiently close to $\widetilde{v}_\ell$ while the Reynolds stress would be localized to the intersection of the consecutive time intervals.

However, even if $\mathring{R}_\ell$ is well localized, the solutions v_i will immediately differ from $\widetilde{v}_\ell$ in the whole space. Furthermore, different solutions v_i, v_{i+1} will also differ in the whole space during the intersection of their temporal domains. If we tried to apply Isett's procedure to them we would obtain a Reynolds stress that spreads throughout the whole space. This is not suitable for our purposes, so we must modify Isett's approach.

What we will do is to glue *in space* the exact solutions v_i to $\widetilde{v}_\ell$ in the region where $\mathring{\widetilde{R}}_\ell$ is small, obtaining subsolutions $(\widetilde{v}_i, \widetilde{p}_i, \mathring{\widetilde{R}}_i)$. The Reynolds stress will no longer be 0, but it will be so small that we may ignore it in the current iteration. Since these subsolutions will coincide far from the turbulent zone, we will be able to glue them in time while keeping the Reynolds stress localized.

The actual process is not so simple because the difference between the exact solutions (v_i, p_i) and the subsolution $(\widetilde{v}_\ell, \widetilde{p}_\ell, \mathring{\widetilde{R}}_\ell)$ is too big, leading to an unacceptably large error if we try to glue them. Our approach consists in producing a series of intermediate subsolutions that act as a sort of interpolation between them. Instead of a single gluing we perform a big number of them, going from v_i to $\widetilde{v}_\ell$ far from the turbulent zone. The difference between two consecutive intermediate subsolutions will be very small so that the error introduced in each of these middle gluings is small.

At the end of this process we will obtain subsolutions $(\widetilde{v}_i, \widetilde{p}_i, \mathring{\widetilde{R}}_i)$ such that

$$(\widetilde{v}_i, \widetilde{p}_i, \mathring{\widetilde{R}}_i) = (v_0, p_0, \mathring{\widetilde{R}}_0) \text{ in } B_3 \times [t_i - \tau_q, t_i + \tau_q], \tag{4.17}$$

$$\widetilde{v}_i(t_i, \cdot) = \widetilde{v}_\ell(t_i, \cdot). \tag{4.18}$$

In addition, for $|t - t_i| \le \tau_q$ and any $N \ge 0$ we will have the following estimates:

$$\|\mathring{\widetilde{R}}_i\|_0 \le \frac{1}{2}\delta_{q+2}\lambda_{q+1}^{-6\alpha}, \tag{4.19}$$

$$\|\widetilde{v}_i - \widetilde{v}_\ell\|_{N+\alpha} \lesssim \tau_q \delta_{q+1} \ell^{-N-1+\alpha}, \tag{4.20}$$

$$\left\|D_{t,\ell}(\widetilde{v}_i - \widetilde{v}_\ell)\right\|_{N+\alpha} \lesssim \delta_{q+1}\ell^{-N-1+\alpha}, \tag{4.21}$$

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where we write

$$D_{t,\ell} := \partial_t + \widetilde{v}_\ell \cdot \nabla$$

for the transport derivative. Furthermore, there exist smooth vector potentials $\widetilde{z}_i$ defined in $[t_i - \tau_q, t_i + \tau_q]$ such that $\widetilde{v}_i = \operatorname{curl} \widetilde{z}_i$ and at the intersection of two intervals we have:

$$\|\widetilde{z}_i - \widetilde{z}_{i+1}\|_{N+\alpha} \lesssim \tau_q \delta_{q+1} \ell^{-N+\alpha}, \tag{4.22}$$

$$\|D_{t,\ell}(\widetilde{z}_i - \widetilde{z}_{i+1})\|_{N+\alpha} \lesssim \delta_{q+1} \ell^{-N+\alpha}. \tag{4.23}$$

These estimates are completely analogous to the ones in [7, Proposition 3.3, Proposition 3.4] but we have the additional benefit of the fields being equal to the initial subsolution far from the turbulent zone. We do have to pay a price because now we have subsolutions instead of solutions of the Euler equations. Nevertheless, the errors $\mathring{\widetilde{R}}_i$ are so small that we may ignore them until the $(q+1)$-th iteration.

All of the steps summarized here will be discussed in full detail in Section 5, where we derive the claimed estimates.

4.4. Overview of the gluing in time

Once we have our subsolutions $(\widetilde{v}_i, \widetilde{p}_i, \mathring{\widetilde{R}}_i)$ we will glue them in time. The goal is to obtain a subsolution defined in all $[0, T]$ that remains close to $\widetilde{v}_\ell$ but in which most of the Reynolds stress is localized to temporally disjoint regions of the appropriate length. This will allow us to correct the error in each region separately using Mikado flows. Let

$$t_i := i\tau_q, \qquad I_i := \left[t_i + \frac{1}{3}\tau_q, t_i + \frac{2}{3}\tau_q\right] \cap [0, T],$$

$$J_i := \left(t_i - \frac{1}{3}\tau_q, t_i + \frac{1}{3}\tau_q\right) \cap [0, T].$$

Note that $\{I_j, J_i\}$ is a pairwise disjoint decomposition of $[0, T]$. We choose a smooth partition of unity $\{\chi_i\}$ such that:

- $\sum_i \chi_i = 1$.
- $\operatorname{supp} \chi_i \cap \operatorname{supp} \chi_{i+2} = \emptyset$. Furthermore,

$$\operatorname{supp} \chi_i \subset \left(t_i - \frac{2}{3}\tau_q, t_i + \frac{2}{3}\tau_q\right), \qquad \chi_i(t) = 1 \quad \forall t \in J_i. \tag{4.24}$$

- For any i and $N \geq 0$ we have

$$\left\|\partial_t^N \chi_i\right\|_0 \lesssim \tau_q^{-N}. \tag{4.25}$$

We define

$$\overline{v}_q := \sum_i \chi_i \widetilde{v}_i, \qquad \overline{p}_q^{(1)} := \sum_i \chi_i \widetilde{p}_i, \qquad \overline{\mathcal{R}}_q^{(1)} := \sum_i \chi_i \mathring{\widetilde{R}}_i.$$

It is clear that $\overline{v}_q$ is divergence-free and it equals v_0 in $B_3 \times [0, T]$ because of (4.17). In addition, it inherits the estimates of $\widetilde{v}_i$:

Proposition 4.1. *The velocity field $\overline{v}_q$ satisfies the following estimates:*

$$\left\|\overline{v}_q - \widetilde{v}_\ell\right\|_\alpha \lesssim \delta_{q+1}^{1/2}\ell^\alpha, \tag{4.26}$$

$$\left\|\overline{v}_q - \widetilde{v}_\ell\right\|_{N+\alpha} \lesssim \tau_q\delta_{q+1}\ell^{-1-N+\alpha}, \tag{4.27}$$

$$\left\|\overline{v}_q\right\|_{1+N} \lesssim \delta_q^{1/2}\lambda_q\ell^{-N}, \tag{4.28}$$

$$\left|\int_\Omega \left|\overline{v}_q\right|^2 - |\widetilde{v}_\ell|^2\right| \lesssim \delta_{q+1}\ell^\alpha, \tag{4.29}$$

for all $N \geq 0$.

The proof is exactly the same as the proof of [7, Proposition 4.3, Proposition 4.5] because our fields $\widetilde{v}_i$ satisfy completely analogous estimates to the solutions v_i in [7].

We conclude that the new velocity $\overline{v}_q$ equals v_0 far from the turbulent zone, it is close to $\widetilde{v}_\ell$ and satisfies suitable bounds. Nevertheless, we must check if it leads to a subsolution. If $t \in J_i$, then in a neighborhood of t we have $\chi_i = 1$ while the rest of the cutoffs vanish. Thus, for all $t \in J_i$ we have

$$\overline{v}_q = \widetilde{v}_i, \qquad \overline{p}_q^{(1)} = \widetilde{p}_i, \qquad \overline{\mathcal{R}}_q^{(1)} = \mathring{\widetilde{R}}_i.$$

Since $(\widetilde{v}_i, \widetilde{p}_i, \mathring{\widetilde{R}}_i)$ is a subsolution, we have

$$\partial_t\overline{v}_q + \operatorname{div}(\overline{v}_q \otimes \overline{v}_q) + \nabla\overline{p}_q^{(1)} = \operatorname{div}\overline{\mathcal{R}}_q^{(1)}$$

On the other hand, if $t \in I_i$, then $\chi_j = 0$ for $j \neq i, i+1$ and $\chi_i + \chi_{i+1} = 1$. Hence, on I_i we have

$$\overline{v}_q = \chi_i\widetilde{v}_i + \chi_{i+1}\widetilde{v}_{i+1}, \qquad \overline{p}_q^{(1)} = \chi_i\widetilde{p}_i + \chi_{i+1}\widetilde{p}_{i+1}, \qquad \overline{\mathcal{R}}_q^{(1)} = \chi_i\mathring{\widetilde{R}}_i + \chi_{i+1}\mathring{\widetilde{R}}_{i+1}.$$

After a tedious computation we obtain

$$\begin{aligned}\partial_t\overline{v}_q + \operatorname{div}(\overline{v}_q \otimes \overline{v}_q) + \nabla\overline{p}_q^{(1)} - \operatorname{div}\overline{\mathcal{R}}_q^{(1)} &= \\ = \partial_t\chi_i(\widetilde{v}_i - \widetilde{v}_{i+1}) - \chi_i(1-\chi_i)\operatorname{div}((\widetilde{v}_i - \widetilde{v}_{i+1}) &\otimes (\widetilde{v}_i - \widetilde{v}_{i+1})),\end{aligned} \tag{4.30}$$

where we have used the fact that $(\widetilde{v}_i, \widetilde{p}_i, \mathring{\widetilde{R}}_i)$ are subsolutions.

In conclusion, for $t \in J_i$ the triplet $(\overline{v}_q, \overline{p}_q^{(1)}, \overline{\mathcal{R}}_q^{(1)})$ is trivially a subsolution, whereas for $t \in I_i$ it suffices to express the right-hand side of Equation (4.30) as the divergence of a symmetric matrix. However, we must do this carefully because it is very important to keep under control the spatial support of the Reynolds stress. Indeed, since $\overline{\mathcal{R}}_q^{(1)}$ equals $\mathring{R}_0$ outside $A_{q+1} \times [0, T]$, any perturbation in the Reynolds stress must be contained within $A_{q+1} \times [0, T]$ in order to satisfy the inductive property (3.13). Fortunately, we will be able to achieve this because the right-hand side of Equation (4.30) is supported in $A_q + B(0, 3\sigma)$ due to (4.17). We see that performing the gluing in space in the previous stage is crucial.

The logical approach would be to apply Lemma 2.9 to the right-hand side of (4.30) and define the new Reynolds stress as the sum of the obtained matrix and $\overline{\mathcal{R}}_q^{(1)}$. Unfortunately, we cannot simply do that because we cannot obtain the necessary estimates for the transport derivative from Lemma 2.9.

Nevertheless, this difficulty can be solved. In Section 6 we will find smooth symmetric matrices $\overline{\mathcal{R}}_q^{(2)}, \overline{\mathcal{R}}_q^{(3)}$ solving the equation

$$\operatorname{div}\left(\overline{\mathcal{R}}_q^{(2)} + \overline{\mathcal{R}}_q^{(3)}\right) = \partial_t\chi_i(\widetilde{v}_i - \widetilde{v}_{i+1})$$

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for $t \in I_i$. We set $\overline{\mathcal{R}}_q^{(2)}, \overline{\mathcal{R}}_q^{(3)} = 0$ for $t \notin \bigcup_i I_i$. Since the source term is supported in $A_q + B(0, 3\sigma)$ due to (4.17), we will be able to choose $\overline{\mathcal{R}}_q^{(2)}, \overline{\mathcal{R}}_q^{(3)}$ supported in $A_q + B(0, 4\sigma)$. The motivation for each matrix is the following:

- $\overline{\mathcal{R}}_q^{(2)}$ solves the equation except for a small error. We have good bounds for the derivative of the material derivative.
- $\overline{\mathcal{R}}_q^{(3)}$ corrects the errors introduced when fixing the support of $\overline{\mathcal{R}}_q^{(2)}$. We do not have good bounds for its material derivative, but its C^0-norm is very small.

Using these auxiliary matrices, we define

$$\mathring{\bar{R}}q^{(1)} := \overline{\mathcal{R}}_q^{(1)} + \overline{\mathcal{R}}_q^{(3)} - \frac{1}{3} \operatorname{tr} \overline{\mathcal{R}}_q^{(3)} \operatorname{Id}, \tag{4.31}$$

$$\mathring{\bar{R}}_q^{(2)} := \overline{\mathcal{R}}_q^{(2)} - \chi_i(1-\chi_i)(\widetilde{v}_i - \widetilde{v}_{i+1}) \mathring{\otimes} (\widetilde{v}_i - \widetilde{v}_{i+1}) - \frac{1}{3} \operatorname{tr} \overline{\mathcal{R}}_q^{(2)} \operatorname{Id}, \tag{4.32}$$

$$\mathring{\bar{R}}_q := \mathring{\bar{R}}_q^{(1)} + \mathring{\bar{R}}_q^{(2)}, \tag{4.33}$$

$$\overline{p}_q := \overline{p}_q^{(1)} - \chi_i(1-\chi_i) \, |\widetilde{v}_i - \widetilde{v}_{i+1}|^2 - \frac{1}{3} \operatorname{tr} \left(\overline{\mathcal{R}}_q^{(2)} + \overline{\mathcal{R}}_q^{(3)} \right). \tag{4.34}$$

By construction of $\overline{\mathcal{R}}_q^{(2)}$ and $\overline{\mathcal{R}}_q^{(3)}$, we see that $(\overline{v}_q, \overline{p}_q, \mathring{\bar{R}}_q)$ is a smooth subsolution. Furthermore, we have

$$(\overline{v}_q, \overline{p}_q, \mathring{\bar{R}}_q) = (v_0, p_0, \mathring{R}_0) \text{ in } B_4 \times [0, T]$$

because of (4.17) and the fact that $\overline{\mathcal{R}}_q^{(2)}, \overline{\mathcal{R}}_q^{(3)}$ are supported in $A_q + B(0, 4\sigma)$. We emphasize again the importance of performing a gluing in space to use $\widetilde{v}_i$ instead of the solutions v_i. Otherwise, we would have no control on the Reynolds stress because $v_i - v_{i+1}$ will in general be spread throughout the whole space.

We summarize the facts about the new Reynolds stress that we will prove in Section 6:

Proposition 4.2. *The smooth symmetric matrices* $\mathring{\bar{R}}_q^{(1)}, \mathring{\bar{R}}_q^{(2)}$ *satisfy*

$$\|\mathring{\bar{R}}_q^{(1)}\|_0 \leq \frac{3}{4} \delta_{q+2} \lambda_{q+1}^{-3\alpha}, \tag{4.35}$$

$$\operatorname{supp} \mathring{\bar{R}}_q^{(2)} \subset [A_q + B(0, 4\sigma)] \times \bigcup_i I_i, \tag{4.36}$$

$$\|\mathring{\bar{R}}_q^{(2)}\|_N \lesssim \delta_{q+1} \ell^{-N+\alpha}, \tag{4.37}$$

$$\|(\partial_t + \overline{v}_q \cdot \nabla) \mathring{\bar{R}}_q^{(2)}\|_N \lesssim \delta_{q+1} \delta_q^{1/2} \lambda_q \ell^{-N-\alpha}. \tag{4.38}$$

Comparing (4.37) with the analogous estimate in [7], we see that we estimate the C^N-norm instead of the $C^{N+\alpha}$-norm. This difference is immaterial because they merely use the $C^{N+\alpha}$-norm to estimate the C^N-norm, which leads to the same bound as (4.37).

In conclusion, $\mathring{\bar{R}}_q^{(1)}$ is so small that we may ignore it for the present iteration, whereas $\mathring{\bar{R}}_q^{(2)}$ is big but it is supported in temporally disjoint regions and it satisfies good estimates. The next stage of the process is aimed at correcting $\mathring{\bar{R}}_q^{(2)}$ by means of highly oscillatory perturbations.

All of the steps summarized here will be discussed in full detail in Section 6, where we derive the claimed estimates.

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4.5. Overview of the perturbation step

We have localized most of the Reynolds stress, that is, $\mathring{\overline{R}}_q{}^{(2)}$, to small disjoint temporal regions but to reduce it we must resort to convex integration.

First of all, we fix a cutoff $\phi_q \in C^\infty(\mathbb{R}^3, [0,1])$ that equals 1 in $A_q + \overline{B}(0, 4\sigma)$ and whose support is contained in $A_q + B(0, 5\sigma)$. In particular, $\phi_q \equiv 1$ on the support of $\mathring{\overline{R}}_q{}^{(2)}(\cdot, t)$ for all $t \in [0, T]$.

We follow the construction of [7] with Mikado flows, but there are some differences:

- We control the support of the perturbation by introducing the cutoff ϕ_q.
- To obtain the desired energy, we must use a slightly different normalization coefficient for the perturbation to account for the presence of the cutoff ϕ_q when integrating.
- We construct the new Reynolds stress using Lemma 2.9 so that we have control on its support. To apply this lemma we must introduce a minor correction w_L to ensure that the perturbation has vanishing angular momentum.

Since $\|\phi_q\|_N$ are much smaller than the C^N-norms of the other maps involved, the presence of the cutoff does not affect the estimates. In addition, w_L will be negligible because its size is determined by an integral quantity, which is very small for a highly oscillating perturbation.

The form of the perturbation $\widetilde{w}_{q+1} = v_{q+1} - v_q$ is $\widetilde{w}_{q+1} = w_0 + w_c + w_L$, where w_0 is the main perturbation term and it is used to cancel the Reynolds stress, w_c is a small correction to ensure that the perturbation is divergence-free and w_L is a tiny correction that ensures that the perturbation has vanishing angular momentum. This term is not present in [7] because they do not control the support of the Reynolds stress, so it suffices to use $w_{q+1} := w_0 + w_c$.

At the end of the process we will obtain a new subsolution $(v_{q+1}, p_{q+1}, \mathring{R}_{q+1})$ that equals $(v_0, p_0, \mathring{R}_0)$ outside $A_{q+1} \times [0, T]$ and satisfying the following estimates:

$$\left\|v_{q+1}\overline{v}_q\right\|_0 + \frac{1}{\lambda_{q+1}}\left\|v_{q+1}\overline{v}_q\right\|_1 \leq \frac{3}{4} M \delta_{q+1}^{1/2}, \tag{4.39}$$

$$\|\mathring{R}_{q+1}\|_0 \leq \delta_{q+2}\lambda_{q+1}^{-6\alpha}, \tag{4.40}$$

$$\delta_{q+2}\lambda_{q+1}^{-7\alpha} \leq e(t) - \int_\Omega \left|v_{q+1}(x,t)\right|^2 dx \leq \delta_{q+1}. \tag{4.41}$$

4.6. Proof of Proposition 3.2

Finally, we are ready to complete the proof of the proposition. The estimate (3.19) is a consequence of (4.10), (4.11), (4.26), (4.28) and (4.39):

$$\left\|v_{q+1} - v_q\right\|_0 + \lambda_{q+1}^{-1}\left\|v_{q+1} - v_q\right\|_1 \leq \frac{3}{4} M \delta_{q+1}^{1/2} + C\delta_{q+1}^{1/2}\ell^\alpha + C\delta_q^{1/2}\lambda_q\lambda_{q+1}^{-1},$$

where the constant C depends on α, β, Ω and $(v_0, p_0, \mathring{R}_0)$, but not on a, b or q. Thus, for any fixed b (3.19) holds for sufficiently large a. Regarding (3.15), we use the inequality at level q to get

$$\left\|v_{q+1}\right\|_1 \leq M\delta_q^{1/2}\lambda_q + \frac{3}{4} M\delta_{q+1}^{1/2} + C\delta_{q+1}^{1/2}\ell^\alpha\lambda_{q+1} + C\delta_q^{1/2}\lambda_q.$$

Hence, if we choose a large enough we obtain (3.15). Finally, (3.16) follows from

$$\left\|v_{q+1}\right\|_0 \leq \left\|v_q\right\|_0 + \left\|v_{q+1} - v_q\right\|_0 \leq 1 - \delta_q^{1/2} + M\delta_{q+1}^{1/2}.$$

The inductive hypotheses (3.13), (3.14) and (3.17) were obtained in the perturbation step, and we are done.

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5. Gluing in space

In this section we develop the second step of the proof of Proposition 3.2, which was summarized in Section 4.3. We do it in three subsections.

5.1. Interpolating sequence

We begin by describing the intermediate subsolutions that we will use. First of all, we recall the following local existence result. It is standard, but we provide a proof for the sake of completeness.

Proposition 5.1. *For any $\alpha > 0$ there exists a constant $c(\alpha) > 0$ with the following property. Given any C^∞ initial data $u_0 \in H^3(\mathbb{R}^3)$, any C^∞ force $f \in L^1_{\mathrm{loc}}(\mathbb{R}, H^3(\mathbb{R}^3))$, let us fix a constant $T > 0$ such that*

$$T \|u_0\|_{1+\alpha} + T^2 \|f\|_{1+\alpha} \le c(\alpha). \tag{5.1}$$

Then there exists a unique solution $u \in C^\infty(\mathbb{R}^3 \times [-T,T]) \cap C([-T,T], H^3(\mathbb{R}^3))$ to the Euler equation

$$\begin{cases} \partial_t u + \operatorname{div}(u \otimes u) + \nabla p = f, \qquad \operatorname{div} u = 0, \\ u(\cdot, 0) = u_0. \end{cases}$$

Moreover, u obeys the bounds

$$\|u\|_{N+\alpha} \lesssim \|u_0\|_{N+\alpha} + T \|f\|_{N+\alpha}$$

for all $N \ge 1$, where the implicit constant depends on N and $\alpha > 0$.

Proof. It is classical [48, 37, 50] that the 3d Euler equation is locally well-posed on $H^3(\mathbb{R}^3)$, and that the solution, which is defined a priori for some time $T = T(\|u_0\|_{H^3(\mathbb{R}^3)}) > 0$, can be continued (and stay smooth, provided that $u_0 \in C^\infty$) as long as the norm $\|u(t)\|_{H^3(\mathbb{R}^3)}$ remains bounded. Furthermore, the weak Beale–Kato–Majda criterion shows [3] that this norm is controlled by $\|\nabla u\|_{L^1 L^\infty}$. More precisely, one has

$$\|u(t)\|_{H^3(\mathbb{R}^3)} \le \|u_0\|_{H^3(\mathbb{R}^3)} + C \int_{-|t|}^{|t|} \|f(\tau)\|_{H^3(\mathbb{R}^3)}\, d\tau \\ + C \int_{-|t|}^{|t|} \|\nabla u(\tau)\|_{L^\infty(\mathbb{R}^3)} \|f(\tau)\|_{H^3(\mathbb{R}^3)}\, e^{C \int_{-|\tau|}^{|\tau|} \|f(\tau')\|_{H^3(\mathbb{R}^3)}\, d\tau'}\, d\tau\,.$$

Therefore, we only need to provide a uniform a priori estimate for $\|\nabla u\|_{L^\infty}$ for all times $t < T$, where the maximum value $T > 0$ is yet to be determined. To this end, note that for any multi-index θ with $|\theta| = N$ we have

$$\partial_t \partial^\theta u + u \cdot \nabla \partial^\theta u + [\partial^\theta, u \cdot \nabla] u + \nabla \partial^\theta p = \partial^\theta f.$$

Since the pressure satisfies the equation $-\Delta p = \operatorname{tr}(\nabla u \nabla u) - \operatorname{div} f$, it follows that

$$\left\|\nabla \partial^\theta p\right\|_\alpha \lesssim \|\operatorname{tr}(\nabla u \nabla u)\|_{N-1+\alpha} + \|f\|_{N+\alpha} \lesssim \|u\|_{1+\alpha} \|u\|_{N+\alpha} + \|f\|_{N+\alpha}\,.$$

Hence, we have

$$\left\|(\partial_t \partial^\theta + u \cdot \nabla \partial^\theta) u\right\|_\alpha \lesssim \|u\|_{1+\alpha} \|u\|_{N+\alpha} + \|f\|_{N+\alpha}\,.$$

Thus, by Lemma B.2:

$$\|u(\cdot, t)\|_{N+\alpha} \lesssim \|u_0\|_{N+\alpha} + T \|f\|_{N+\alpha} + \int_0^t \|u(\cdot, s)\|_{1+\alpha} \|u(\cdot, s)\|_{N+\alpha}\, ds. \tag{5.2}$$

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Specializing to the case $N = 1$, we note that for all $|t| < T$ one has

$$\|u(\cdot,t)\|_{1+\alpha} \le \tilde{c}(\alpha)\left(\|u_0\|_{1+\alpha} + T\,\|f\|_{1+\alpha}\right) + \tilde{c}(\alpha)\int_0^t \|u(\cdot,s)\|_{1+\alpha}^2\,ds, \tag{5.3}$$

for some constant $\tilde{c}(\alpha) > 0$. We define the constant $c(\alpha)$ that appears in the statement of the proposition as $c(\alpha) := [2\tilde{c}(\alpha)]^{-2}$ and then assume that $T > 0$ satisfies (5.1). Let y_0 be the first term on the right-hand side of (5.3) and let $y(t)$ be the solution to the ODE $y' = \tilde{c}(\alpha)y^2$ with $y(0) = y_0$. One finds

$$\|u\|_{1+\alpha} \le y(t) = \frac{y_0}{1 - \tilde{c}(\alpha)y_0 t}.$$

Since $y(t) \lesssim 1$ for all $|t| < T$, by our choice of T, we conclude that the solution is well defined for $|t| < T$. Furthermore, inserting this estimate in (5.2) and applying Grönwall's inequality, we obtain the desired Hölder bounds for $N > 1$. □

To construct the desired sequence of subsolutions, we define

$$m := \left\lceil \lambda_q^{1/2} \right\rceil, \tag{5.4}$$

$$\mathring{R}_i^k := \frac{k}{m}\tilde{\mathring{R}}_\ell. \tag{5.5}$$

For $i \ge 0$ and $0 \le k \le m$ we consider the smooth solutions of the forced Euler equations

$$\begin{cases} \partial_t v_i^k + \operatorname{div}(v_i^k \otimes v_i^k) + \nabla p_i^k = \operatorname{div} \mathring{R}_i^k, \\ \operatorname{div} v_i^k = 0, \\ v_i^k(\cdot, t_i) = \widetilde{v}_\ell(\cdot, t_i). \end{cases} \tag{5.6}$$

Thus, the triplet $({v_i}^k, p_i^k, \mathring{R}_i^k)$ is a subsolution. The pressure is recovered as the unique solution to the equation

$$-\Delta p_i^k = \operatorname{tr}(\nabla v_i^k \nabla v_i^k) - \operatorname{div}\operatorname{div} {\mathring{R}_i}^k$$

with the normalization

$$\int_{\mathbb{T}^3} p_i^k(x,t)\,dx = \int_{\mathbb{T}^3} \widetilde{p}_\ell(x,t)\,dx.$$

Note that (in their common interval of existence) the pair (v_i^0, p_i^0) equals the solutions (v_i, p_i) considered in (4.15), while the pair (v_i^m, p_i^m) equals $(\widetilde{v}_\ell, \widetilde{p}_\ell)$. The rest of the subsolutions form a sort of interpolating sequence between them.

We claim that (v_i^k, p_i^k) are defined in the time interval $[t_i - \tau_q, t_i + \tau_q]$, where τ_q was defined in (4.14). Indeed, it follows from (4.11) that

$$\tau_q \|\widetilde{v}_\ell\|_{1+\alpha} \lesssim \tau_q \delta_q^{1/2} \lambda_q \ell^{-\alpha} \lesssim \ell^\alpha.$$

On the other hand, by (4.12) we have

$$\tau_q^2 \|\tilde{\mathring{R}}_\ell\|_{2+\alpha} \lesssim \tau_q^2 \delta_{q+1} \ell^{-2+3\alpha} \lesssim \left(\frac{\ell^{4\alpha}}{\delta_q \lambda_q^2}\right)\left(\delta_{q+1}\ell^{3\alpha}\frac{\delta_q \lambda_q^{2+6\alpha}}{\delta_{q+1}}\right) = \ell^{7\alpha}\lambda_q^{6\alpha} \le \ell^\alpha.$$

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Hence, for sufficiently large a we have

$$\tau_q\|\widetilde{v}_\ell\|_{1+\alpha}+\tau_q^2\|\mathring{\widetilde{R}}_\ell\|_{2+\alpha}<\frac{1}{2}.$$

By Proposition 5.1, we conclude that solutions (v_i^k,p_i^k) of the corresponding forced Euler equations exist in the claimed interval and they are unique. Furthermore, we have the following bounds:

Corollary 5.2. *If a is sufficiently large, for $|t-t_i|\le\tau_q$ and any $N\ge 1$ we have*

$$\left\|v_i^k\right\|_{N+\alpha}\lesssim\delta_q^{1/2}\lambda_q\ell^{1-N-\alpha}\lesssim\tau_q^{-1}\ell^{1-N+\alpha}. \tag{5.7}$$

Proof. It follows from Proposition 5.1 and estimates (4.11) and (4.12) that

$$\left\|v_i^k\right\|_{N+\alpha}\lesssim\|\widetilde{v}_\ell(t_i)\|_{N+\alpha}+\tau_q\|\mathring{\widetilde{R}}_\ell\|_{N+1+\alpha}\lesssim\delta_q^{1/2}\lambda_q\ell^{1-N-\alpha}+\tau_q\delta_{q+1}\ell^{-N-1+\alpha}$$

Using definitions (4.1) and (4.14) and the comparison (4.2), we see that

$$\frac{\tau_q\delta_{q+1}\ell^{-2+\alpha}}{\delta_q^{1/2}\lambda_q}=\frac{\delta_{q+1}\ell^{-2+3\alpha}}{\delta_q\lambda_q^2}=\lambda_q^{3\alpha}\ell^{3\alpha}\le 1,$$

from which the first inequality follows. For the second one, we use (4.14) again. □

5.2. Estimates for the interpolating sequence

Now that we have defined our subsolutions, we must ensure that they remain close to $\widetilde{v}_\ell$ and to each other. Taking into account that $(\widetilde{v}_\ell,\widetilde{p}_\ell,\mathring{\widetilde{R}}_\ell)$ and $(v_i^k,p_i^k,\mathring{R}_i^k)$ are subsolutions, we see that the difference satisfies the following transport equation:

$$\partial_t(\widetilde{v}_\ell-v_i^k)+\widetilde{v}_\ell\cdot\nabla(\widetilde{v}_\ell-v_i^k)=(v_i^k-\widetilde{v}_\ell)\cdot\nabla v_i^k-\nabla(\widetilde{p}_\ell-p_i^k)+\left(1-\frac{k}{m}\right)\operatorname{div}\mathring{\widetilde{R}}_\ell.$$

Hence, the difference $\widetilde{v}_\ell-v_i^k$ satisfies the same equation as $v_\ell-v_i$ in [7] except for a factor multiplying $\operatorname{div}\mathring{\widetilde{R}}_\ell$, but it is less than or equal to 1. Furthermore, by (4.11) and (5.7) the fields $\widetilde{v}_\ell$ and v_i^k satisfy the same estimates as v_ℓ and v_i. Therefore, we may argue as in [7, Proposition 3.3], obtaining:

Proposition 5.3. *If a is sufficiently large, for $|t-t_i|\le\tau_q$ and any $N\ge 0$ we have*

$$\left\|v_i^k-\widetilde{v}_\ell\right\|_{N+\alpha}\lesssim\tau_q\delta_{q+1}\ell^{-N-1+3\alpha}, \tag{5.8}$$

$$\left\|\nabla(\widetilde{p}_\ell-p_i^k)\right\|_{N+\alpha}\lesssim\delta_{q+1}\ell^{-N-1+3\alpha}, \tag{5.9}$$

$$\left\|D_{t,\ell}(v_i^k-\widetilde{v}_\ell)\right\|_{N+\alpha}\lesssim\delta_{q+1}\ell^{-N-1+3\alpha}, \tag{5.10}$$

where we write

$$D_{t,\ell}:=\partial_t+\widetilde{v}_\ell\cdot\nabla$$

for the transport derivative.

Note that our extra factor $\ell^{2\alpha}$ in (4.12) is inherited by these estimates. Next, let us consider the vector potential associated to the field v_i^k:

$$z_i^k=\mathcal{B}v_i^k:=(-\Delta)^{-1}\operatorname{curl}v_i^k,$$

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where $\mathcal{B}$ is the Biot-Savart operator. We have

$$\operatorname{div} z_i^k = 0 \qquad \text{and} \qquad \operatorname{curl}, z_i^k = v_i^k - \fint_{\mathbb{T}^3} v_i^k .$$

Since $(v_i^k, p_i^k, \frac{k}{m}\mathring{\widetilde{R}}_\ell)$ is a subsolution, we have

$$\frac{d}{dt}\int_{\mathbb{T}^3} v_i^k = -\int_{\mathbb{T}^3}\left(\operatorname{div}(v_i^k \otimes v_i^k) + \nabla p_i^k - \frac{k}{m}\operatorname{div}\mathring{\widetilde{R}}_\ell\right) = 0.$$

On the other hand, the average of v_i^k at time $t = t_i$ is the average of $\widetilde{v}_\ell$ at that time, which vanishes because $\widetilde{v}_\ell(\cdot, t_i)$ is divergence-free and its support is contained in $(0,1)^3$. Therefore, v_i^k has zero mean and we have $\operatorname{curl} z_i^k = v_i^k$.

Since our fields satisfy the same estimates as the fields in [7], we can again argue as in [7, Proposition 3.4], obtaining:

Proposition 5.4. *For $|t_i - \tau_q| \leq \tau_q$ and any $N \geq 0$ we have*

$$\left\|z_i^k - z_i^m\right\|_{N+\alpha} \lesssim \tau_q \delta_{q+1} \ell^{-N+3\alpha}, \tag{5.11}$$

$$\|D_{t,\ell}(z_i^k - z_i^m)\|_{N+\alpha} \lesssim \delta_{q+1}\ell^{-N+3\alpha}, \tag{5.12}$$

where $D_{t,\ell} = \partial_t + \widetilde{v}_\ell \cdot \nabla$.

In summary, the difference $v_i^k - \widetilde{v}_\ell$ satisfies the same stability estimates as the ones in [7] plus an additional factor $\ell^{2\alpha}$ due to the difference in the definition of ℓ and in the inductive estimate (3.14). Furthermore, we will see that the difference between consecutive fields is much smaller, which will allow us to obtain suitable bounds for the gluing. We argue as in [7, Proposition 3.3]. Subtracting the equation for each field and rearranging we obtain

$$\partial_t(v_i^{k+1} - v_i^k) + v_i^{k+1}\cdot\nabla(v_i^{k+1} - v_i^k) = (v_i^k - v_i^{k+1})\cdot\nabla v_i^k - \nabla(p_i^{k+1} - p_i^k) + \frac{1}{m}\operatorname{div}\mathring{\widetilde{R}}_\ell. \tag{5.13}$$

Taking the divergence, we have

$$\Delta(p_i^{k+1} - p_i^k) = \operatorname{div}[\nabla v_i^{k+1}(v_i^k - v_i^{k+1})] + \operatorname{div}[\nabla v_i^k(v_i^k - v_i^{k+1})] + \frac{1}{m}\operatorname{div}\operatorname{div}\mathring{\widetilde{R}}_\ell.$$

Since $(-\Delta)^{-1}\operatorname{div}\operatorname{div}$ is a Calderón-Zygmund operator, we obtain

$$\left\|\nabla(p_i^{k+1} - p_i^k)\right\|_\alpha \lesssim \tau_q^{-1}\left\|v_i^k - v_i^{k+1}\right\|_\alpha + \frac{1}{m}\delta_{q+1}\ell^{-1+\alpha}$$

where we have used (4.12) and (5.7). The additional factor $\ell^{2\alpha}$ is not needed here, so we just omit it. Inserting this estimate into Equation (5.13) and using (4.12) and (5.7) again, we obtain

$$\left\|\partial_t(v_i^{k+1} - v_i^k) + v_i^{k+1}\cdot\nabla(v_i^{k+1} - v_i^k)\right\|_\alpha \lesssim \frac{1}{m}\delta_{q+1}\ell^{-1+\alpha} + \tau_q^{-1}\left\|v_i^k - v_i^{k+1}\right\|_\alpha .$$

Applying Lemma B.2 yields

$$\left\|(v_i^{k+1} - v_i^k)(\cdot,t)\right\|_\alpha \lesssim \frac{1}{m}|t - t_i|\,\delta_{q+1}\ell^{-1+\alpha} + \int_{t_i}^t \tau_q^{-1}\left\|(v_i^{k+1} - v_i^k)(\cdot,s)\right\|_\alpha ds.$$

Using Grönwall's inequality and the assumption $|t - t_i| \le \tau_q$ we conclude

$$\left\|v_i^{k+1} - v_i^k\right\|_\alpha \lesssim \frac{1}{m}\tau_q \delta_{q+1} \ell^{-1+\alpha}.$$

If we carry on arguing as in [7, Proposition 3.3] we obtain the following higher-order estimates:

$$\left\|v_i^{k+1} - v_i^k\right\|_{N+\alpha} \lesssim \frac{1}{m}\tau_q \delta_{q+1} \ell^{-N-1+\alpha}. \tag{5.14}$$

Let us use this bound to estimate the other fields. We may rewrite the equation for the pressure as

$$\begin{aligned}\Delta(p_i^{k+1} - p_i^k) = \operatorname{div}\operatorname{div}\bigg(&\frac{1}{2}(v_i^k + v_i^{k+1}) \otimes (v_i^k - v_i^{k+1}) \\ &+ \frac{1}{2}(v_i^k - v_i^{k+1}) \otimes (v_i^k + v_i^{k+1}) + \frac{1}{m}\mathring{\widetilde{R}}_\ell\bigg)\end{aligned} \tag{5.15}$$

because

$$v_i^k \otimes v_i^k - v_i^{k+1} \otimes v_i^{k+1} = \frac{1}{2}(v_i^k + v_i^{k+1}) \otimes (v_i^k - v_i^{k+1}) + \frac{1}{2}(v_i^k - v_i^{k+1}) \otimes (v_i^k + v_i^{k+1}).$$

Interpolating between (3.14) and (3.15) and between (4.10) and (4.11) we have

$$\begin{aligned}\left\|v_i^k + v_i^{k+1}\right\|_\alpha &\le 2\left\|v_q\right\|_\alpha + 2\left\|v_q - \widetilde{v}_\ell\right\|_\alpha + \left\|v_i^k - \widetilde{v}_\ell\right\|_\alpha + \left\|v_i^{k+1} - \widetilde{v}_\ell\right\|_\alpha \\ &\lesssim (\delta_q^{1/2}\lambda_q)^\alpha + (\delta_{q+1}\lambda_q^{-\alpha})^{1-\alpha}(\delta_q^{1/2}\lambda_q)^\alpha + \frac{1}{m}\tau_q\delta_{q+1}\ell^{-1+\alpha} \lesssim \lambda_q^\alpha.\end{aligned}$$

For the higher-order bounds we use Corollary 5.2. Therefore, from (5.14) we conclude

$$\begin{aligned}\left\|v_i^k \otimes v_i^k - v_i^{k+1} \otimes v_i^{k+1}\right\|_\alpha &\lesssim \lambda_q^\alpha(m^{-1}\tau_q\delta_{q+1}\ell^{-N-1+\alpha}) \\ &\quad + \sum_{j=1}^N (\tau_q^{-1}\ell^{1-j+\alpha})(m^{-1}\tau_q\delta_{q+1}\ell^{-(N-j)-1+\alpha}) \\ &\lesssim \frac{1}{m}\tau_q\delta_{q+1}\ell^{-N-1} \lesssim \frac{1}{m}\delta_{q+1}^{1/2}\ell^{-N}.\end{aligned} \tag{5.16}$$

Introducing this estimate and (4.12) in Equation (5.15), we finally obtain

$$\left\|p_i^{k+1} - p_i^k\right\|_{N+\alpha} \lesssim \frac{1}{m}\tau_q\delta_{q+1}\ell^{-N-1} \lesssim \frac{1}{m}\delta_{q+1}^{1/2}\ell^{-N}. \tag{5.17}$$

Let us now estimate the difference in the vector potentials. We recall the identity curl curl $= -\Delta + \nabla$ div and that div $z_i^k = 0$. Hence, taking the curl in Equation (5.13) and rearranging we arrive at

$$\begin{aligned}-\Delta\left[\partial_t(z_i^{k+1} - z_i^k)\right] = \operatorname{curl}\operatorname{div}\bigg(&\frac{1}{2}(v_i^k + v_i^{k+1}) \otimes (v_i^k - v_i^{k+1}) \\ &+ \frac{1}{2}(v_i^k - v_i^{k+1}) \otimes (v_i^k + v_i^{k+1}) + \frac{1}{m}\mathring{\widetilde{R}}_\ell\bigg).\end{aligned} \tag{5.18}$$

Reasoning as in the case of the pressure, we conclude

$$\left\|\partial_t(z_i^{k+1} - z_i^k)\right\|_{N+\alpha} \lesssim \frac{1}{m}\tau_q\delta_{q+1}\ell^{-N-1} \lesssim \frac{1}{m}\delta_{q+1}^{1/2}\ell^{-N}. \tag{5.19}$$

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Since $z_i^k(\cdot,t_i) = z_i^{k+1}(\cdot,t_i) = \mathcal{B}\widetilde{v}_\ell(\cdot,t_i)$, the difference vanishes at $t = t_i$. Using the assumption $|t-t_i| \leq \tau_q$ we deduce

$$\left\|z_i^{k+1} - z_i^k\right\|_{N+\alpha} \lesssim \frac{1}{m}\tau_q^2\delta_{q+1}\ell^{-N-1} \lesssim \frac{1}{m}\tau_q\delta_{q+1}^{1/2}\ell^{-N}. \tag{5.20}$$

5.3. Gluing the interpolating sequence

Now that we have the appropriate estimates, we will start gluing the subsolutions $(v_i^k, p_i^k, \mathring{R}_i^k)$ to one another to construct a subsolution that equals $(v_i, p_i, 0)$ in $A_q + B(0,2\sigma)$ and $(v_0, p_0, \mathring{R}_0)$ in B_3. For the sake of clarity, we will do it inductively. Let

$$r := \lambda_q^{-3/5}$$

and for $k \geq 0$ consider the sets

$$\begin{aligned}\Omega^k &:= A_q + B(0, 2\sigma + kr),\\ U^k &:= \{x \in \mathbb{R}^3 : 2\sigma + kr < \operatorname{dist}(x, A_q) < 2\sigma + (k+1)r\}\end{aligned}$$

and we fix smooth cutoff functions $\varphi^k \in C^\infty(\Omega^{k+1}, [0,1])$ that equal 1 in a neighborhood of Ω^k. By Lemma B.1, we may assume the bounds $\|\varphi^k\|_N \lesssim r^{-N}$.

We will construct a sequence of subsolutions $(\widetilde{v}_i^k, \widetilde{p}_i^k, \mathring{\widetilde{R}}_i{}^k)$ and potentials $\widetilde{z}_i^k$ with $\widetilde{v}_i^k = \operatorname{curl}\widetilde{z}_i^k$ such that

$$(\widetilde{v}_i^k, \widetilde{p}_i^k, \mathring{\widetilde{R}}_i{}^k, \widetilde{z}_i^k)(x,t) = (v_i^k, p_i^k, \mathring{R}_i^k, z_i^k)(x,t) \quad \forall x \notin \Omega^k,\ |t-t_i| \leq \tau_q, \tag{5.21}$$

$$\widetilde{v}_i^k(\cdot,t_i) = \widetilde{v}_\ell(\cdot,t_i), \tag{5.22}$$

Furthermore, for $|t-t_i| \leq \tau_q$ and any $N \geq 0$ they satisfy

$$\|\mathring{\widetilde{R}}_i{}^k\|_0 \leq \frac{1}{2}\delta_{q+2}\lambda_{q+1}^{-6\alpha}, \tag{5.23}$$

$$\left\|\widetilde{v}_i^k - \widetilde{v}_\ell\right\|_N \lesssim \tau_q\delta_{q+1}\ell^{-N-1+3\alpha}, \tag{5.24}$$

$$\left\|D_{t,\ell}(\widetilde{v}_i^k - \widetilde{v}_\ell)\right\|_N \lesssim \delta_{q+1}\ell^{-N-1+3\alpha}, \tag{5.25}$$

$$\left\|\widetilde{z}_i^k - z_i^m\right\|_N \lesssim \tau_q\delta_{q+1}\ell^{-N+3\alpha}, \tag{5.26}$$

$$\|D_{t,\ell}(\widetilde{z}_i^k - z_i^m)\|_N \lesssim \delta_{q+1}\ell^{-N+3\alpha}, \tag{5.27}$$

where we write

$$D_{t,\ell} := \partial_t + \widetilde{v}_\ell \cdot \nabla$$

for the transport derivative.

If we could construct such a sequence, setting $(\widetilde{v}_i, \widetilde{p}_i, \mathring{\widetilde{R}}_i, \widetilde{z}_i) := (\widetilde{v}_i^m, \widetilde{p}_i^m, \mathring{\widetilde{R}}_i{}^m, \widetilde{z}_i^m)$ would yield the subsolution and potential claimed in Section 4.3. Indeed, (4.18) and (4.19) are just (5.22) and (5.23). The estimates (4.20) and (4.21) follow from (5.24) and (5.25) by interpolation, that is, we use (A.3) to estimate the $C^{N+\alpha}$ seminorm using the C^0 and C^{N+1} norms:

$$\begin{aligned}
\left[\widetilde{v}_i^k - \widetilde{v}_\ell\right]_{N+\alpha} &\lesssim \left\|\widetilde{v}_i^k - \widetilde{v}_\ell\right\|_0^{1-\frac{N+\alpha}{N+1}} \left\|\widetilde{v}_i^k - \widetilde{v}_\ell\right\|_{N+1}^{\frac{N+\alpha}{N+1}} \\
&\lesssim \left(\tau_q \delta_{q+1} \ell^{-1+3\alpha}\right)^{1-\frac{N+\alpha}{N+1}} \left(\tau_q \delta_{q+1} \ell^{-1-(N+1)+3\alpha}\right)^{\frac{N+\alpha}{N+1}} \\
&\lesssim \tau_q \delta_{q+1} \ell^{-1-N+2\alpha}.
\end{aligned}$$

This, combined with the estimate for the C^N norm, yields (4.20). Note that here we have obtained an extra factor ℓ^α, but we discard it because it will not be necessary. Estimate (4.19) follows from (5.25) by an analogous argument. Meanwhile, to obtain (4.22) and (4.23) from (5.26) and (5.27), we also need to apply the triangle inequality and use the fact that $z_{i+1}^m - z_i^m = 0$. Recall that both vector potentials are just the restriction of $\mathcal{B}\widetilde{v}_\ell$ to their respective intervals. Finally, (4.17) follows from (5.21) because $mr \lesssim \lambda_q^{-1/10}$, so by (4.5) it must be smaller than σ for sufficiently large a.

Let us then construct this sequence. We define the initial term as

$$(\widetilde{v}_i^0, \widetilde{p}_i^0, \mathring{\widetilde{R}}_i{}^0, \widetilde{z}_i^0) := (v_i^0, p_i^0, \mathring{R}_i^0) = (v_i, p_i, 0, z_i^0).$$

It follows from Corollary 5.2, Proposition 5.3 and Proposition 5.4 that this term satisfies Equations (5.21) to (5.27). Next, let us suppose that we have defined the k-th term $(\widetilde{v}_i^k, \widetilde{p}_i^k, \mathring{\widetilde{R}}_i{}^k, \widetilde{z}_i^k)$ satisfying these inductive hypotheses. We will construct the $(k+1)$-th term satisfying them, too. To do so, we will apply Lemma 2.11 to glue $(\widetilde{v}_i^k, \widetilde{p}_i^k, \mathring{\widetilde{R}}_i{}^k)$ and $(v_i^{k+1}, p_i^{k+1}, \mathring{R}_i^{k+1})$ in the region U^k. Since these subsolutions are defined in the whole space, by Remark 2.12 we do not need to check the compatibility conditions (2.16) and (2.17).

Note that in Lemma 2.11 we use skew-symmetric matrices instead of potential vectors because it is stated in any dimension $n \geq 2$. However, it is completely equivalent: given a potential vector z, we simply define $A_{ij} = \varepsilon_{ijk} z_k$, where ε_{ijk} is the usual Levi-Civita symbol. It is easy to check that A is skew-symmetric and $\operatorname{curl} z = \operatorname{div} A$.

Hence, applying Lemma 2.11 we obtain a subsolution satisfying:

$$(\widetilde{v}_i^{k+1}, \widetilde{p}_i^{k+1}, \mathring{\widetilde{R}}_i{}^{k+1})(\cdot, t) = \begin{cases} (\widetilde{v}_i^k, \widetilde{p}_i^k, \mathring{\widetilde{R}}_i{}^k) & \text{in } \Omega^k, \\ (v_i^{k+1}, p_i^{k+1}, \mathring{R}_i^{k+1}) & \text{outside } \Omega^{k+1} \end{cases}$$

for $|t - t_i| \leq \tau_q$. In addition, there exists a smooth vector potential $\widetilde{z}_i^{k+1}$ with $\widetilde{v}_i^{k+1} = \operatorname{curl} \widetilde{z}_i^{k+1}$ and such that $\widetilde{z}_i^{k+1}(\cdot, t) \equiv z_i^{k+1}(\cdot, t)$ outside Ω^{k+1}. Thus, (5.21) is satisfied.

Furthermore, by definition of v_i^{k+1} and the inductive hypothesis (5.22) we have

$$v_i^{k+1}(\cdot, t_i) = \widetilde{v}_\ell(\cdot, t_i) = \widetilde{v}_i^k(\cdot, t_i).$$

In addition, by (5.21) we know that $\widetilde{z}_i^k$ equals z_i^k outside Ω^k. Since z_i^k and z_i^{k+1} both equal $\mathcal{B}\widetilde{v}_\ell$ at time $t = t_i$, we see that the difference $z_i^{k+1} - \widetilde{z}_i^k$ vanishes outside Ω^k at $t = t_i$. Inspecting Lemma 2.11 and replacing skew-symmetric matrices by the equivalent potential vectors, we see that

$$(\widetilde{v}_i^{k+1} - w_L)(\cdot, t_i) = \left[\varphi^k \widetilde{v}_i^k + (1 - \varphi^k) v_i^{k+1} + \nabla\varphi^k \times (z_i^{k+1} - \widetilde{z}_i^k)\right](\cdot, t_i) = \widetilde{v}_\ell(\cdot, t_i).$$

Since only the time derivative of w_L matters, we may assume that it vanishes at $t = t_i$, that is, we start the integration in (2.26) at $t = t_i$. Hence, (5.22) holds.

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Concerning the estimates, we see in Lemma 2.11 that we only need to consider the bounds in U^k, so by (5.21) we only need to study the difference between $(v_i^k, p_i^k, \mathring{R}_i^k, z_i^k)$ and $(v_i^{k+1}, p_i^{k+1}, \mathring{R}_i^{k+1}, z_i^{k+1})$. Since

$$\|\mathring{R}_i^k - \mathring{R}_i^{k+1}\|_0 = \frac{1}{m}\|\mathring{\widetilde{R}}_\ell\|_0 \lesssim \frac{1}{m}\delta_{q+1},$$

it follows from Equations (5.16), (5.17) and (5.19) that the matrix M that appears in Lemma 2.11 satisfies

$$\|M\|_{0;U^k} \leq \frac{1}{m}\delta_{q+1}^{1/2}.$$

With this bound we may use the estimates from Lemma 2.11. Let us focus on the vector potential:

$$\left\|\widetilde{z}_i^{k+1} - (\varphi^k \widetilde{z}_i^k + (1-\varphi^k) z_i^{k+1})\right\|_N \lesssim \frac{1}{m}\tau_q \delta_{q+1}^{1/2} r^{-N}, \tag{5.28}$$

$$\left\|\partial_t (\widetilde{z}_i^{k+1} - \varphi^k \widetilde{z}_i^k - (1-\varphi^k) z_i^{k+1})\right\|_N \lesssim \frac{1}{m}\delta_{q+1}^{1/2} r^{-N}, \tag{5.29}$$

To use these estimates, the following inequality will be useful:

$$m\delta_{q+2}\delta_{q+1}^{-1/2} = \lambda_q^{1/2-2b^2\beta+b\beta} \geq \lambda_q^{1/20}, \tag{5.30}$$

where we have used that the exponent in the middle term is greater than $1/20$ for any $0 < \beta < 1/3$ and $1 < b < 11/10$. We compute

$$\begin{aligned}
\left\|\widetilde{z}_i^{k+1} - z_i^m\right\|_N &\leq \left\|\widetilde{z}_i^{k+1} - (\varphi^k \widetilde{z}_i^k + (1-\varphi^k) z_i^{k+1})\right\|_N \\
&\quad + \left\|\varphi^k (\widetilde{z}_i^k - z_i^m)\right\|_N + \left\|(1-\varphi^k)(z_i^{k+1} - z_i^m)\right\|_N \\
&\lesssim \frac{1}{m}\tau_q \delta_{q+1}^{1/2} r^{-N} + \sum_{j=0}^N r^{-j}\left(\left\|\widetilde{z}_i^k - z_i^m\right\|_{N-j} + \left\|z_i^{k+1} - z_i^m\right\|_{N-j}\right) \\
&\lesssim \frac{1}{m}\tau_q \delta_{q+1}^{1/2} r^{-N} + \tau_q \delta_{q+1}\ell^{-N+3\alpha} \lesssim \tau_q \delta_{q+1}\ell^{-N+3\alpha},
\end{aligned}$$

where we have used (5.30) and we have assumed α to be sufficiently small. Hence, (5.26) holds for the $(k+1)$-th term. In addition, (5.24) clearly follows from it.

Note that we have used (5.11) to estimate $\|\cdot\|_N$, that is, we lose an α. This is clearly not optimal, but it cannot be avoided for $N = 0$. Thus, we pay the price of losing a factor ℓ^α in the estimates for $\|\cdot\|_{N+\alpha}$. To compensate this, we gain an extra factor $\ell^{2\alpha}$ from (3.14) and (4.1), which are different from their counterparts in [7].

Let us now estimate the material derivative. By the triangle inequality:

$$\begin{aligned}
\left\|D_{t,\ell}(\widetilde{z}_i^{k+1} - z_i^m)\right\|_N &\leq \left\|\partial_t (\widetilde{z}_i^{k+1} - \varphi^k \widetilde{z}_i^k - (1-\varphi^k) z_i^{k+1})\right\|_N \\
&\quad + \left\|\widetilde{v}_\ell \cdot \nabla (\widetilde{z}_i^{k+1} - \varphi^k \widetilde{z}_i^k - (1-\varphi^k) z_i^{k+1})\right\|_N \\
&\quad + \left\|D_{t,\ell}[\varphi^k (\widetilde{z}_i^k - z_i^m)]\right\|_N \\
&\quad + \left\|D_{t,\ell}[(1-\varphi^k)(z_i^{k+1} - z_i^m)]\right\|_N .
\end{aligned} \tag{5.31}$$

The last two terms can be estimated in the same manner, so we just study one:

$$\begin{aligned}
\left\|D_{t,\ell}[\varphi^k (\widetilde{z}_i^k - z_i^m)]\right\|_N &\lesssim \sum_{j=0}^N \Big(\left\|\varphi^k\right\|_{j+1} \left\|\widetilde{z}_i^k - z_i^m\right\|_{N-j} \\
&\quad + \left\|\varphi^k\right\|_j \left\|D_{t,\ell}(\widetilde{z}_i^k - z_i^m)\right\|_{N-j} \Big) \\
&\lesssim r^{-1}\tau_q \delta_{q+1}\ell^{-N+3\alpha} + \delta_{q+1}\ell^{-N+3\alpha} \lesssim \delta_{q+1}\ell^{-N+3\alpha},
\end{aligned}$$

where we have used $\|\varphi^k\|_N \lesssim r^{-N} \lesssim \ell^{-N}$ and $\tau_q r^{-1} \lesssim 1$. Next, taking into account that $\|\widetilde{v}_\ell\|_N \lesssim \ell^{-N}$, we have

$$\|\widetilde{v}_\ell \cdot \nabla(\widetilde{z}_i^{k+1} - \varphi^k \widetilde{z}_i^k - (1-\varphi^k) z_i^{k+1})\|_N \lesssim \sum_{j=0}^{N} \ell^{-j} \left\|\widetilde{z}_i^{k+1} - \varphi^k \widetilde{z}_i^k - (1-\varphi^k) z_i^{k+1}\right\|_{N+1-j}$$
$$\lesssim \frac{1}{mr} \tau_q \delta_{q+1}^{1/2} \ell^{-N} \lesssim \delta_{q+1} \ell^{-N+3\alpha},$$

where we have used $\tau_q r^{-1} \lesssim 1$, the inequality (5.30) and we have assumed α to be sufficiently small. Using (5.29) along with the same tricks, we obtain the same estimates for the first term in Equation (5.31), and we conclude that (5.27) holds for the $(k+1)$-th term.

To obtain (5.25), we estimate the commutator $[D_{t,\ell}, \mathrm{curl}]$. We fix an arbitrary vector field u and we compute

$$\left(\mathrm{curl}(D_{t,\ell} u) - D_{t,\ell}\, \mathrm{curl}\, u\right)_i = \varepsilon_{ijk} \partial_j (\widetilde{v}_\ell)_l \partial_l u_k .$$

Therefore, we have

$$\left\|[D_{t,\ell}, \mathrm{curl}](\widetilde{z}_i^{k+1} - z_i^m)\right\|_N \lesssim \sum_{j=0}^{N} \|\widetilde{v}_\ell\|_{j+1} \left\|\widetilde{z}_i^{k+1} - z_i^m\right\|_{N-j+1},$$
$$\lesssim \sum_{j=0}^{N} (\delta_q^{1/2} \lambda_q \ell^{-j})(\tau_q \delta_{q+1} \ell^{-N-1+j+3\alpha})$$
$$\lesssim \delta_q^{1/2} \lambda_q \tau_q \delta_{q+1} \ell^{-N-1+3\alpha} \lesssim \delta_{q+1} \ell^{-N-1+3\alpha}$$

where we have used (4.11) and that $\delta_q^{1/2} \lambda_q \tau_q = \ell^{2\alpha} \lesssim 1$ by definition of τ_q. Hence,

$$\left\|D_{t,\ell}(\widetilde{v}_i^{k+1} - \widetilde{v}_\ell)\right\|_N \le \left\|\mathrm{curl}[D_{t,\ell}(\widetilde{z}_i^{k+1} - z_i^m)]\right\|_N + \left\|[D_{t,\ell}, \mathrm{curl}](\widetilde{z}_i^{k+1} - z_i^m)\right\|_N$$
$$\lesssim \delta_{q+1} \ell^{-N-1+3\alpha}.$$

Finally, let us consider the size of the new Reynolds stress. Taking into account that $\widetilde{z}_i^k$ equals z_i^k in U^k, it follows from (5.20) and Lemma 2.11 that

$$\|w\|_0 = \left\|\widetilde{v}_i^{k+1} - \varphi^k \widetilde{v}_i^k - (1-\varphi^k) v_i^{k+1}\right\|_0 \lesssim \tau_q r^{-1} \frac{1}{m} \delta_{q+1}^{1/2} \lesssim \frac{1}{m} \delta_{q+1}^{1/2},$$

where we have used $\tau_q r^{-1} \lesssim 1$. Taking into account that $\widetilde{v}_i^k$ equals v_i^k in U^k, it follows from (5.14) and Lemma 2.11 that

$$\|\mathring{\widetilde{R}}_i{}^{k+1} - (\varphi^k \mathring{\widetilde{R}}_i{}^k + (1-\varphi^k) \mathring{R}_i^{k+1})\|_0 \lesssim \frac{1}{m} \delta_{q+1}^{1/2}.$$

Taking into account (5.30), we see that for sufficiently small α and sufficiently large a we have

$$\|\mathring{\widetilde{R}}_i{}^{k+1} - (\varphi^k \mathring{\widetilde{R}}_i{}^k + (1-\varphi^k) \mathring{R}_i^{k+1})\|_0 \le \frac{1}{4} \delta_{q+2} \lambda_{q+1}^{-3\alpha}.$$

If $x \in \Omega^k$, then $\varphi^k(x) = 1$ and by (5.23) we have

$$|\mathring{\widetilde{R}}_i{}^{k+1}(x,t)| = |\mathring{\widetilde{R}}_i{}^k(x,t)| \le \frac{1}{2} \delta_{q+2} \lambda_{q+1}^{-3\alpha}.$$

On the other hand, if $x \notin \Omega^k$, then $\mathring{\widetilde{R}}_i{}^k(x,t) = \frac{k}{m}\mathring{\widetilde{R}}_\ell(x,t)$ by (4.17). Furthermore, we also have $\mathring{\widetilde{R}}_\ell(x,t) = \mathring{R}_0(x,t)$ because $\operatorname{dist}(x, A_q) \geq 2\sigma$. Since $x \notin A_q$, it follows from (3.11) that $|\mathring{R}_0(x,t)| \leq \frac{1}{4}\delta_{q+2}\lambda_{q+1}^{-6\alpha}$, so

$$\begin{aligned} |\mathring{\widetilde{R}}_i{}^{k+1}(x,t)| &\leq \varphi^k \|\mathring{\widetilde{R}}_i{}^k\|_0 + (1-\varphi^k)\|\mathring{R}_i^{k+1}\|_0 + \|\mathring{\widetilde{R}}_i{}^{k+1} - (\varphi^k \mathring{\widetilde{R}}_i{}^k + (1-\varphi^k)\mathring{R}_i^{k+1})\|_0 \\ &\leq \varphi^k \frac{1}{4}\delta_{q+2}\lambda_{q+1}^{-6\alpha} + (1-\varphi^k)\frac{1}{4}\delta_{q+2}\lambda_{q+1}^{-6\alpha} + \frac{1}{4}\delta_{q+2}\lambda_{q+1}^{-6\alpha} \\ &\leq \frac{1}{2}\delta_{q+2}\lambda_{q+1}^{-6\alpha} \end{aligned}$$

for $x \notin \Omega^k$. We conclude that (5.23) holds for the $(k+1)$-th term.

6. Gluing in time

In this short section we develop the third step of the proof of Proposition 3.2, which was summarized in Section 4.4.

By [7, Proposition 4.4] the matrix $S := \mathcal{R}[\partial\chi_i(\widetilde{v}_i - \widetilde{v}_{i+1})]$ satisfies the following bounds for any $N \geq 0$:

$$\begin{aligned} \|S\|_{N+\alpha} &\lesssim \delta_{q+1}\ell^{-N+\alpha}, \\ \|(\partial_t + \overline{v}_q \cdot \nabla)S\|_{N+\alpha} &\lesssim \delta_{q+1}\delta_q^{1/2}\lambda_q\ell^{-N-\alpha}. \end{aligned}$$

We define

$$m := \lceil \lambda_q^{1/2} \rceil, \qquad r = \lambda_q^{-3/5}$$

and we fix smooth cutoff functions θ_j such that

- $\theta_j \equiv 1$ in a neighborhood of $A_q + B(0, 3\sigma + (j-1)r)$,
- the support of θ_j is contained in $A_q + B(0, 3\sigma + jr)$

for $1 \leq j \leq m$. We define

$$\overline{\mathcal{R}}_q^{(2)} := \sum_{i=1}^m \frac{1}{m}\theta_j S.$$

Note that $\operatorname{supp} \overline{\mathcal{R}}_q^{(2)}(\cdot,t)$ is contained in $A_q + B(0, 4\sigma)$ because $mr < \sigma$ for a sufficiently large. Since $r^{-1} \lesssim \ell^{-1}$, we have

$$\|\overline{\mathcal{R}}_q^{(2)}\|_N \lesssim \delta_{q+1}\ell^{-N+\alpha}. \tag{6.1}$$

Let us estimate the material derivative. We compute

$$(\partial_t + \overline{v}_q \cdot \nabla)\overline{\mathcal{R}}_q^{(2)} = \sum_{i=1}^m \frac{1}{m}\theta_j(\partial_t + \overline{v}_q \cdot \nabla)S + \sum_{i=1}^m \frac{1}{m}\overline{v}_q \cdot \nabla\theta_j S.$$

Regarding the second term, it follows from (4.27) and $\|\widetilde{v}_\ell\|_N \lesssim \ell^{-N}$ that the new field also satisfies $\|\overline{v}_q\|_N \lesssim \ell^{-N}$. Thus,

$$\|\overline{v}_q \cdot \nabla\theta_j\|_N \lesssim r^{-1}\ell^{-N}.$$

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Since the support of the $\nabla\theta_j$ are pairwise disjoint, we have

$$\begin{aligned}\|(\partial_t + \overline{v}_q \cdot \nabla)\overline{\mathcal{R}}_q^{(2)}\|_N &\lesssim \delta_{q+1}\delta_q^{1/2}\lambda_q \ell^{-N-\alpha} + \frac{1}{m} r^{-1}\delta_{q+1}\ell^{-N+\alpha} \\ &\lesssim \delta_{q+1}\delta_q^{1/2}\lambda_q \ell^{-N-\alpha}.\end{aligned} \tag{6.2}$$

We conclude that the matrix $\overline{\mathcal{R}}_q^{(2)}$ satisfies the right estimates but we have changed the equation:

$$\operatorname{div} \overline{\mathcal{R}}_q^{(2)} = \operatorname{div} S + \sum_{j=1}^{m} \frac{1}{m} \nabla\theta_j \cdot S,$$

where we have used that the support of $\operatorname{div} S$ is contained in $A_q + B(0, 3\sigma)$ by (5.21). Let us correct this. We define $\rho_j := \nabla\theta_j \cdot S$, whose support is contained in

$$\{x \in \mathbb{R}^3 : 3\sigma + (i-1)r < \operatorname{dist}(x, A_q) < 3\sigma + ir\}.$$

Therefore, by Lemma B.4 we have

$$\|\rho_j\|_{B^{-1+\alpha}_{\infty,\infty}} \lesssim r^{1-\alpha} \|\nabla\theta_j \cdot S\|_0 \lesssim \delta_{q+1}.$$

We wish to apply Lemma 2.9, so we have to check the compatibility conditions. We fix a Killing field ξ and we compute

$$\int \xi \cdot \rho_j = \int \xi \cdot \operatorname{div}(\theta_j S) - \int \xi \cdot \theta_j \operatorname{div} S = -\int \xi \cdot \operatorname{div} S = -\partial_t \chi_i \int \xi \cdot (\widetilde{v}_i - \widetilde{v}_{i+1}),$$

where we have used (2.6) and the fact that $\theta_j = 1$ on the support of $\operatorname{div} S$. To show that this integral vanishes, we first compute

$$\frac{d}{dr} \int \xi \cdot (\widetilde{v}_i - \widetilde{v}_\ell) = \int \xi \cdot \operatorname{div}\left(\widetilde{v}_\ell \otimes \widetilde{v}_\ell - \widetilde{v}_i \otimes \widetilde{v}_i + \widetilde{p}_\ell \operatorname{Id} - \widetilde{p}_i \operatorname{Id} + \mathring{\widetilde{R}}_i - \mathring{\widetilde{R}}_\ell\right) = 0$$

because of (2.6) and the fact that the matrix in parentheses is compactly supported due to (4.9) and (4.17). Since $\widetilde{v}_i(\cdot, t_i) = \widetilde{v}_\ell(\cdot, t_i)$, we see that $\int \xi \cdot (\widetilde{v}_i - \widetilde{v}_\ell) = 0$. Repeating this for $\widetilde{v}_{i+1}$ and subtracting, we conclude

$$\int \xi \cdot \rho_j = -\partial_t \chi_j \int \xi \cdot (\widetilde{v}_i - \widetilde{v}_{i+1}) = 0$$

for any Killing field ξ. Therefore, by Lemma 2.9 there exists a smooth symmetric matrix M_j such that $\operatorname{div} M_j = \rho_j$ and whose support is contained in

$$\{x \in \mathbb{R}^3 : 3\sigma + (j-1)r < \operatorname{dist}(x, A_q) < 3\sigma + jr\}.$$

Furthermore, we have the estimate

$$\|M_j\|_0 \lesssim r^{-\alpha}\delta_{q+1}.$$

We define

$$\operatorname{div} \overline{\mathcal{R}}_q^{(3)} := -\frac{1}{m} \sum_{j=1}^{m} M_j.$$

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By construction of the M_j we have

$$\operatorname{div}\left(\overline{\mathcal{R}}_q^{(2)} + \overline{\mathcal{R}}_q^{(3)}\right) = \partial_t \chi_j (\widetilde{v}_i - \widetilde{v}_{i+1}),$$

as we wanted. Since the supports of the M_j are pairwise disjoint, we see that

$$\|\overline{\mathcal{R}}_q^{(3)}\|_0 = \frac{1}{m} \max_j \left\| M_j \right\|_0 \lesssim \frac{1}{m} r^{-\alpha} \delta_{q+1}.$$

It follows from our assumption $b - 1 < 1/10$ that

$$\frac{\delta_{q+1}}{\delta_{q+2}} = \lambda_q^{\beta b(b-1)} < \lambda_q^{1/10}.$$

Therefore, for a is sufficiently large and α sufficiently small the matrix $\mathring{\bar{R}}_q^{(1)}$ defined in (4.31) satisfies

$$\|\mathring{\bar{R}}_q^{(1)}\|_0 \leq \frac{3}{4} \delta_{q+2} \lambda_{q+1}^{-6\alpha},$$

where we have used that

$$\|\overline{\mathcal{R}}_q^{(3)}\|_0 \leq \frac{1}{2} \delta_{q+2} \lambda_{q+1}^{-6\alpha}$$

due to (5.23). Thus, $\mathring{\bar{R}}_q^{(1)}$ is so small that it may be ignored until the next iteration.

Finally, let us conclude the estimates for $\mathring{\bar{R}}_q^{(2)}$, defined in (4.32). It follows from [7, Proposition 4.4] that

$$\|\chi_i(1-\chi_i)(\widetilde{v}_i - \widetilde{v}_{i+1}) \mathring{\otimes} (\widetilde{v}_i - \widetilde{v}_{i+1})\|_{N+\alpha} \lesssim \delta_{q+1} \ell^{-N+\alpha},$$
$$\|(\partial_t + \overline{v}_q \cdot \nabla)[\chi_i(1-\chi_i)(\widetilde{v}_i - \widetilde{v}_{i+1}) \mathring{\otimes} (\widetilde{v}_i - \widetilde{v}_{i+1})]\|_{N+\alpha} \lesssim \delta_{q+1} \delta_q^{1/2} \lambda_q \ell^{-N+\alpha}.$$

Combining this with (6.1) and (6.2), we infer (4.37) and (4.38).

7. The perturbation step

In this section we complete the final step in the proof of Proposition 3.2, which is done in Subsections 7.1 to 7.5. The last part, Subsection 7.6, contains the proof of Lemma 3.3.

For simplicity, we will assume that Ω is connected. If it had several connected components Ω^j and we wanted to fix an energy profile e^j in each of them, we would simply carry out the construction of this section in each connected component, taking into account Remark 3.1. Note that ϕ_q can be split into cutoffs ϕ_q^j associated to each Ω^j.

7.1. Squiggling stripes and the stress $\widetilde{R}_{q,i}$

Before we can define the perturbation, we need to introduce several objects. By [7, Subsection 5.2] there exist nonnegative cutoff functions η_i with the following properties:

(i) $\eta_i \in C^\infty(\mathbb{T}^3 \times [0,T], [0,1])$.
(ii) $\operatorname{supp} \eta_i \cap \operatorname{supp} \eta_j = \varnothing$.
(iii) $\eta_i(x,t) = 1$ for any x and $t \in I_i$.
(iv) $\operatorname{supp} \eta_i \subset \mathbb{T}^3 \times (t_i - \frac{1}{3}\tau_q, t_{i+1} + \frac{1}{3}\tau_q) \cap [0,T]$.

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(v) There exists a positive geometric constant $c_0 > 0$ such that for any $t \in [0, T]$

$$\sum_i \int_{\mathbb{T}^3} \eta_i(x,t)^2\, dx \geq c_0.$$

(vi) For any $k, N \geq 0$ there exists constants depending on k, N such that

$$\left\|\partial_t^k \eta_i\right\|_N \lesssim \tau_q^{-N}.$$

We replace $\eta_i(x,t)$ by $\eta_i(mx,t)$ for sufficiently large $m \in \mathbb{N}$, so that we may assume that the cutoffs η_i satisfy

$$\widetilde{c}_0 \leq \sum_i \int_{\mathbb{T}^3} \phi_q(x)^2 \eta_i(x,t)^2\, dx \leq 2\,|\Omega| \tag{7.1}$$

for some constant $\widetilde{c}_0$ depending on Ω. This can be done because A_0 will contain one of the cubes of a grid of sidelength m^{-1} for sufficiently large m. This settles the first inequality. Regarding the second inequality, note that at most 2 of the cutoffs are nonzero at any given time. In addition, the new cutoffs will still satisfy (i) – (vi) but the constants that appear in (vi) will now depend on Ω, too.

We proceed analogously to [7], defining the amplitudes

$$\rho_q(t) := \frac{1}{3}\left(e(t) - \frac{1}{2}\delta_{q+2} - \int_\Omega \left|\overline{v}_q\right|^2 dx\right), \tag{7.2}$$

$$\rho_{q,i}(x,t) := \frac{\eta_i(x,t)^2}{\sum_i \int_{\mathbb{T}^3} \phi_q(x)^2 \eta_i(x,t)^2 dx} \rho_q(t). \tag{7.3}$$

Note that our definition of $\rho_{q,i}$ differs from the one in [7] in the normalization. Next, we define the backwards flows Φ_i for the velocity field $\overline{v}_q$ as the solution of the transport equation

$$\begin{cases} (\partial_t + \overline{v}_q \cdot \nabla)\Phi_i = 0, \\ \Phi_i(x,t_i) = x. \end{cases}$$

Finally, we define

$$R_{q,i} := \rho_{q,i}\,\mathrm{Id} - \eta_i^2 \mathring{\overline{R}}_q{}^{(2)}, \tag{7.4}$$

$$\widetilde{R}_{q,i} := \frac{\nabla\Phi_i R_{q,i} (\nabla\Phi_i)^t}{\rho_{q,i}}. \tag{7.5}$$

It follows from properties (i) – (iv) of η_i that

- $\operatorname{supp} R_{q,i} \subset \operatorname{supp} \eta_i$,
- we have $\sum_i \eta_i^2 = 1$ on $\operatorname{supp} \mathring{\overline{R}}_q{}^{(2)}$,
- $\operatorname{supp} \widetilde{R}_{q,i} \subset \mathbb{T}^3 \times (t_i - \frac{1}{3}\tau_q, t_{i+1} + \frac{1}{3}\tau_q)$,
- $\operatorname{supp} \widetilde{R}_{q,i} \cap \operatorname{supp} \widetilde{R}_{q,j} = \varnothing$ for all $i \neq j$.

We collect some other useful estimates:

Lemma 7.1. *For $a \gg 1$ sufficiently large we have*

$$\|\nabla\Phi_i - \mathrm{Id}\|_0 \leq \frac{1}{2} \quad \textit{for } t \in \operatorname{supp}(\eta_i). \tag{7.6}$$

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Furthermore, for any $N \geq 0$

$$\frac{\delta_{q+1}}{8\lambda_q^\alpha} \leq \left|\rho_q(t)\right| \leq \delta_{q+1} \quad \forall t, \tag{7.7}$$

$$\left\|\rho_{q,i}\right\|_0 \leq \frac{\delta_{q+1}}{\widetilde{c}_0}, \tag{7.8}$$

$$\left\|\rho_{q,i}\right\|_N \lesssim \delta_{q+1}, \tag{7.9}$$

$$\left\|\partial_t \rho_q\right\|_0 \lesssim \delta_{q+1}\delta_q^{1/2}\lambda_q, \tag{7.10}$$

$$\left\|\partial_t \rho_{q,i}\right\|_N \lesssim \tau_q^{-1}\delta_{q+1}, \tag{7.11}$$

$$\|\widetilde{R}_{q,i}\|_N \lesssim \ell^{-N}, \tag{7.12}$$

$$\|D_{t,q}\widetilde{R}_{q,i}\|_N \lesssim \tau_q^{-1}\ell^{-N}, \tag{7.13}$$

where $D_{t,q} := \partial_t + \overline{v}_q \cdot \nabla$. Moreover, for all (x,t) we have $\widetilde{R}_{q,i}(x,t) \in B\left(\mathrm{Id}, \frac{1}{2}\right) \subset \mathcal{S}^3$.

Proof. Since our subsolution $(\overline{v}_q, \overline{p}_q, \mathring{\overline{R}}_q)$ satisfies analogous estimates to the ones in [7], we may use the same argument to infer (7.6), (7.7) and (7.10). Estimate (7.8) then follows from (7.7) and (7.1). In fact, since our $\rho_{q,i}$ only differs from the one in [7] in a time-dependent normalization coefficient that is bounded above and below, the bounds for $\left\|\rho_{q,i}\right\|_N$ are the same except for the implicit constant. Next, it follows from the property (vi) of η_i that

$$\left|\frac{d}{dt}\sum_i \int_\Omega \phi_q(y,t)^2 \eta_i(y,t)^2 dy\right| \lesssim \tau_q^{-1}.$$

Using this estimate and arguing as in [7] we obtain (7.11). Finally, taking into account these estimates for $\rho_{q,i}$ and the bounds (4.37) and (4.38), the facts regarding $\widetilde{R}_{q,i}$ follow as in [7]. □

7.2. Definition of the perturbation

The building blocks of the perturbation are Mikado flows, introduced in [21, Lemma 2.3]:

Lemma 7.2. *For any compact subset of positive-definite matrices $\mathcal{N} \subset \mathcal{S}^3$ there exists a smooth vector field*

$$W : \mathcal{N} \times \mathbb{T}^3 \to \mathbb{R}^3$$

such that, for every $R \in \mathcal{N}$

$$\begin{cases} \operatorname{div}_\xi [W(R,\xi) \otimes W(R,\xi)] = 0, \\ \operatorname{div}_\xi W(R,\xi) = 0 \end{cases} \tag{7.14}$$

and

$$\fint_{\mathbb{T}^3} W(R,\xi)\, d\xi = 0,$$

$$\fint_{\mathbb{T}^3} W(R,\xi) \otimes W(R,\xi)\, d\xi = R.$$

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Since $W(R,\cdot)$ is $\mathbb{T}^3$-periodic and has zero mean, we may write

$$W(R,\xi)=\sum_{k\in\mathbb{Z}^3\setminus\{0\}} a_k(R)e^{ik\cdot\xi} \tag{7.15}$$

for some $a_k\in C^\infty(\mathcal{N},\mathbb{R}^3)$. Similarly, for some $C_k\in C^\infty(\mathcal{N},\mathcal{S}^3)$ we have

$$W(R,\xi)\otimes W(R,\xi)=R+\sum_{k\neq 0} C_k(R)e^{ik\cdot\xi}. \tag{7.16}$$

It follows from (7.14) that

$$a_k(R)\cdot k=0,\qquad C_k(R)k=0. \tag{7.17}$$

In addition, because of the smoothness we have

$$\sup_{R\in\mathcal{N}}\left|D_R^N a_k(R)\right|+\sup_{R\in\mathcal{N}}\left|D_R^N C_k(R)\right|\le \frac{C(\mathcal{N},N,m)}{|k|^m}. \tag{7.18}$$

In the construction these estimates are used with a particular choice of $\mathcal{N}$ (namely $B(\mathrm{Id},1/2)\subset\mathcal{S}^3$) and m. This choice determines the constant M appearing in Proposition 3.2.

With these building blocks, we define the main perturbation term w_0 as

$$w_0:=\sum_i \phi_q(x,t)(\rho_{q,i}(x,t))^{1/2}(\nabla\Phi_i)^{-1}W(\widetilde{R}_{q,i},\lambda_{q+1}\Phi_i)=\sum_i w_{0,i}.$$

Recall that in Lemma 7.1 we saw that $\widetilde{R}_{q,i}$ takes values in the compact subset of positive-definite matrices $\mathcal{N}:=B(\mathrm{Id},1/2)\subset\mathcal{S}^3$. Therefore, the previous expression is well-defined. To shorten the notation, we set

$$b_{i,k}(x,t):=\phi_q(x,t)(\rho_{q,i}(x,t))^{1/2}a_k(\widetilde{R}_{q,i}(x,t)).$$

Thus, using (7.15) we may write

$$w_0=\sum_{i,k\neq 0}(\nabla\Phi_i)^{-1}b_{i,k}e^{i\lambda_{q+1}k\cdot\Phi_i}.$$

Although Mikado flows are divergence-free, the perturbation w_0 will not be solenoidal, in general, due to the other factor. We must add a small correction term w_c so that $w_{q+1}:=w_0+w_c$ is divergence-free. We set

$$w_{q+1}=\frac{-1}{\lambda_{q+1}}\operatorname{curl}\left(\sum_{i,k\neq 0}(\nabla\Phi)^t\left(\frac{ik\times b_{k,i}}{|k|^2}\right)e^{i\lambda_{q+1}k\cdot\Phi_i}\right).$$

It is clear that w_{q+1} is divergence-free and the correction $w_{q+1}-w_0$ can be seen to be a lower-order term in λ_{q+1}.

Unlike in [7], w_{q+1} is not the final correction. We must add another small perturbation w_L to ensure the final correction $\widetilde{w}_{q+1}=w_{q+1}+w_L$ has no angular momentum. We define $L\in C^\infty([0,T],\mathbb{R}^3)$ as

$$L(t):=\frac{1}{\lambda_{q+1}}\sum_{i,k\neq 0}\int(\nabla\Phi_i)^t\left(\frac{ik\times b_{k,i}}{|k|^2}\right)e^{i\lambda_{q+1}k\cdot\Phi_i}\,dx,$$

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where the integration is undestood to be component-wise. We fix a ball $B \subset A_0$ and we choose $\psi \in C_c^\infty(B, \mathbb{R})$ such that $\int \psi = 1$. We define the correction w_L as

$$w_L := \operatorname{curl}(\psi L).$$

By construction, the correction $\widetilde{w}_{q+1} = w_{q+1} + w_L$ is given by $\widetilde{w}_{q+1} = \operatorname{curl} z$ for a potential vector z such that $\int z\, dx = 0$ component-wise. Using the vector identity $\operatorname{div}(\xi \times z) = z \cdot \operatorname{curl} \xi - \xi \cdot \operatorname{curl} z$, we have

$$\int \xi \cdot \widetilde{w}_{q+1} = \int \xi \cdot \operatorname{curl} z = -\int z \cdot \operatorname{curl} \xi.$$

Since the curl of any Killing field is constant, we conclude

$$\int \xi \cdot \widetilde{w}_{q+1}\, dx = 0 \qquad \forall t \in [0, T],\ \forall \xi \in \ker \nabla_{\text{sym}}. \tag{7.19}$$

7.3. Estimates on the perturbation

Aside from w_L, our perturbation differs from the one in [7] in the presence of the factor ϕ_q. Nevertheless, the factor ϕ_q appears multiplying $a_k(\widetilde{R}_{q,i})$. Thus, if we show that $\phi_q a_k(\widetilde{R}_{q,i})$ satisfies the same bounds as $a_k(\widetilde{R}_{q,i})$, the same estimates that are derived in [7] will apply here. It follows from (7.18), (7.12) and (7.13) that

$$\|a_k(\widetilde{R}_{q,i})\|_N \lesssim \frac{\ell^{-N}}{|k|^6}, \qquad \|D_{t,q} a_k(\widetilde{R}_{q,i})\|_N \lesssim \frac{\tau_q^{-1}\ell^{-N}}{|k|^6}.$$

Since $\|\phi_q\|_N \lesssim \lambda_q^{-N/10} \lesssim \tau_q^{-N} \lesssim \ell^{-N}$ we also have

$$\|\phi_q a_k(\widetilde{R}_{q,i})\|_N \lesssim \frac{\ell^{-N}}{|k|^6}, \qquad \|D_{t,q}[\phi_q a_k(\widetilde{R}_{q,i})]\|_N \lesssim \frac{\tau_q^{-1}\ell^{-N}}{|k|^6}.$$

We conclude that all of the estimates in [7] are also valid here. In particular, we have

Lemma 7.3. *Assuming a is sufficiently large, the perturbations w_0, w_c and w_q satisfy the following estimates:*

$$\|w_0\|_0 + \frac{1}{\lambda_{q+1}} \|w_0\|_1 \le \frac{M}{4}\delta_{q+1}^{1/2},$$
$$\|w_c\|_0 + \frac{1}{\lambda_{q+1}} \|w_c\|_1 \lesssim \delta_{q+1}^{1/2}\ell^{-1}\lambda_{q+1}^{-1},$$
$$\|w_{q+1}\|_0 + \frac{1}{\lambda_{q+1}} \|w_{q+1}\|_1 \le \frac{M}{2}\delta_{q+1}^{1/2},$$

where the constant M depends solely on the constant $\widetilde{c}_0$ in (7.1).

We carry out the estimates for w_L with more detail. First, due to (7.18) we have $\|b_{i,k}\|_0 \lesssim \|\rho_{i,q}\|_0^{1/2} |k|^{-6} \lesssim \delta_{q+1}^{1/2} |k|^{-6}$. Next, it follows from (7.6) that $\|\nabla\Phi\|_0 \lesssim 1$. Introducing these estimates in the definition of L we obtain

$$|L(t)| \lesssim \sum_{k\neq 0} \frac{1}{\lambda_{q+1}} \|\nabla\Phi_i\|_0 \|b_{i,k}\|_0 \lesssim \sum_{k\neq 0} \frac{\delta_{q+1}^{1/2}}{|k|^6 \lambda_{q+1}} \lesssim \delta_{q+1}^{1/2}\lambda_{q+1}^{-1},$$

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where we have used that at most two of the $b_{i,q}$ are nonzero at any given time. Since ψ is fixed throughout the iterative process, we conclude

$$\|w_L\|_N \lesssim \delta_{q+1}^{1/2}\lambda_{q+1}^{-1}. \tag{7.20}$$

Therefore, the correction w_L is really small. We see that for sufficiently large a the perturbation $\widetilde{w}_{q+1}$ then satisfies

$$\left\|w_{q+1}\right\|_0 + \frac{1}{\lambda_{q+1}}\left\|w_{q+1}\right\|_1 \leq \frac{3}{4}M\delta_{q+1}^{1/2}.$$

Regarding $\partial_t w_L$, we must first estimate

$$\begin{aligned}\|D_{t,q}b_{i,k}\|_0 &\lesssim \|\partial\rho_{q,i} + \overline{v}_q\cdot\nabla\rho_{q,i}\|_0\,\|\phi_q a_k(\widetilde{R}_{q,i})\|_0 + \left\|\rho_{q,i}\right\|_0\,\|D_{t,q}[\phi_q a_k(\widetilde{R}_{q,i})]\|_0\\ &\lesssim \tau_q^{-1}\delta_{q+1}^{1/2}\,|k|^{-6}.\end{aligned}$$

Next, we compute

$$\begin{aligned}L'(t) &= \frac{1}{\lambda_{q+1}}\sum_{i,k\neq 0}\int D_{t,q}\left[(\nabla\Phi)^t\left(\frac{ik\times b_{k,i}}{|k|^2}\right)e^{i\lambda_{q+1}k\cdot\Phi_i}\right]dx\\ &= \frac{1}{\lambda_{q+1}}\sum_{i,k\neq 0}\int\left[-\nabla\overline{v}_q(\nabla\Phi)^t\left(\frac{ik\times b_{k,i}}{|k|^2}\right) + (\nabla\Phi)^t\left(\frac{ik\times D_{t,q}b_{k,i}}{|k|^2}\right)\right]e^{i\lambda_{q+1}k\cdot\Phi_i}\,dx,\end{aligned}$$

where we have computed the material derivative of $\nabla\Phi_i$ taking into account that Φ_i solves the transport equation. Since $\left\|\nabla\overline{v}_q\right\|_0 \lesssim \delta_q^{1/2}\lambda_q$ by (4.28), we have

$$|L'(t)| \lesssim \lambda_{q+1}^{-1}\delta_{q+1}^{1/2}(\delta_q^{1/2}\lambda_q + \tau_q^{-1}) \lesssim \frac{\delta_{q+1}^{1/2}\delta_q^{1/2}\lambda_q}{\lambda_{q+1}^{1-3\alpha}}$$

because $\tau_q^{-1} = \delta_q^{1/2}\lambda_q\ell^{-2\alpha} \lesssim \delta_q^{1/2}\lambda_q\lambda_{q+1}^{-3\alpha}$ by (4.2). Since ψ is fixed, we conclude

$$\|\partial_t w_L\|_N \lesssim \frac{\delta_{q+1}^{1/2}\delta_q^{1/2}\lambda_q}{\lambda_{q+1}^{1-3\alpha}}. \tag{7.21}$$

7.4. The final Reynolds stress

Taking into account that ϕ_q and $\sum_i \eta_i^2$ equal 1 on supp $\mathring{\bar{R}}_q{}^{(2)}$, it follows from the definition of $R_{q,i}$ that

$$\sum_i \phi_q^2 R_{q,i} = -\mathring{\bar{R}}_q{}^{(2)} + \sum_i \phi_q^2\rho_{q,i}\,\mathrm{Id}\,.$$

Using this along with the fact that $\left(\overline{v}_q, \overline{p}_q, \mathring{\bar{R}}_q{}^{(1)} + \mathring{\bar{R}}_q{}^{(2)}\right)$ is a subsolution, we obtain:

$$\begin{aligned}&\partial_t v_{q+1} + \operatorname{div}(v_{q+1}\otimes v_{q+1}) + \nabla\overline{p}_q\\ &\quad= \operatorname{div}\left(\mathring{\bar{R}}_q{}^{(1)} + w_L\otimes\overline{v}_q + \overline{v}_q\otimes w_L + w_L\otimes w_L\right)\\ &\qquad+ \nabla\left(\sum_i\phi_q^2\rho_{q,i}\right) + (\partial_t w_{q+1} + \overline{v}_q\cdot\nabla w_{q+1}) + w_{q+1}\cdot\nabla\overline{v}_q + \operatorname{div}\left(w_{q+1}\otimes w_{q+1} - \sum_i\phi_q^2 R_{q,i}\right).\end{aligned}$$

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Hence, we conclude that we may construct a new subsolution $(v_{q+1}, p_{q+1}, \mathring{R}_{q+1})$ by setting

$$\mathring{R}_{q+1} := \mathring{\overline{R}}_q^{(1)} + w_L \mathring{\otimes} \overline{v}_q + \overline{v}_q \mathring{\otimes} w_L + w_L \mathring{\otimes} w_L + S - \frac{1}{3}(\operatorname{tr} S)\,\mathrm{Id},$$
$$p_{q+1} := \overline{p}_q - \sum_i \phi_q^2 \rho_{q,i} - |w_L|^2 - 2\,\overline{v}_q \cdot w_L - \frac{1}{3}\operatorname{tr} S,$$

where the smooth symmetric matrix S satisfies

$$\operatorname{div} S = \partial_t w_L + \underbrace{(\partial_t w_{q+1} + \overline{v}_q \cdot \nabla w_{q+1})}_{\text{transport error}} + \underbrace{w_{q+1} \cdot \nabla \overline{v}_q}_{\text{Nash error}} + \underbrace{\operatorname{div}\left(w_{q+1} \otimes w_{q+1} - \sum_i \phi_q^2 R_{q,i}\right)}_{\text{oscillation error}}$$

and the support of $S(\cdot, t)$ is contained in $A_q + B(0, 5\sigma)$ for all $t \in [0, T]$. If such a matrix existed, the new subsolution $(v_{q+1}, p_{q+1}, \mathring{R}_{q+1})$ would equal $(\overline{v}_q, \overline{p}_q, \mathring{\overline{R}}_q)$ in $A_{q+1} \times [0, T]$ because the support of ϕ_q and $\widetilde{w}_{q+1}(\cdot, t)$ is contained in $A_q + B(0, 5\sigma)$. Therefore, it equals $(v_0, p_0, \mathring{R}_0)$ in $A_{q+1} \times [0, T]$.

We will show that it is possible to construct S and we will derive the necessary estimates. Let $f := \operatorname{div} S$. Note that the support of $f(\cdot, t)$ is contained in $A_q + B(0, 5\sigma)$ for all $t \in [0, T]$ because so is the support of ϕ_q and the perturbation. Next, we see that for any Killing field ξ we have

$$\int \xi \cdot f\, dx = \frac{d}{dt} \int \xi \cdot \widetilde{w}_{q+1} + \int \xi \cdot \operatorname{div}\left(\overline{v}_q \otimes w_{q+1} + w_{q+1} \otimes \overline{v}_q + w_{q+1} \otimes w_{q+1} - \sum_i \phi_q^2 R_{q,i}\right)$$
$$= 0$$

because of (7.19) and (2.6). Therefore, by Lemma 2.9 there exists a symmetric matrix $S \in C^\infty(\mathbb{R}^3 \times [0, T], \mathcal{S}^3)$ such that $\operatorname{div} S = f$ and with the stated support. We will now estimate the C^α-norm of the potential theoretic solution of the equation. Arguing as in Lemma 2.9, this yields a bound for the C^0-norm of S.

We begin by studying the first term in f. It follows from (7.21) and the fact that $\mathcal{R}$ is bounded on Hölder spaces that

$$\|\mathcal{R}(\partial_t w_L)\|_\alpha \le \|\mathcal{R}(\partial_t w_L)\|_{1+\alpha} \lesssim \|\partial_t w_L\|_\alpha \lesssim \frac{\delta_{q+1}^{1/2} \delta_q^{1/2} \lambda_q}{\lambda_{q+1}^{1-3\alpha}}. \tag{7.22}$$

The remaining three error terms are analogous to the ones in [7]. Since our fields satisfy the same estimates as in [7], the estimates for the potential-theoretic solution are completely analogous. We do have to take into account that the oscillation error has a slightly different expression than the one in [7]:

$$\operatorname{div}\left(w_{q+1} \otimes w_{q+1} - \sum_i \phi_q^2 R_{q,i}\right)$$
$$= \operatorname{div}\left(w_0 \otimes w_0 - \sum_i \phi_q^2 R_{q,i}\right) + \operatorname{div}(w_0 \otimes w_c + w_c \otimes w_0 + w_c \otimes w_c) \equiv \mathcal{O}_1 + \mathcal{O}_2.$$

The second term $\mathcal{O}_2$ is the same as in [7]. Regarding $\mathcal{O}_1$, it follows from the fact that the cutoffs η_i have pairwise disjoint support that

$$\mathcal{O}_1 = \sum_i \operatorname{div}(w_{0,i} \otimes w_{0,i} - \phi_q^2 R_{q,i}).$$

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We use the definition of $w_{0,i}$ and (7.16) to write

$$\begin{aligned} w_{0,i} \otimes w_{0,i} &= \phi_q^2 \rho_{q,i} \nabla\Phi_i^{-1} (W \otimes W)(\widetilde{R}_{q,i}, \lambda_{q+1}\Phi_i) \nabla\Phi_i^{-t} \\ &= \phi_q^2 \nabla\Phi_i^{-1} \widetilde{R}_{q,i} \nabla\Phi_i^{-t} + \sum_{k\neq 0} \phi_q^2 \rho_{q,i} \nabla\Phi_i^{-1} C_k(\widetilde{R}_{q,i}) \nabla\Phi_i^{-t} e^{i\lambda_{q+1} k\cdot\Phi_i} \\ &= \phi_q^2 R_{q,i} + \sum_{k\neq 0} \phi_q^2 \rho_{q,i} \nabla\Phi_i^{-1} C_k(\widetilde{R}_{q,i}) \nabla\Phi_i^{-t} e^{i\lambda_{q+1} k\cdot\Phi_i}. \end{aligned} \tag{7.23}$$

On the other hand, it follows from (7.17) that

$$\nabla\Phi_i^{-1} C_k \nabla\Phi_i^{-t} \nabla\Phi_i^t k = 0,$$

so

$$\mathcal{O}_1 = \sum_{i,k\neq 0} \operatorname{div}(\phi_q^2 \rho_{q,i} \nabla\Phi_i^{-1} C_k(\widetilde{R}_{q,i}) \nabla\Phi_i^{-t}) e^{i\lambda_{q+1} k\cdot\Phi_i},$$

which is the same as in [7] except for the presence of ϕ_q^2. Nevertheless, it is easy to check, as in Section 7.3, that $\phi_q^2 C_k(\widetilde{R}_{q,i})$ satisfies the same estimates as $C_k(\widetilde{R}_{q,i})$. Hence, we obtain the same bounds for $\mathcal{O}_1$.

Aside from the presence of the cutoff, there is another subtle difference that we have to take into account. The proof in [7] uses that the following inequality holds for a suitable choice of the parameters:

$$\frac{1}{\lambda_{q+1}^{N-\alpha} \ell^{N+\alpha}} \leq \frac{1}{\lambda_{q+1}^{1-\alpha}}.$$

Since our definition of ℓ is slightly different, we must check that this inequality holds. Remember that $\lambda_{q+1} \gtrsim \lambda_q^b$ by (4.4). Hence, we have

$$\frac{\lambda_{q+1}^{N-\alpha} \ell^{N+\alpha}}{\lambda_{q+1}^{1-\alpha}} = \lambda_{q+1}^{N-1-\beta(N+\alpha)} \lambda_q^{-(N+\alpha)(1-\beta+3\alpha)} \gtrsim \lambda_q^{[(b-1)(1-\beta)-3\alpha]N - b(1+\beta\alpha) - \alpha(1-\beta+3\alpha)}.$$

Note that $(b-1)(1-\beta) > 0$ so by choosing α sufficiently small we can ensure that the coefficient multiplying N is positive. Thus, for sufficiently large N the exponent is positive and choosing a sufficiently large beats any geometrical constant. We conclude that with this choice of parameters the claimed inequality holds.

A similar argument is used several times, for instance, to obtain the inequality $\lambda_{q+1}\ell \geq 1$. Nevertheless, in all of them the difference in the definition of ℓ is quite harmless and it only leads to choosing a smaller α and a slightly larger N.

In conclusion, the estimates from [7] apply to our case. Combining them with (7.22) we obtain

$$\|\mathcal{R} f\|_\alpha \lesssim \frac{\delta_{q+1}^{1/2} \delta_q^{1/2} \lambda_q}{\lambda_{q+1}^{1-4\alpha}}.$$

Since $\int \xi \cdot f \, dx = 0$ for all $t \in [0, T]$ and any Killing field ξ, we may use the construction of Lemma 2.9 to modify $\mathcal{R} f$ into a smooth symmetric matrix S such that

- $\operatorname{div} S = f$,
- $\operatorname{supp} S(\cdot, t) \subset A_q + B(0, 5\sigma)$ for all $t \in [0, T]$,
- $\|S\|_0 \lesssim \delta_{q+1}^{1/2} \delta_q^{1/2} \lambda_q \lambda_{q+1}^{-(1-4\alpha)}$.

Combining this with (7.20) we have

$$\left\| w_L \mathring{\otimes} \overline{v}_q + \overline{v}_q \mathring{\otimes} w_L + w_L \mathring{\otimes} w_L + S - \frac{1}{3}(\operatorname{tr} S)\,\mathrm{Id} \right\|_0 \lesssim \frac{\delta_{q+1}^{1/2}\delta_q^{1/2}\lambda_q}{\lambda_{q+1}^{1-4\alpha}}. \tag{7.24}$$

We claim that with a suitable choice of the parameters

$$\frac{\delta_{q+1}^{1/2}\delta_q^{1/2}\lambda_q}{\lambda_{q+1}} \leq \delta_{q+2}\lambda_{q+1}^{-11\alpha}. \tag{7.25}$$

In that case, (4.35) and (7.24) would yield (4.40) for sufficiently large a, as we wanted. To prove the claimed inequality, we compute

$$\frac{\lambda_{q+1}^{1-11\alpha}\delta_{q+2}}{\delta_{q+1}^{1/2}\delta_q^{1/2}\lambda_q} = \lambda_{q+1}^{1+\beta-11\alpha}\lambda_{q+2}^{-2\beta}\lambda_q^{-1+\beta} \gtrsim \lambda_q^{b(1+\beta-11\alpha)-2b^2\beta-1+\beta}.$$

The condition (3.18) ensures that

$$b - 1 + \beta + b\beta - 2b^2\beta > 0,$$

so the exponent is positive for sufficiently small $\alpha > 0$. Thus, choosing a sufficiently large beats any numerical constant and (7.25) follows.

7.5. The new energy profile

By definition

$$\int_\Omega |v_{q+1}|^2 dx = \int_\Omega |\overline{v}_q|^2 dx + 2\int_\Omega \overline{v}_q \cdot \widetilde{w}_{q+1} dx + \int_\Omega |\widetilde{w}_{q+1}|^2 dx.$$

Note that in the last two terms we can integrate on the whole $\mathbb{T}^3$ because the perturbation is supported in Ω. Arguing as in [7] yields the estimate

$$\left| \int_{\mathbb{T}^3} \left(2\,\overline{v}_q \cdot w_{q+1} + 2w_0 \cdot w_c + |w_c|^2 \right) dx \right| \lesssim \frac{\delta_q^{1/2}\delta_{q+1}^{1/2}\lambda_q^{1+2\alpha}}{\lambda_{q+1}}.$$

On the other hand, any term containing w_L will be smaller than this bound by (7.20). The remaining term is

$$\int_{\mathbb{T}^3} |w_0|^2 dx = \sum_i \int_{\mathbb{T}^3} \phi_q^2 \operatorname{tr} R_{q,i} dx + \int_{\mathbb{T}^3} \sum_{i,k\neq 0} \phi_q^2 \rho_{q,i} \nabla\Phi_i^{-1} \operatorname{tr} C_k(\widetilde{R}_{q,i}) \nabla\Phi_i^{-t} e^{i\lambda_{q+1}k\cdot\Phi_i},$$

where we have used (7.23). The second term can be estimated as in [7] because $\phi_q^2 C_k(\widetilde{R}_{q,i})$ satisfies the same estimates as $C_k(\widetilde{R}_{q,i})$, as argued several times. Regarding the first term:

$$\sum_i \int_{\mathbb{T}^3} \phi_q^2(x) \operatorname{tr} R_{q,i}(x,t)\, dx = 3\sum_i \int_{\mathbb{T}^3} \phi_q^2(x)\rho_{q,i}(x,t)\, dx = 3\rho_q(t) = e(t) - \frac{1}{2}\delta_{q+2} - \int_\Omega |\overline{v}_q|^2.$$

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We conclude that

$$\left|e(t) - \int_\Omega |v_{q+1}|^2 dx - \frac{\delta_{q+2}}{2}\right| \lesssim \frac{\delta_q^{1/2}\delta_{q+1}^{1/2}\lambda_q^{1+2\alpha}}{\lambda_{q+1}} \overset{(7.25)}{\lesssim} \delta_{q+2}\lambda_{q+1}^{-9\alpha},$$

which yields (4.41) for sufficiently large a.

7.6. Proof of Lemma 3.3

This lemma is just a simplified version of the construction presented in the previous subsections, so we will just sketch its proof. Since the initial Reynolds stress $\mathring{R}_0$ and its derivatives vanish at $\partial\Omega \times [0, T]$, for any $k \in \mathbb{N}$ there exist a constant C_k such that for any $x \in \Omega$ we have

$$|\mathring{R}_0(x, t)| \leq C_k \operatorname{dist}(x, \partial\Omega)^k.$$

Therefore, if $\operatorname{dist}(x, \partial\Omega) < 3\lambda^{-1/12}$ we have

$$|\mathring{R}_0(x, t)| \leq C_6(3\lambda^{-1/12})^6 \lesssim \frac{1}{\lambda^{1/2}}.$$

We fix a smooth cutoff function $\phi \in C_c^\infty(\Omega, [0, 1])$ such that

$$\phi(x) = \begin{cases} 1 & \text{if } \operatorname{dist}(x, \partial\Omega) \geq 3\lambda^{-1/12}, \\ 0 & \text{if } \operatorname{dist}(x, \partial\Omega) \leq 2\lambda^{-1/12}. \end{cases}$$

It can be chosen so that $\|\phi\|_N \lesssim \lambda^{N/12}$. This function will control the support of the perturbation.

Next, we define the backwards flows Φ for the velocity field v_0 as the solution of the transport equation

$$\begin{cases} (\partial_t + v_0 \cdot \nabla)\Phi_i = 0, \\ \Phi_i(x, t_i) = x \end{cases}$$

and we define

$$\widetilde{R} := \nabla\Phi \left(\operatorname{Id} - (2\|\mathring{R}_0\|_0)^{-1}\mathring{R}_0\right) \nabla\Phi^t.$$

Let $\mathcal{N} \subset \mathcal{S}^3$ be the compact subset of positive definite matrices whose eigenvalues take values between $1/2$ and $3/2$. We see that $\widetilde{R}$ takes values in $\mathcal{N}$. Thus, we may apply Lemma 7.2 and define

$$w_0 := (2\|\mathring{R}_0\|_0)^{1/2}\phi\,(\nabla\Phi)^{-1}W(\widetilde{R}, \lambda\Phi) = \sum_{k\neq 0}(\nabla\Phi)^{-1}b_k e^{i\lambda k\cdot\Phi},$$

with $b_k := (2\|\mathring{R}_0\|_0)^{1/2}\phi\, a_k(\widetilde{R})$. We also have

$$w_0 \otimes w_0 = 2\|\mathring{R}_0\|_0\, \phi^2\, \widetilde{R} + \sum_{k\neq 0} 2\|\mathring{R}_0\|_0\, \phi^2 \nabla\Phi^{-1} C_k(\widetilde{R})\nabla\Phi^{-t} e^{i\lambda k\cdot\Phi}. \tag{7.26}$$

Next, the correction w_c is then defined so that $w := w_0 + w_c$ is divergence-free:

$$w_0 + w_c = \frac{-1}{\lambda}\operatorname{curl}\left(\sum_{k\neq 0}(\nabla\Phi)^t \left(\frac{ik \times b_k}{|k|^2}\right) e^{i\lambda k\cdot\Phi}\right). \tag{7.27}$$

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Regarding the angular momentum, we define

$$L(t) := \frac{1}{\lambda} \sum_{k \neq 0} \int (\nabla \Phi)^t \left(\frac{ik \times b_k}{|k|^2} \right) e^{i\lambda_{q+1} k \cdot \Phi} \, dx.$$

We fix a ball $B \Subset \Omega$ and we choose $\psi \in C_c^\infty(B)$ such that $\int \psi = 1$. We add the correction w_L so that the perturbation has vanishing angular momentum:

$$w_L := \operatorname{curl}(\psi L).$$

If Ω has several connected components Ω^j, we will have to consider the partial angular momentum L^j. We will need to add one such vortex to each Ω^j to cancel the angular momentum in each connected component of Ω. We still denote the total correction as w_L.

The new velocity field is $v := v_0 + w_0 + w_c + w_L$. Note that by taking a larger λ we may force $B \subset \operatorname{supp} \phi$, so the perturbation vanishes for $\operatorname{dist}(x, \partial\Omega) \leq 2\lambda^{-1/12}$.

Since $\|\phi\|_N \lesssim \lambda^{N/12}$, the dominant term is the exponential. Hence, from (7.27) and the definition of $L(t)$ we conclude

$$\|v\|_N \lesssim \lambda^N.$$

Let us denote $D_t := \partial_t + v_0 \cdot \nabla$. Since $D_t \Phi = 0$, we see that $|L'(t)| \lesssim \lambda^{-(1-1/12)}$. We conclude

$$\|\partial_t w_L\|_N \lesssim \lambda^{-(1-1/12)}.$$

Finally, we define

$$\mathring{R} := (1 - \phi^2)\mathring{R}_0 + v \otimes w_L + w_L \otimes v - w_L \otimes w_L + S - \frac{1}{3}\left[2v \cdot w_L - |w_L|^2 + \operatorname{tr}(S)\right] \mathrm{Id},$$

$$p := p_0 - 2\|\mathring{R}_0\|_0 \phi^2 - \frac{1}{3}\left[2v \cdot w_L - |w_L|^2 + \operatorname{tr}(S)\right],$$

where the smooth symmetric matrix S satisfies

$$\operatorname{div} S = \partial_t w_L + \left[D_t w + w \cdot \nabla v_0 + \operatorname{div}\left(w \otimes w - 2\|\mathring{R}_0\|_0 \phi^2 \widetilde{R} \right) \right] \equiv f. \tag{7.28}$$

It is easy to check that $(v, p, \mathring{R})$ is a subsolution. Furthermore, the fact that the perturbation has vanishing angular momentum ensures that we may choose S with support contained in A_* by using Lemma 2.9. Therefore, the $(v, p, \mathring{R})$ equals the initial subsolution outside A_*.

Regarding the estimates, by (7.26) we may write the term in brackets in (7.28) as

$$\sum_{k \neq 0} c_k e^{i\lambda k \cdot \Phi}$$

for certain vectors c_k such that $\|c_k\|_N \lesssim |k|^{-6} \lambda^{(N+1)/12}$. The standard stationary phase lemma (see [7]) yields

$$\|\mathcal{R} f\|_{1/4} \lesssim \frac{\lambda^{2/12}}{\lambda^{1-1/4}} \leq \lambda^{-1/2}.$$

Continuing the construction of Lemma 2.9, the claimed bound for $\mathring{R}$ follows.

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Regarding the energy, (7.26) and a standard stationary phase lemma (see [7]) yield:

$$\begin{aligned}\int_\Omega |v|^2 dx &= \int_\Omega |v_0|^2 dx + \int_\Omega |w_0|^2 dx + O\left(\frac{1}{\lambda^{1-1/12}}\right)\\ &= \int_\Omega |v_0|^2 dx + \int_\Omega 2\|\mathring{R}_0\|\phi^2 \operatorname{tr}(\widetilde{R})\, dx.\end{aligned}$$

Since $\mathring{R}_0$ is trace-free, $\operatorname{tr}\widetilde{R} = 3$. We conclude that (3.20) holds for sufficiently large λ.

8. Proof of Theorem 1.1

We are ready to prove our main theorem using Theorems 1.6 and 1.7. First, we show that the conditions are necessary. Suppose that such a weak solution (v, p) exists. We fix a connected component Σ of $\partial\Omega$ and $a \in \mathbb{R}^3$. In order to give us some room to mollify the subsolution, we also fix a smooth surface $\Sigma' \subset \Omega$ that will be used to approximate Σ from the inside of Ω. Given $\varepsilon > 0$, if follows from the smoothness of (v_0, p_0) that Σ' can be chosen sufficiently close to Σ so that

$$\left|\int_\Sigma v_0\cdot\nu - \int_{\Sigma'} v_0\cdot\nu\right| < \varepsilon,$$

$$\left|\int_\Sigma [(a\cdot x)\partial_t v_0 + (a\cdot v)v_0 + p_0 a]\cdot\nu - \int_{\Sigma'} [(a\cdot x)\partial_t v_0 + (a\cdot v)v_0 + p_0 a]\cdot\nu\right| < \varepsilon$$

for any $t \in [0, T]$.

Next, we fix a mollification kernel $\psi \in C_c^\infty(\mathbb{R}^3 \times \mathbb{R})$ whose support is contained in the unit ball and for $0 < \ell < \varepsilon$ we define

$$\begin{aligned} v_\ell &:= v * \psi_\ell,\\ p_\ell &:= p * \psi_\ell + \left|v_q\right|^2 * \psi_\ell - |v_\ell|^2,\\ \mathring{R}_\ell &:= v_\ell \mathring{\otimes} v_\ell - (v\mathring{\otimes} v) * \psi_\ell,\end{aligned}$$

where $f\mathring{\otimes} g$ denotes the traceless part of the tensor $f\otimes g$. Since (v, p) is a weak solution, it is easy to see that $(v_\ell, p_\ell, \mathring{R}_\ell)$ is a smooth subsolution in $\mathbb{R}^3\times(\varepsilon, T-\varepsilon)$. On the other hand, the values of $(v_\ell, p_\ell, \mathring{R}_\ell)$ on Σ' depend only on (v_0, p_0) for $\ell < \operatorname{dist}(\Sigma', \partial\Omega)$ because (v, p) equals (v_0, p_0) on $\overline{\Omega}\times[0, T]$. In particular, we have,

$$\lim_{\ell\to 0}(v_\ell, p_\ell, \mathring{R}_\ell)(x,t) = (v_0, p_0, 0)(x,t) \qquad \text{uniformly in } (x,t)\in\Sigma'\times[\varepsilon, T-\varepsilon],$$

$$\lim_{\ell\to 0}\partial_t v_\ell(x,t) = \partial_t v_0(x,t) \qquad \text{uniformly in } (x,t)\in\Sigma'\times[\varepsilon, T-\varepsilon].$$

Hence, for sufficiently small ℓ, for any $t \in [\varepsilon, T-\varepsilon]$ we have

$$\left|\int_{\Sigma'} v_0\cdot\nu - \int_{\Sigma'} v_\ell\cdot\nu\right| < \varepsilon,$$

$$\left|\int_{\Sigma'} [(a\cdot x)\partial_t v_0 + (a\cdot v)v_0 + p_0 a]\cdot\nu - \int_{\Sigma'} \left[(a\cdot x)\partial_t v_\ell + (a\cdot v)v_\ell + p_\ell a - a^t\mathring{R}_\ell\right]\cdot\nu\right| < \varepsilon.$$

However, these integrals vanish:

$$\int_{\Sigma'} v_\ell\cdot\nu = \int_{\Sigma'}\left[(a\cdot x)\partial_t v_\ell + (a\cdot v)v_\ell + p_\ell a - a^t\mathring{R}_\ell\right]\cdot\nu = 0 \qquad \forall t\in(\varepsilon, T-\varepsilon)$$

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because $(v_\ell, p_\ell, \mathring{R}_\ell)$ is a smooth subsolution (see the proof of Lemma 2.11, equations (2.18) and (2.19)). We conclude that for all $t \in (\varepsilon, T-\varepsilon)$ we have

$$\left|\int_\Sigma v_0 \cdot \nu\right| + \left|\int_\Sigma [(a\cdot x)\partial_t v_0 + (a\cdot \nu)v_0 + p_0 a]\cdot \nu\right| < 4\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary and (v_0, p_0) is smooth up to the endpoints of the interval, we deduce

$$\int_\Sigma v_0 \cdot \nu = \int_\Sigma [(a\cdot x)\partial_t v_0 + (a\cdot \nu)v_0 + p_0 a]\cdot \nu = 0 \qquad \forall t \in [0,T].$$

Taking into account that $a \in \mathbb{R}^3$ and the connected component Σ of $\partial\Omega$ are arbitrary, we see that the conditions in Theorem 1.1 are, indeed, necessary.

Next, we prove that the conditions are also sufficient. First of all, we may assume that $\Omega' \supset \overline{\Omega}$ is a bounded set with smooth boundary and with a finite number of connected components. Next, by Theorem 1.6 there exists a subsolution $(\widetilde{v}_0, \widetilde{p}_0, \mathring{\widetilde{R}}_0) \in C^\infty(\mathbb{R}^3\times[0,T])$ that extends $(v_0, p_0, 0)$ outside $\overline{\Omega}$ and whose spatial support is contained in Ω'. In particular, supp $\mathring{\widetilde{R}}_0(\cdot,t)$ will be contained in $\overline{\Omega}'\backslash\Omega$. Applying Theorem 1.7, we obtain a weak solution of the Euler equations (v,p) that equals $(\widetilde{v}_0, \widetilde{p}_0)$ outside $\overline{\Omega}'\backslash\Omega$. Therefore, its support is contained in $\overline{\Omega}'$ and it extends (v_0, p_0). On the other hand, when applying Theorem 1.7 we may prescribe any energy profile $e \in C^\infty([0,T])$ such that

$$e(t) > \int_{\mathbb{R}^3} |\widetilde{v}_0(x,t)|^2 dx + 6\|\mathring{\widetilde{R}}_0\|_0\,|\Omega'\backslash\Omega|\,.$$

Hence, we can define the constant e_0 that appears in the statement of Theorem 1.1 as any number greater than the right-hand side of the previous inequality. Finally, $v \in C^\beta(\mathbb{R}^3\times[0,T])$, as we wanted, thus completing the proof of the theorem.

Sketch of the proof of Remark 1.2. If we construct the spatial extension $(\widetilde{v}_0, \widetilde{p}_0, \mathring{\widetilde{R}}_0) \in C^\infty(\mathbb{R}^3\times[0,T])$ so that $\widetilde{v}_0$ has vanishing total angular momentum, we can easily extend in time to a subsolution $(\hat{v}_0, \hat{p}_0, \mathring{\hat{R}}_0) \in C^\infty(\mathbb{R}^3\times[0,+\infty)$ whose temporal support is contained in $[0,T')$. By taking the energy profile e larger, if necessary, we may extend it to $[0,T']$ maintaining an analogue of the condition (1.3).

We then carry out the same construction as in Theorem 1.7 with some minor modifications:

- in Lemma 3.3 and in the perturbation step of Proposition 3.2 we introduce a temporal cutoff so that we do not modify the subsolution at the times when the Reynolds stress is identically 0 and
- in the perturbation step of Proposition 3.2 we define $\rho_{q,i} := \eta_i^2\delta_{q+1}$ instead of (7.3) if the interval $(t_i - \frac{1}{3}\tau_q, t_{i+1} + \frac{1}{3}\tau_q)$ is disjoint from $[0,T]$.

With such an scheme we only prescribe the energy profile in $[0,T]$. However, the energy profile in $[T,T']$ does not differ much (depending on the initial Reynolds stress) from $\int |\hat{v}_0(x,t)|^2 dx$. Hence, if we choose $e(0)$ sufficiently large, the final weak solution will be admissible.

9. Open time interval

In this section we study what happens when the fields are defined in an open interval $(0,T)$ and there is some singular behavior at the endpoints of the interval. So far we have always considered the supremum in time of the spatial Hölder norms of our fields. This is not an option for the situation that we have in mind, which is the setting for our applications.

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9.1. Main result

Our approach consists in decomposing $(0,T)$ as a countable union of closed intervals $\{\mathcal{I}_k\}_{k=0}^{\infty}$ meeting only at their endpoints and trying to work in each of them independently. While some parts of the iterative scheme that we have discussed in the previous sections depend only on what is happening at the current time (solving the symmetric divergence equation, for instance), many others do not (whenever we have dealt with transport). Therefore, if the Reynolds stress is nonzero at the endpoints of the intervals, when we try to correct it the subsolution in one interval will affect its neighbor. This propagates the bad estimates from near $t = 0$ and $t = T$ to any $\mathcal{I}_k$ after enough iterations of the scheme.

Hence, if we want to isolate the closed intervals and work in each of them independently, we must ensure that the Reynolds stress vanishes identically at their endpoints:

Lemma 9.1. *Let* $(v, p, R) \in C^\infty(\mathbb{R}^3 \times [0,T])$ *be a subsolution of the Euler equations. Let* $t_0 \in (0,T)$ *and let* $s > 0$ *be sufficiently small. Suppose that the support of* $\mathring{R}(\cdot, t_0)$ *is contained in an open set* Ω *and that* $\operatorname{div}\operatorname{div} R(\cdot, t_0) = 0$*. Then, there exists a smooth subsolution* $(\widetilde{v}, \widetilde{p}, \widetilde{R})$ *such that* $\widetilde{R}(\cdot, t_0) \equiv 0$ *and such that* $(\widetilde{v}, \widetilde{p}, \widetilde{R}) = (v, p, R)$ *outside* $\Omega \times (t_0 - s, t_0 + s)$*. Furthermore, we have the following estimates:*

$$\|\widetilde{v} - v\|_N \le C(N)\, s\, \|R(\cdot, t_0)\|_{C^{N+1}},$$
$$\|\widetilde{R} - R\|_0 \le C\left(\|R(\cdot, t_0)\|_{C^0} + s\,\|R(\cdot, t_0)\|_{C^1}\,\|v\|_0 + s^2\,\|R(\cdot, t_0)\|_{C^1}^2\right)$$

for some constants independent of s and (v, p, R)*.*

Proof. We fix a smooth cutoff function $\chi \in C_c^\infty((-1,1), \mathbb{R})$ that equals 1 in a neighborhood of the origin. Consider the field

$$\widetilde{v}(x,t) = v(x,t) + w(x,t) := v(x,t) - (t - t_0)\,\chi\left(\frac{t - t_0}{s}\right) \operatorname{div} R(x, t_0).$$

The condition $\operatorname{div}\operatorname{div} R(\cdot, t_0) = 0$ ensures that $\widetilde{v}$ is divergence-free. By definition of χ, we see that w vanishes unless $|t - t_0| < s$ so we deduce $\|w\|_N \le C\, s\, \|R(\cdot, t_0)\|_{C^{N+1}}$. Next, we define

$$\begin{aligned}\widetilde{R}(x,t) :=& R(x,t) - \chi\left(\frac{t-t_0}{s}\right) R(x,t_0) - \frac{t-t_0}{s}\,\chi'\left(\frac{t-t_0}{s}\right) R(x,t_0)\\ &+ w \otimes v + v \otimes w + w \otimes w.\end{aligned}$$

It is easy to see that $(\widetilde{v}, p, \mathring{R})$ is a subsolution. Furthermore, it follows from our choice of χ and the fact that w vanishes at $t = t_0$ that $\mathring{R}(\cdot, t_0)$ is identically 0. On the other hand, the bound for $\|w\|_0$ yields the claimed estimate for $\widetilde{R}$.

On the other hand, since the support of χ is contained in $(-1,1)$ and the support of $R(\cdot, t_0)$ is contained in Ω, we see that the support of $\widetilde{v} - v$ and $\widetilde{R} - R$ is contained in $\Omega \times (t_0 - s, t_0 + s)$.

Finally, we absorb the trace of $\mathring{R}$ into the pressure, which preserves the other properties that we have discussed. □

Remark 9.2. The condition $\operatorname{div}\operatorname{div} R(\cdot, t_0) = 0$ is quite restrictive, but it can be removed if one is willing to relinquish spatial control of the velocity field. Indeed, we may decompose $\operatorname{div} R(\cdot, t_0)$ as the sum of a divergence-free field and a gradient, which we absorb into the pressure. The divergence-free part is canceled using the previous lemma. The issue is that the divergence-free component of $\operatorname{div} R(\cdot, t_0)$ does not have compact support, in general. Thus, we modify the subsolution outside $\operatorname{supp} R$ and we loose the spatial control, which has been our main concern so far. This approach could yield interesting applications in $\mathbb{T}^3$, nevertheless. However, in $\mathbb{R}^3$ we would have to modify the construction to address the fact that we have to add perturbations in the whole space. We do not pursue this path here.

Since our construction relies on keeping the velocity fixed at the endpoints of the intervals $\mathcal{I}_k$, we cannot expect to obtain a nonincreasing energy profile for the final weak solution. Indeed, by weak-strong

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uniqueness it should equal the smooth solution with that initial data (in its domain of definition) and that is not what will will obtain with the convex integration scheme.

Thus, instead of trying to fix the energy with this construction, we will focus on keeping the changes small after each iteration. Our goal is to ensure that the energy profile can be extended to a continuous function in $[0,T]$. In that case, we may use Theorem 1.7 to add a (nonsingular) perturbation elsewhere so that the total energy achieves the desired profile.

Hence, the main result that we will prove in this section, which is a nontrivial variation of Theorem 1.7, is:

Theorem 9.3. *Let $0<\beta<1/3$. Let $T>0$ and let $\{\mathcal{I}_k\}_{k=0}^{\infty}$ be a sequence of closed intervals meeting only at their endpoints and such that $(0,T)=\bigcup_{k\geq 0}\mathcal{I}_k$. Let $\Omega_k \Subset (0,1)^3$ be a bounded domain with smooth boundary for $k\geq 0$. Let $(v_0,p_0,\mathring{R}_0)\in C^\infty(\mathbb{R}^3\times(0,T))$ be a subsolution of the Euler equations such that* $\operatorname{supp}\mathring{R}_0(\cdot,t)\subset\overline{\Omega}_k$ *for $t\in\mathcal{I}_k$. In addition, assume that* $\operatorname{div}\operatorname{div}\mathring{R}_0$ *vanishes. Then, there exists a weak solution of the Euler equations $v\in C^{\beta}_{loc}(\mathbb{R}^3\times(0,T))$ that equals v_0 in $(\mathbb{R}^3\backslash\Omega_k)\times\mathcal{I}_k$ for any $k\geq 0$. In addition, $v=v_0$ at the endpoints of the intervals $\mathcal{I}_k$. Furthermore,*

$$\|(v-v_0)(\cdot,t)\|_{C^0}\leq C\sup_{t\in\mathcal{I}_k}\|\mathring{R}(\cdot,t)\|_{C^0}^{1/2}\qquad\forall k\geq 0$$

for some universal constant C.

9.2. The iterative process

As in Proposition 3.2, we are given an initial subsolution $(v_0,p_0,\mathring{R}_0)\in C^\infty(\mathbb{R}^3\times[0,T])$ and we will iteratively construct a sequence of subsolutions $\{(v_q,p_q,\mathring{R}_q)\}_{q=0}^{\infty}$ whose limit will be the desired weak solution. To construct the subsolution at step q from the one in step $q-1$, we will add an oscillatory perturbation with frequency λ_q. Meanwhile, the size of the Reynolds stress will be measured by an amplitude δ_q. These parameters are given by

$$\lambda_q=2\pi\lceil a^{b^q}\rceil,\tag{9.1}$$

$$\delta_q=\lambda_q^{-2\beta},\tag{9.2}$$

The parameters $a,b>1$ are very large and very close to 1, respectively. They will be chosen depending on the exponent $0<\beta<1/3$ that appears in Theorem 9.3, on Ω and on the initial subsolution. We introduce another parameter $\alpha>0$ that will be very small. The necessary size of all the parameters will be discovered in the proof.

We will assume that the support of $(v_0,p_0,\mathring{R}_0)(\cdot,t)$ is contained in $(0,1)^3$. Meanwhile, the support of $\mathring{R}$ is contained in $\overline{\Omega}\times[0,T]$ for a potentially smaller domain Ω with smooth boundary. The main difference with respect to Proposition 3.2 is that we also assume that $\mathring{R}_0(\cdot,t)$ vanishes for $t=0$ and $t=T$.

It will be convenient to do an additional rescaling in our problem. In the rescaled problem the initial subsolution will depend on a, but we assume that nevertheless there exists a sequence $\{y_N\}_{N=0}^{\infty}$ independent of the parameters such that

$$\|v_0\|_N+\|\partial_t v_0\|_N\leq y_N,\tag{9.3}$$

$$\|p_0\|_N\leq y_N,\tag{9.4}$$

$$\|\mathring{R}_0\|_N+\|\partial_t\mathring{R}_0\|_N\leq y_N.\tag{9.5}$$

Since the initial Reynolds stress $\mathring{R}_0$ and its derivatives vanish at $\partial\Omega\times[0,T]$, for any $k\in\mathbb{N}$ there exist a constant C_k such that for any $x\in\Omega$ we have

$$|\mathring{R}_0(x,t)|\leq C_k\operatorname{dist}(x,\partial\Omega)^k.$$

Note that the constants C_k are independent of a by (9.5). We define

$$d_q := \left(\frac{\delta_{q+2}\lambda_{q+1}^{-6\alpha}}{4C_{10}}\right)^{1/10}, \tag{9.6}$$

$$A_q := \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) \geq d_q\}. \tag{9.7}$$

Hence, we have

$$|\mathring{R}_0(x,t)| \leq \frac{1}{4}\delta_{q+2}\lambda_{q+1}^{-6\alpha} \qquad \text{if } x \notin A_q.$$

On the other hand, since $\mathring{R}_0(\cdot, t)$ vanishes for $t = 0$ and $t = T$ there exists a constant C_t depending on $\|\partial_t \mathring{R}_0\|_0$ such that

$$|\mathring{R}_0(x,t)| \leq C_t \min\{t, T-t\}.$$

Again, the constant C_t does not depend on a because of (9.5). It depends only on the initial subsolution (before the rescaling). We define

$$s_q := \frac{\delta_{q+2}\lambda_{q+1}^{-6\alpha}}{4C_t}, \tag{9.8}$$

$$\mathcal{J}_q := [s_q, T - s_q] \tag{9.9}$$

so that

$$|\mathring{R}_0(x,t)| \leq \frac{1}{4}\delta_{q+2}\lambda_{q+1}^{-6\alpha} \qquad \text{if } t \in [0,T]\backslash\mathcal{J}_q. \tag{9.10}$$

At step q the perturbation will be localized in $A_q \times \mathcal{J}_q$ so that $(v_q, p_q, \mathring{R}_q)$ equals the initial subsolution in $(\mathbb{R}^3 \backslash A_q) \times \big([0,T]\backslash\mathcal{J}_q\big)$. In this region the Reynolds stress is so small that we will ignore it. We will focus on reducing the error in $A_q \times \mathcal{J}_q$.

The complete list of inductive estimates is the following:

$$(v_q, p_q, \mathring{R}_q) = (v_0, p_0, \mathring{R}_0) \qquad \text{outside } A_q \times \mathcal{J}_q, \tag{9.11}$$

$$\|\mathring{R}_q\|_0 \leq \delta_{q+1}\lambda_q^{-6\alpha}, \tag{9.12}$$

$$\left\|v_q\right\|_1 \leq M\delta_q^{1/2}\lambda_q, \tag{9.13}$$

$$\left\|v_q\right\|_0 \leq 1 - \delta_q^{1/2}, \tag{9.14}$$

where M is a geometric constant that depends on Ω and is fixed throughout the iterative process. The following instrumental result is key to the proof of Theorem 9.3, and is analogous to Proposition 3.2:

Proposition 9.4. *There exists a universal constant M with the following property: Let $T \geq 1$ and let $\Omega \subset (0,1)^3 \subset \mathbb{R}^3$ be a bounded domain with smooth boundary. Let $(v_0, p_0, \mathring{R}_0) \in C^\infty(\mathbb{R}^3 \times [0,T])$ be a subsolution whose support is contained in $(0,1)^3 \times [0,T]$ and such that* $\operatorname{supp} \mathring{R}_0 \subset \overline{\Omega} \times [0,T]$.

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Furthermore, assume that (9.3)–(9.5) are satisfied for some sequence of positive numbers $\{y_N\}_{N=0}^{\infty}$. *Let* $0 < \beta < 1/3$ *and*

$$1 < b^2 < \min\left\{\frac{1-\beta}{2\beta}, \frac{11}{10}\right\}. \tag{9.15}$$

Then, there exists an α_0 *depending on* β *and* b *such that for any* $0 < \alpha < \alpha_0$ *there exists an* a_0 *depending on* β, b, α, Ω *and* $\{y_N\}_{N=0}^{\infty}$ *such that for any* $a \geq a_0$ *the following holds: Given a subsolution* $(v_q, p_q, \mathring{R}_q)$ *satisfying (9.11)–(9.14), there exists a subsolution* $(v_{q+1}, p_{q+1}, \mathring{R}_{q+1})$ *satisfying (9.11)–(9.14) with* q *replaced by* $q+1$. *Furthermore, we have*

$$\left\|v_{q+1} - v_q\right\|_0 + \frac{1}{\lambda_{q+1}}\left\|v_{q+1} - v_q\right\|_1 \leq M\delta_{q+1}^{1/2}. \tag{9.16}$$

As in Theorem 1.7, we need an auxiliary lemma to start the iterative process. This is the analogue of Lemma 3.3:

Lemma 9.5. *Let* $T > 0$ *and let* $\Omega \subset (0,1)^3 \subset \mathbb{R}^3$ *be a bounded domain with smooth boundary. Let* $(v_0, p_0, \mathring{R}_0) \in C^\infty(\mathbb{R}^3 \times [0,T])$ *be a subsolution whose support is contained in* $(0,1)^3 \times [0,T]$ *and such that* $\operatorname{supp} \mathring{R}_0 \subset \overline{\Omega} \times [0,T]$. *Let* $\lambda > 0$ *be sufficiently large. There exists a subsolution* $(v, p, \mathring{R}) \in C^\infty(\mathbb{R}^3 \times [0,T])$ *that equals the initial subsolution outside the set*

$$\left\{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \lambda^{-1/12}\right\} \times (\lambda^{-1/3}, T - \lambda^{-1/3})$$

and such that

$$\begin{aligned} \|v\|_N &\lesssim \lambda^N \qquad \forall N \geq 0, \\ \|\mathring{R}\|_0 &\leq \lambda^{-1/2}, \end{aligned}$$

where the implicit constants are independent of λ. *Furthermore, there exist geometric constants* K_1, K_2 *such that if* $K_1 T \|v_0\|_1 \leq 1$, *then*

$$\|v - v_0\|_0 \leq K_2 \|\mathring{R}_0\|_0^{1/2}.$$

9.3. Proof of Proposition 9.4

The proof is very similar to the proof of Proposition 3.2, but we will only perturb the subsolution at times $t \in \mathcal{J}_{q+1}$. It will be convenient to define

$$\begin{aligned} \gamma &:= \frac{1}{2}(s_q - s_{q+1}), \\ \widetilde{\mathcal{J}}_q &:= [s_q - \gamma, T - s_q + \gamma]. \end{aligned}$$

The parameters ℓ and τ_q are defined as in Proposition 3.2. Let us compare the time parameters:

$$\frac{s_q}{\tau_q} \gtrsim \delta_{q+2}\lambda_{q+1}^{-6\alpha}\delta_q^{1/2}\lambda_q \ell^{-2\alpha} \gtrsim (\lambda_{q+1}^3 \ell)^{-2\alpha}\lambda_q^{1-\beta-2b^2\beta}.$$

Note that the exponent of λ_q is greater than 0 by our assumption on b. Since we may assume that $2d_{q+1} \leq d_q$, we conclude that for sufficiently small α and sufficiently large a we have

$$\gamma \gg \tau_q.$$

Hence, the temporal cutoffs that we will need to use will not be too sharp.

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After these definitions, we can prove the result in a similar manner to Proposition 3.2. As before, we divide the proof in four steps:

1. Preparing the subsolution. The beginning of the iterative process is identical to the one in Proposition 3.2: we use a convolution kernel in space ψ_ℓ to mollify $(v_q, p_q, \mathring{R}_q)$ into $(v_\ell, p_\ell, \mathring{R}_\ell)$ and then we glue it in space to $(v_0, p_0, \mathring{R}_0)$, obtaining a subsolution $(\widetilde{v}_\ell, \widetilde{p}_\ell, \widetilde{\mathring{R}}_\ell)$ that equals $(v_0, p_0, \mathring{R}_0)$ in B_2.
   We must, however, introduce a minor modification: we add a correction w_L to ensure that $\widetilde{v}_\ell + w_L$ has the same angular momentum as v_0. Note that v_q has the same angular momentum as v_0 because they are equal at $t = 0$ and subsolutions preserve angular momentum, as argued several times. In addition, it is easy to check that mollifying does not change the total angular momentum, so $v_\ell - v_0$ has 0 total angular momentum. This may not be true for $\widetilde{v}_\ell - v_0$. Nevertheless, since we are gluing in a (small) region where v_ℓ is very similar to v_0 (in terms of the parameters), the change in the angular momentum will be very small. Therefore, we may add a small correction w_L to $\widetilde{v}_\ell$ in A_q while keeping the desired estimates. Of course, we modify the pressure and the Reynolds stress accordingly to obtain a subsolution, which we still denote as $(\widetilde{v}_\ell, \widetilde{p}_\ell, \widetilde{\mathring{R}}_\ell)$, for simplicity.
2. Gluing in space. In the intervals such that $[t_i - \tau_q, t_i + \tau_q] \cap \mathcal{J}_q \neq \varnothing$ the process is exactly the same as in Proposition 3.2. In the rest of the intervals it suffices to take $(\widetilde{v}_i, \widetilde{p}_i, \widetilde{\mathring{R}}_i) = (v_0, p_0, \mathring{R}_0)$ because outside $\mathcal{J}_q$ we have $|\mathring{R}|_0 \leq \frac{1}{4}\delta_{q+2}\lambda_{q+1}^{-6\alpha}$, as required by (4.19).
   Regarding the estimates (4.20)–(4.23), remember that $(v_q, p_q, \mathring{R}_q)$ equals the initial subsolution for $t \notin \mathcal{J}_q$. Therefore, it follows from (9.3)–(9.5) and standard estimates for mollifiers that for $t \notin \mathcal{J}_q$ we have
$$\begin{aligned} \|(v_\ell - v_0)(\cdot,t)\|_{C^N} + \|(\partial_t v_\ell - \partial_t v_0)(\cdot,t)\|_{C^N} &\lesssim \ell^2, \\ \|(p_\ell - p_0)(\cdot,t)\|_{C^N} &\lesssim \ell^2, \\ \|(\mathring{R}_\ell - \mathring{R}_0)(\cdot,t)\|_{C^N} &\lesssim \ell^2. \end{aligned}$$
   The same bounds hold for $(\widetilde{v}_\ell, \widetilde{p}_\ell, \widetilde{\mathring{R}}_\ell)$. Therefore, this choice of $(\widetilde{v}_i, \widetilde{p}_i, \widetilde{\mathring{R}}_i)$ for the rest of the intervals satisfies (4.20)–(4.23). Although (4.18) is not satisfied, its only use in the following step is to ensure that $\widetilde{v}_i$ and $\widetilde{v}_\ell$ have the same angular momentum. This is exactly what we did at the end of the previous step.
3. Gluing in time. The construction is the same as in Proposition 3.2 but due to our choice of $(\widetilde{v}_i, \widetilde{p}_i, \widetilde{\mathring{R}}_i)$ we are actually gluing only in $\widetilde{\mathcal{J}}_q$ (remember that $\tau_q \ll \gamma$). We see that $(\overline{v}_q, \overline{v}_q, \overline{\mathring{R}}_q)$ equals $(v_0, p_0, \mathring{R}_0)$ for $t \notin \widetilde{\mathcal{J}}_q$ and the support of $\overline{\mathring{R}}_q^{(2)}$ is contained in $[A_q + B(0, 4\sigma)] \times \widetilde{\mathcal{J}}_q$.
4. Perturbation. There are only two changes with respect to Proposition 3.2:
   - We define the amplitudes $\rho_{q,i}$ as $\rho_{q,i}(x,t) := \eta_i(x,t)^2\delta_{q+1}$.
   - Instead of the cutoff ϕ_q we use $\widetilde{\phi}_q(x,t) := \phi_q(x)\theta_q(t)$ for some smooth cutoff $\theta_q \in C_c^\infty(I_{q+1})$ that equals 1 in $\widetilde{\mathcal{J}}_q$. Hence, $\widetilde{\phi}_q = 1$ in the support of $\overline{\mathring{R}}_q^{(2)}$.

   The amplitudes $\rho_{q,i}(x,t)$ clearly satisfy the same estimates as in Proposition 3.2. In particular, $\mathring{R}_{q,i}$ takes values in $B(\mathrm{Id}, 1/2)$. On the other hand, $\widetilde{\phi}_q a(\mathring{R}_{q,i})$ will satisfy the same estimates as $a(\mathring{R}_{q,i})$ because we may choose θ_q so that $|\partial_t\theta_q| \lesssim \gamma^{-1} \leq \tau_q^{-1}$.
   We conclude that we may carry out the same construction as in the perturbation step of Proposition 3.2 (except for fixing the energy). The new subsolution will satisfy (9.12)–(9.16). In addition, the cutoff $\widetilde{\phi}_q$ ensures that the support of the perturbation is contained in $A_{q+1} \times \mathcal{J}_{q+1}$, as required by the inductive hypothesis (9.11).
   Finally, we emphasize an important difference: the constant M in this case is universal. In Proposition 3.2 it depended on Ω because so did the amplitudes $\rho_{q,i}$. Since here they are independent of Ω, arguing as in [7] yields a universal M.

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9.4. Proof of Lemma 9.5

We fix a cutoff $\theta \in C_c^\infty((0,T))$ such that

$$\theta(t) = \begin{cases} 1 & \text{if } t \in (2\lambda^{-1/3}, T - 2\lambda^{-1/3}), \\ 0 & \text{if } t \notin (\lambda^{-1/3}, T - \lambda^{-1/3}). \end{cases}$$

Since $\mathring{R}_0$ vanishes at $t = 0$ and $t = T$, it is clear that $|(1-\theta)\mathring{R}_0| \lesssim \lambda^{-1/2}$.

We then carry out the same construction as in Lemma 3.3 but replacing the cutoff $\phi(x)$ by $\widetilde{\phi}(x,t) := \phi(x)\theta(t)$. This ensures that the perturbation vanishes if $t \notin (\lambda^{-1/3}, T - \lambda^{-1/3})$. It does worsen the estimates, but it is not catastrophic. The most significant change is that when we write the term in brackets in (7.28) as $\sum_{k\neq 0} c_k e^{i\lambda k\cdot\Phi}$, the vectors c_k now satisfy the estimates

$$\|c_k\|_N \lesssim |k|^{-6}\lambda^{1/3+N/12}.$$

Applying the stationary phase lemma now yields

$$\|\mathscr{R}f\|_{1/15} \lesssim \frac{\lambda^{1/3+(1+1/15)/12}}{\lambda^{1-1/15}} < \lambda^{-1/2},$$

which, continuing the construction in Lemma 3.3, yields the desired bound for $\mathring{R}$. Since the exponent is actually smaller than $-1/2$, we can expend the extra factor in beating any geometric constant.

Finally, let us derive a precise estimate for $v - v_0$. Recall that

$$w_0 = 2\|\mathring{R}_0\|_0\, \widetilde{\phi}^2 (\nabla\Phi)^{-1} W(\widetilde{R}, \lambda\Phi).$$

The last term can be bounded by a geometric constant depending on the compact subset $\mathcal{N}$ used when applying Lemma 7.2. On the other hand, by standard estimates for the transport equation there exists a geometric constant K_1 such that

$$\|\nabla\Phi - \mathrm{Id}\|_0 \le \frac{1}{2} K_1 T\, \|v_0\|_1 .$$

By our assumption on $T\,\|v_0\|_1$, we have $\|\nabla\Phi - \mathrm{Id}\|_0 \le 1/2$, so $\|(\nabla\Phi)^{-1}\|_0 \le 2$. We conclude that

$$\|w_0\|_0 \le \frac{1}{2} K_2 \|\mathring{R}_0\|_0$$

for some geometric constant K_2. Since $w_c + w_L = O(\lambda^{-1})$, it is clear that the required estimate holds for sufficiently large λ.

9.5. Proof of Theorem 9.3

To simplify the notation of this proof, given a map f defined in $\mathbb{R}^3 \times I$ for some interval I, we will write

$$\|f\|_{N,I} := \mathrm{supp}_{t\in I}\, \|f(\cdot,t)\|_{C^N} .$$

By dividing in half the intervals $\mathcal{I}_k$ as many times as necessary, we may assume that

$$2K_1\, |\mathcal{I}_k|\, \|v_0\|_{1,\mathcal{I}_k} \le 1,$$

where $|\mathcal{I}_k|$ is the length of the interval $\mathcal{I}_k$ and K_1 is the constant that appears in Lemma 9.5.

We apply Lemma 9.1 at the endpoints of each interval $\mathcal{I}_k$, obtaining a new subsolution $(\widetilde{v}_0, \widetilde{p}_0, \mathring{\widetilde{R}}_0)$ in which the Reynolds stress vanishes at the endpoints of all the intervals $\mathcal{I}_k$. In addition, we have

$(\widetilde{v}_0, \widetilde{p}_0, \widetilde{\mathring{R}}_0) = (v_0, p_0, \mathring{R}_0)$ outside Ω_k for $t \in \mathcal{I}_k$. By taking s sufficiently small in each application of Lemma 9.1, we may assume that

$$\begin{aligned}\|\widetilde{v}_0 - v_0\|_{0,\mathcal{I}_k} &\leq \|\mathring{R}_0\|_{0,\mathcal{I}_k},\\ \|\widetilde{v}_0\|_{1,\mathcal{I}_k} &\leq 2\,\|v_0\|_{1,\mathcal{I}_k}\,,\\ \|\widetilde{\mathring{R}}_0\|_{0,\mathcal{I}_k} &\leq 2\|\mathring{R}_0\|_{0,\mathcal{I}_k}.\end{aligned}$$

Once we have a subsolution in which the Reynolds stress vanishes at the endpoints of the intervals $\mathcal{I}_k$, we may work in each of them independently. Indeed, our constructions keep the subsolution fixed near the endpoints of the intervals.

We fix $k \geq 0$, and we fix a sequence of positive numbers $\{y_{N,k}\}_{N=0}^{\infty}$ such that

$$\|\widetilde{v}_0\|_{N,\mathcal{I}_k} + \|\partial_t \widetilde{v}_0\|_{N,\mathcal{I}_k} \leq y_N, \tag{9.17}$$

$$\|\widetilde{p}_0\|_{N,\mathcal{I}_k} \leq y_N, \tag{9.18}$$

$$\|\widetilde{\mathring{R}}_0\|_{N,\mathcal{I}_k} + \|\partial_t \widetilde{\mathring{R}}_0\|_{N,\mathcal{I}_k} \leq y_N. \tag{9.19}$$

We choose b satisfying (9.15) and we choose α smaller than the threshold given by Proposition 9.4. Let $a_{0,k}$ be the threshold given by Proposition 9.4 when applied to Ω_k and our sequence $\{y_{N,k}\}_{N=0}^{\infty}$. For $a_k \geq a_{0,k}$ we consider the parameters $\lambda_{q,k}$ and $\delta_{q,k}$ defined as in (9.1) and (9.2).

Taking a_k larger if necessary, we apply Lemma 9.5 with the parameter $\lambda_{1,k}^{12\alpha}$, obtaining a subsolution $(v_1, p_1, \mathring{R}_1)$ that equals $(\widetilde{v}_0, \widetilde{p}_0, \widetilde{\mathring{R}}_0)$ outside

$$\left\{(x,t) \in \Omega_k \times \mathcal{I}_k : \operatorname{dist}((x,t), \partial(\Omega_k \times \mathcal{I}_k)) > \lambda_{1,k}^{-4\alpha}\right\}$$

and satisfying the following estimates:

$$\|v_1\|_{N,\mathcal{I}_k} \leq C_{N,k}\lambda_{1,k}^{12N\alpha}, \qquad \|\mathring{R}_1\|_{0,\mathcal{I}_k} \leq \lambda_{1,k}^{-6\alpha},$$

where the constants $C_{N,k}$ are independent of $\lambda_{1,k}$ but they will depend on Ω_k, $\mathcal{I}_k$ and the initial subsolution. Furthermore, we have

$$\|v_1 - \widetilde{v}_0\|_{0,\mathcal{I}_k} \leq K_2\|\widetilde{\mathring{R}}_0\|_{0,\mathcal{I}_k} \leq 2K_2\|\mathring{R}_0\|_{0,\mathcal{I}_k},$$

where we have used that $K_1\,|\mathcal{I}_k|\,\|\widetilde{v}_0\|_{1,\mathcal{I}_k} \leq 1$.

Next, we consider the scale invariance of the Euler equations and subsolutions:

$$v(x,t) \mapsto \Gamma v(x,\Gamma t), \quad p(x,t) \mapsto \Gamma^2 p(x,\Gamma t), \quad \mathring{R}(x,t) \mapsto \Gamma^2 \mathring{R}(x,\Gamma t).$$

We choose $\Gamma = \delta_{2,k}^{1/2}$ and we begin to work in this rescaled setting, which we will indicate with a superscript r. Note that $(\widetilde{v}_0^r, \widetilde{p}_0^r, \widetilde{\mathring{R}}_0^r)$ still satisfies (9.3)–(9.5) with the same sequence $\{y_{N,k}\}_{N=0}^{\infty}$. Regarding $(v_1^r, p_1^r, \mathring{R}_1^r)$, it follows from the definition of the rescaling that

$$\|\mathring{R}_1^r\|_{0,\mathcal{I}_k^r} \leq \delta_{2,k}\lambda_{1,k}^{-6\alpha}.$$

On the other hand, since the constants $C_{N,k}$ are independent of $\lambda_{1,k}$, for sufficiently large a_k we have

$$\begin{aligned}\left\|v_1^r\right\|_{0,\mathcal{I}_k^r} &= \delta_{2,k}^{1/2}\,\|v_1\|_{0,\mathcal{I}_k} \leq \delta_{2,k}^{1/2} C_{0,k} \leq 1 - \delta_{1,k}^{1/2},\\ \left\|v_1^r\right\|_{1,\mathcal{I}_k^r} &= \delta_{2,k}^{1/2}\,\|v_1\|_{1,\mathcal{I}_k} \leq \delta_{2,k}^{1/2} C_{1,k}\lambda_{1,k}^{12\alpha} \leq M\delta_{1,k}^{1/2}\lambda_{1,k}.\end{aligned}$$

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Finally, $(v_1^r, p_1^r, \mathring{R}_1^r) = (\widetilde{v}_0^r, \widetilde{p}_0^r, \mathring{\widetilde{R}}_0^r)$ outside

$$\left\{(x,t) \in \Omega_k \times \mathcal{I}_k^r : \operatorname{dist}((x,t), \partial(\Omega_k \times \mathcal{I}_k^r)) > \lambda_1^{-4\alpha}\right\}.$$

Let us consider the sets $A_{q,k}$ and $\mathcal{J}_{q,k}$ that we obtain when we apply the definition of (9.7) and (9.9) to Ω_k and $\mathcal{I}_k^r$. We see that $(v_1^r, p_1^r, \mathring{R}_1^r) = (\widetilde{v}_0^r, \widetilde{p}_0^r, \mathring{\widetilde{R}}_0^r)$ outside $A_{1,k} \times \mathcal{J}_{1,k}$ for sufficiently small α and sufficiently large a_k.

We conclude that $(v_1^r, p_1^r, \mathring{R}_1^r)$ satisfies the inductive hypotheses (9.11)–(9.14) in the interval $\mathcal{I}_k^r$ with initial subsolution $(\widetilde{v}_0^r, \widetilde{p}_0^r, \mathring{\widetilde{R}}_0^r)$. In addition, for $t \in \mathcal{I}_k^r$ the initial subsolution satisfies (9.3)–(9.5) with the sequence $\{y_{N,k}\}_{N=0}^{\infty}$ and the support of $\mathring{\widetilde{R}}_0^r(\cdot,t)$ is contained in $\overline{\Omega}_k$. Finally, by taking a_k even larger, we may assume that $|\mathcal{I}_k^r| \geq 1$.

Applying Proposition 9.4 in each interval $\mathcal{I}_k^r$ and undoing the scaling, we obtain a sequence of subsolutions $\{(v_q, p_q, \mathring{R}_q)\}_{q=1}^{\infty} \in C^\infty(\mathbb{R}^3 \times (0,T))$ that equal $(v_0, p_0, 0)$ in $(\mathbb{R}^3 \backslash \Omega_k) \times \mathcal{I}_k$ for any $k \geq 0$. In addition, for $t \in \mathcal{I}_k$ we have:

$$\|\mathring{R}_q(\cdot,t)\|_{C^0} \leq \delta_{q+1,k}, \tag{9.20}$$

$$\left\|(v_{q+1} - v_q)(\cdot,t)\right\|_{C^0} + \frac{1}{\lambda_{q+1,k}} \left\|(v_{q+1} - v_q)(\cdot,t)\right\|_{C^1} \leq M\delta_{1,k}^{-1/2}\delta_{q+1,k}^{1/2}. \tag{9.21}$$

We see that v_q converges uniformly in compact subsets of $\mathbb{R}^3 \times (0,T)$ to some continuous map v. On the other hand, note that the pressure is the only compactly supported solution of

$$\Delta p_q = \operatorname{div}\operatorname{div}(-v_q \otimes v_q + \mathring{R}_q).$$

Therefore, p_q also converges to some pressure $p \in L^s(\mathbb{R}^3)$ for any $1 \leq s < \infty$. Since $\mathring{R}_q$ converges to 0 uniformly in compact subsets of $\mathbb{R}^3 \times (0,T)$, we conclude that (v,p) is a weak solution of the Euler equations in $\mathbb{R}^3 \times (0,T)$.

Furthermore, using (9.21) we obtain

$$\begin{aligned}\sum_{q=1}^{\infty} \left\|v_{q+1} - v_q\right\|_{\beta',\mathcal{I}_k} &\leq \sum_{q=1}^{\infty} C(\beta',\beta) \left\|v_{q+1} - v_q\right\|_{0,\mathcal{I}_k}^{1-\beta'} \left\|v_{q+1} - v_q\right\|_{1,\mathcal{I}_k}^{\beta'} \\ &\leq C(\beta',\beta) \sum_{q=1}^{\infty} (M\delta_{1,k}^{-1/2}\delta_{q,k}^{1/2})^{1-\beta'} (M\delta_{1,k}^{-1/2}\delta_{q,k}^{1/2})^{\beta'} \\ &\leq M\, C(\beta',\beta)\delta_{1,k}^{-1/2} \sum_{q=1}^{\infty} \lambda_{q,k}^{\beta'-\beta},\end{aligned}$$

so $\{v_q\}_{q=1}^{\infty}$ is uniformly bounded in $C_t^0 C_x^{\beta'}$ in any compact subset $I \subset (0,T)$ for all $\beta' < \beta$. Arguing as in [7] we obtain (local) time regularity. We conclude that $v \in C^{\beta''}_{\text{loc}}(\mathbb{R}^3 \times (0,T))$, with $\beta'' < \beta' < 1/3$ arbitrary.

Finally, we compute the difference between v_0 and v. We write $b = 1+\gamma$, so that $b^q - 1 \geq \gamma q$. Taking a_k sufficiently large, we have

$$\begin{aligned}\|v - v_1\|_{0,\mathcal{I}_k} &\leq \sum_{q=1}^{\infty} \left\|v_{q+1} - v_q\right\|_{0,\mathcal{I}_k} \leq \sum_{q=1}^{\infty} M\delta_{1,k}^{-1/2}\delta_{q+1,k}^{1/2} \lesssim \sum_{q=1}^{\infty} a_k^{\beta b(1-b^q)} \lesssim \sum_{q=1}^{\infty} a_k^{-\beta b\gamma q} \\ &\lesssim a_k^{-\beta b\gamma} \sum_{q=0}^{\infty} (a_k^{\beta b\gamma})^{-q} \lesssim a_k^{-\beta b\gamma}.\end{aligned}$$

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Thus, by further increasing a_k, we have $\|v - v_1\|_{0,\mathcal{I}_k} \leq \|\mathring{R}_0\|_{0,\mathcal{I}_k}$. We conclude

$$\|v - v_0\|_{0,\mathcal{I}_k} \leq \|v - v_1\|_{0,\mathcal{I}_k} + \|v_1 - \widetilde{v}_0\|_{0,\mathcal{I}_k} + \|\widetilde{v}_0 - v_0\|_{0,\mathcal{I}_k} \leq (2 + 2K_2)\|\mathring{R}_0\|_{0,\mathcal{I}_k}.$$

10. Vortex sheet

This section is devoted to the proof of Theorem 1.4. The key ingredient is Theorem 9.3, which is applied to an initial subsolution with the appropriate behavior. We fix a parameter $0 < \lambda < (4T)^{-1}$. We choose an even function $f \in C_c^\infty((-1,1))$ such that $\int f = 2$ and we define

$$F(x) := \int_{-1}^{x} f(s)\,ds, \qquad G(x) := \int_{-1}^{x} sf(s)\,ds.$$

Let v_0 and R_0 be the periodic extension of

$$v_0(x,t) := \left[1 - F\left(\frac{x_3 - 1/4}{\lambda t}\right) + F\left(\frac{x_3 - 3/4}{\lambda t}\right)\right] e_1,$$
$$\mathring{R}_0(x,t) := \lambda\left[G\left(\frac{x_3 - 1/4}{\lambda t}\right) - G\left(\frac{x_3 - 3/4}{\lambda t}\right)\right](e_1 \otimes e_3 + e_3 \otimes e_1).$$

Direct computation shows that $\partial_t v_0 = \operatorname{div} \mathring{R}_0$. It is also clear that $v_0 \cdot \nabla v_0 = 0$. Therefore, the triplet $(v_0, 0, \mathring{R}_0)$ is a subsolution.

Since f is even, the support of G is contained in $(-1,1)$. For $k \geq 0$ let us define $\mathcal{I}_k := [2^{-(k+1)}T, 2^{-k}T]$ and

$$\Omega_k := \mathbb{T}^2 \times \left[\left(\frac{1}{4} - 2^{-k}\lambda T,\ \frac{1}{4} + 2^{-k}\lambda T\right) \cup \left(\frac{3}{4} - 2^{-k}\lambda T,\ \frac{3}{4} + 2^{-k}\lambda T\right) + \mathbb{Z}\right].$$

We see that $(v_0, 0, \mathring{R}_0)$ equals $(u_0, 0, 0)$ outside Ω_k for $t \in \mathcal{I}_k$. Since $\operatorname{div}\operatorname{div} \mathring{R}_0 = 0$, we may apply Theorem 9.3, obtaining a weak solution of the Euler equations $v \in C^\beta_{\mathrm{loc}}(\mathbb{T}^3 \times (0,T))$ that equals $(u_0, 0, 0)$ outside Ω_k for $t \in \mathcal{I}_k$ (in fact, for $t \in (0, 2^{-k}T)$ because $\Omega_j \subset \Omega_k$ for all $j \leq k$). In particular, the initial datum is u_0. Furthermore, there exists a universal constant C such that for any $k \geq 0$ we have

$$\|v - v_0\|_{0,\mathcal{I}_k} \leq C\|\mathring{R}\|_{0,\mathcal{I}_k} \leq C'\lambda. \tag{10.1}$$

Let us estimate the energy. First, we define

$$\delta := 2 - \int_{-1}^{1} [1 - F(s)]^2 ds.$$

Note that $\delta > 0$ because the continuous function F takes values in $[0,2]$ and it satisfies $F(0) = 1$, since f is even. We compute

$$\int_{\mathbb{T}^3} |v_0(x,t)|^2\,dx = 1 - 4\lambda t + \int_{1/4-\lambda t}^{1/4+\lambda t} \left[1 - F\left(\frac{x_3 - 1/4}{\lambda t}\right)\right]^3 dx_3$$
$$+ \int_{3/4-\lambda t}^{3/4+\lambda t} \left[-1 + F\left(\frac{x_3 - 3/4}{\lambda t}\right)\right]^3 dx_3 = 1 - 2\delta\lambda t,$$

where we have used that $v_0 = u_0$ except close to $x_3 = 1/4$ and $x_3 = 3/4$. Next, we write

$$\int_{\mathbb{T}^3} |v|^2 = \int_{\mathbb{T}^3} |v_0|^2 + \int_{\mathbb{T}^3} |v - v_0|^2 + 2\int_{\mathbb{T}^3} v_0 \cdot (v - v_0).$$

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Let us choose λ sufficiently small so that $\lambda < (32C')^{-1}\delta$, where C' is the constant in (10.1). Hence, for $t \in \mathcal{I}_k$ we have

$$\left|\int_{\mathbb{T}^3} |v|^2 - \int_{\mathbb{T}^3} |v_0|^2\right| \le [(C'\lambda)^2 + 2(C'\lambda)]\ |\Omega_k| < [\delta/8](4 \cdot 2^{-k}\lambda T) = 2^{-(k+1)}\delta\lambda T,$$

where we have used that $v = v_0$ outside Ω_k for $t \in \mathcal{I}_k$. By definition of $\mathcal{I}_k$, we see that $2^{-(k+1)}T \le t$ for any $t \in \mathcal{I}_k$. We conclude that

$$1 - 3\delta\lambda t < \int_{\mathbb{T}^3} |v(x,t)|^2\, dx < 1 - \delta\lambda t.$$

Therefore, the weak solution v is admissible. In addition, we see that we obtain a sequence of different solutions $\{v_i\}_{i=i_0}^{\infty}$ by repeating the construction with $\lambda_i = 4^{-i}$ for sufficiently large i_0.

11. Blowup

In this final section we prove Theorem 1.5 on the existence of Hölder continuous weak solutions of the 3d Euler equations that exhibit a singular set of maximal dimension. The proof makes use of some technical lemmas that are presented in Subsections 11.1 and 11.2.

11.1. Building blocks

The fundamental element in our construction is the following simple blowup, whose proof is postponed to Subsection 11.2:

Lemma 11.1. *Let $0 < \beta < 1/3$ and let $q > 2$. Let $a \in C^\infty(\mathbb{R}, \mathbb{R}^3)$ be a bounded map. Given $\varepsilon > 0$, there exists a weak solution of the Euler equations $(v_\varepsilon, p_\varepsilon)$ in $\mathbb{R}^3 \times \mathbb{R}$ such that:*

- *$(v_\varepsilon, p_\varepsilon) = (a, -\partial_t a \cdot x)$ outside $B(0,\varepsilon) \times (0,\varepsilon)$,*
- *the q-singular set of v_ε is $\mathcal{S}^q_{v_\varepsilon} = \{(0,\varepsilon)\}$,*
- *$v_\varepsilon \in C^\beta_{loc}(\mathbb{R}^3 \times \mathbb{R} \backslash \mathcal{S}^q_{v_\varepsilon})$,*
- *the relative energy $e_\varepsilon(t) := \|v_\varepsilon(\cdot,t) - a\|^2_{L^2(\mathbb{R}^3)}$ is continuous and so is the map $t \mapsto \int a \cdot (v_\varepsilon - a)dx$.*

Furthermore, there exists a constant $C > 0$ depending on $\|a\|_{L^\infty} < \infty$ but not on ε such that

$$\|v_\varepsilon(\cdot,t) - a(t)\|^2_{L^2(\mathbb{R}^3)} + \|p_\varepsilon(\cdot,t) + \partial_t a(t) \cdot x\|_{L^1(\mathbb{R}^3)} \le C\varepsilon^3 \qquad \forall t \in \mathbb{R}. \tag{11.1}$$

Once we known how to construct a single blowup, as stated in the previous lemma, we will use the following result to glue many of them together. Its proof is completely independent of Lemma 11.1.

Lemma 11.2. *Let $a \in C^\infty(\mathbb{R}, \mathbb{R}^3)$. Let $\{(v_i, p_i)\}_{i=0}^\infty$ be a sequence of weak solutions of the Euler equations in $\mathbb{R}^3 \times \mathbb{R}$. Suppose that the sets F_i given by the closure of*

$$\{(x,t) \in \mathbb{R}^4 : (v_i, p_i)(x,t) \ne (a(t), -\partial_t a(t) \cdot x)\}$$

are pairwise disjoint and that

$$\sum_{i=0}^{\infty} \left(\|v_i - a\|_{L^2(\mathbb{R}^4)} + \|p_i + \partial_t a \cdot x\|_{L^1(\mathbb{R}^4)}\right) < \infty.$$

Then, the pair (v, p) *given by*

$$v := a + \sum_{i=0}^{\infty} (v_i - a), \qquad p := -\partial_t a \cdot x + \sum_{i=1}^{\infty} (p_i + \partial_t a \cdot x)$$

is a weak solution of the Euler equations.

Proof. The hypotheses readily yield $v \in L^2_{\mathrm{loc}}(\mathbb{R}^4)$ and $p \in L^1_{\mathrm{loc}}(\mathbb{R}^4)$. Furthermore, it is easy to see that any partial sum

$$\widetilde{v}_k := a + \sum_{i=0}^{k} (v_i - a), \qquad \widetilde{p}_k := \partial_t a \cdot x + \sum_{i=0}^{k} (p_i - \partial_t a \cdot x)$$

is a subsolution. Indeed, fix $\chi_i \in C^\infty(\mathbb{R}^4)$ such that $\chi_i^{-1}(\{0\}) = F_i$ and consider $\theta_1 := \chi_1/(\chi_1 + \chi_2)$ and $\theta_2 := 1 - \theta_1$. They are well defined because $F_1 \cap F_2 = \emptyset$. We see that θ_1 vanishes in F_2 and θ_2 vanishes in θ_1, so for any $\phi \in C_c^\infty(\mathbb{R}^4, \mathbb{R}^3)$ we have

$$\begin{aligned}
\int_{\mathbb{R}^4} (\partial_t \phi \cdot \widetilde{v}_2 + \nabla \phi : (\widetilde{v}_2 \otimes \widetilde{v}_2 + \widetilde{p}_2 \,\mathrm{Id})) &= \int_{\mathbb{R}^4} (\partial_t(\theta_1 \phi) \cdot \widetilde{v}_2 + \nabla(\theta_1 \phi) : (\widetilde{v}_2 \otimes \widetilde{v}_2 + \widetilde{p}_2 \,\mathrm{Id})) \\
&\quad + \int_{\mathbb{R}^4} (\partial_t(\theta_2 \phi) \cdot \widetilde{v}_2 + \nabla(\theta_2 \phi) : (\widetilde{v}_2 \otimes \widetilde{v}_2 + \widetilde{p}_2 \,\mathrm{Id})) \\
&= \int_{\mathbb{R}^4} (\partial_t(\theta_1 \phi) \cdot v_1 + \nabla(\theta_1 \phi) : (v_1 \otimes v_1 + p_1 \,\mathrm{Id})) \\
&\quad + \int_{\mathbb{R}^4} (\partial_t(\theta_2 \phi) \cdot v_2 + \nabla(\theta_2 \phi) : (v_2 \otimes v_2 + p_2 \,\mathrm{Id})) \\
&= 0,
\end{aligned}$$

which follows from the definition of (v_i, p_i) being a weak solution of the Euler equations using the test-function $\theta_i \phi$. An analogous argument proves that $\widetilde{v}_2$ is weakly divergence-free. Therefore, $(\widetilde{v}_2, \widetilde{p}_2)$ is a weak solution of the Euler equations. Furthermore, iterating this argument we conclude that $(\widetilde{v}_k, \widetilde{p}_k)$ is a weak solution of the Euler equations for any $k \geq 1$. Since $\widetilde{v}_k$ converges to v in $L^2(K)$ and $\widetilde{p}_k$ converges to p in $L^1(K)$ for any compact subset $K \subset \mathbb{R}^4$, it is easy to see that (v, p) is a weak solution of the Euler equations. □

11.2. Proof of Lemma 11.1

Note it suffices to prove the result for $\varepsilon = 1$ because for $\varepsilon \neq 1$ we could simply take

$$(v_\varepsilon, p_\varepsilon)(x, t) := (v_1, p_1)(x/\varepsilon, t/\varepsilon)$$

with a rescaled accordingly. It is clear that this scaling preserves the q-singular set, that is,

$$\mathcal{S}^q_{v_\varepsilon} = \{(x, t) \in \mathbb{R}^3 \times \mathbb{R} : (x/\varepsilon, t/\varepsilon) \in \mathcal{S}^q_{v_1}\} = \{(0, \varepsilon)\}.$$

Furthermore, since $p_1 + \partial_t a \cdot x$ will be the only compactly supported solution of

$$\begin{aligned}
-\Delta(p_1 + \partial_t a \cdot x) &= \operatorname{div}\operatorname{div}(v_1 \otimes v_1) = \operatorname{div}\operatorname{div}(v_1 \otimes v_1 - a \otimes a) \\
&= \operatorname{div}\operatorname{div}[(v_1 - a) \otimes (v_1 - a) + (v_1 - a) \otimes a + a \otimes (v_1 - a)],
\end{aligned}$$

standard Calderón-Zygmund estimates yield

$$\|p_1(\cdot,t)+\partial_t a(t)\cdot x\|_{L^1(\mathbb{R}^3)} \le C\,\|v_1(\cdot,t)-a(t)\|^2_{L^2(\mathbb{R}^3)}\,.$$

Thus, to prove (11.1) it suffices to show that $(v_1-a)\in L^\infty_t L^2_x$.

From now on, we will assume $\varepsilon=1$ and drop the subscript 1 for simplicity. By [32] there exists a nontrivial steady solution of the Euler equations with compact support $(u,\pi)\in C^\infty_c(\mathbb{R}^3)$. By rescaling, we may assume that its support is contained in the ball $B(0,1/4)$. We define

$$\begin{aligned} U(x) &:= u\,(x-e_1/4) - u\,(x+e_1/4)\,,\\ P(x) &:= \pi\,(x-e_1/4) - \pi\,(x+e_1/4)\,. \end{aligned}$$

Therefore, $(U,P)\in C^\infty_c(B(0,1))$ is a nontrivial steady solution of the Euler equations such that

$$\int \xi\cdot U = 0 \qquad \forall \xi\in\ker\nabla_{\mathrm{sym}}.$$

Hence, by Lemma 2.9 there exists $S_0\in C^\infty_c(B(0,1),\mathcal{S}^3)$ such that $\operatorname{div} S_0 = U$. We introduce a parameter $\alpha\in(-1,0)$ that will be fixed later and we define

$$S := (1+\alpha)(x\otimes U + U\otimes x) - (4+5\alpha)S_0.$$

Since $\operatorname{div}(x\otimes U+U\otimes x) = 4U + x\cdot\nabla U$ and U is divergence-free, we have

$$\operatorname{div}\operatorname{div} S = \operatorname{div}\left(-\alpha\,U + (1+\alpha)x\cdot\nabla U\right) = 0. \tag{11.2}$$

Furthermore, we see that the triplet (U,P,S) satisfies

$$-\alpha\,U + (1+\alpha)x\cdot\nabla U + \operatorname{div}(U\otimes U + P\,\mathrm{Id}) = \operatorname{div} S. \tag{11.3}$$

Consider the self-similar ansatz

$$\begin{aligned} \widetilde{v}_0(x,t) &:= (1-t)^{\alpha}\,U\left(\frac{x}{(1-t)^{1+\alpha}}\right),\\ \widetilde{p}_0(x,t) &:= (1-t)^{2\alpha}P\left(\frac{x}{(1-t)^{1+\alpha}}\right),\\ \widetilde{R}_0(x,t) &:= (1-t)^{2\alpha}S\left(\frac{x}{(1-t)^{1+\alpha}}\right). \end{aligned}$$

It follows from Equation (11.3) that the triplet $(\widetilde{v}_0,\widetilde{p}_0,\widetilde{R}_0)$ is a subsolution of the Euler equations in $\mathbb{R}^3\times[0,1)$ (in which the Reynolds stress is not normalized to be trace-free, yet). Regarding the scaling, we see that

$$\|\widetilde{v}_0(\cdot,t)\|_{L^r} = (1-t)^{\alpha+3(1+\alpha)/r}\,\|U\|_{L^r}\,.$$

We choose

$$\alpha := -\frac{3}{10} - \frac{3}{2(3+q)},$$

which ensures that

$$\lim_{t\to1}\|\widetilde{v}_0(\cdot,t)\|_{L^2} = 0, \qquad \lim_{t\to1}\|\widetilde{v}_0(\cdot,t)\|_{L^q} = +\infty.$$

Next, we fix $\chi \in C^\infty([0,1])$ that vanishes in a neighborhood of 0 and is identically 1 in a neighborhood of 1. Since $(\widetilde{v}_0, \widetilde{p}_0, \widetilde{R}_0)$ is a subsolution, is is easy to see that the triplet (v_0, p_0, R_0) given by

$$\begin{aligned} v_0(x,t) &:= a(t) + \chi(t)\widetilde{v}_0(x,t), \\ p_0(x,t) &:= -\partial_t a \cdot x + \chi(t)^2 \widetilde{p}_0(x,t), \\ R_0(x,t) &:= \chi(t)\widetilde{R}_0 + \chi'(t)(1-t)^{-1} S_0((1-t)^{-1/2}x) + \chi(t)(a \otimes \widetilde{v}_0 + \widetilde{v}_0 \otimes a)(x,t) \end{aligned}$$

is also a subsolution and that $\operatorname{div}\operatorname{div} R_0$ vanishes. Let $\mathcal{I}_k := [1-2^{-k}, 1-2^{-(k+1)}]$ and $\Omega_k := B(0, 2^{-(1+\alpha)k})$. We see that the support of $R_0(\cdot,t)$ is contained in Ω_k for $t \in \mathcal{I}_k$. In fact, so is the support of $(v_0, p_0, R_0)(\cdot,t)$. In addition, we see that

$$\|v_0 - a\|_{0,\mathcal{I}_k} \lesssim 2^{-\alpha(k+1)}, \qquad \|R_0\|_{0,\mathcal{I}_k} \lesssim 2^{-2\alpha(k+1)},$$

where the implicit constant depends on $\|a\|_{L^\infty}$ but not on k.

We would like to apply Theorem 9.3. While our current situation does not exactly meet the hypotheses of Theorem 9.3, this is not really an issue. In spite of the fact that our subsolution is not compactly supported, what matters is the support of the Reynolds stress, as argued in Subsection 3.3. Furthermore, although R_0 is not trace-free, this can be easily solved in the proof of Theorem 9.3. After applying Lemma 9.1, one simply has to replace $(\widetilde{v}_0, \widetilde{p}_0, \widetilde{R}_0)$ by

$$\left(\widetilde{v}_0,\ \widetilde{p}_0 - \frac{1}{3}\operatorname{tr}(\widetilde{R}_0),\ \widetilde{R}_0 - \frac{1}{3}\operatorname{tr}(\widetilde{R}_0)\operatorname{Id}\right).$$

Therefore, there exists a weak solution (v,p) of the Euler equations in $\mathbb{R}^3 \times (0,1)$ with $v \in C^\beta_{\mathrm{loc}}(\mathbb{R}^3 \times (0,1))$ and $(v,p) = (a,0)$ in $(\mathbb{R}^3 \backslash \Omega_k) \times \mathcal{I}_k$ for $k \geq 0$. Furthermore, since $(v_0, p_0, R_0) = (a,0,0)$ for t sufficiently close to 0, a careful revision of Theorem 9.3 will convince us that $(v,p) = (a,0)$ in a neighborhood of $t=0$. Thus, we may extend (v,p) to the interval $(-\infty, 1)$.

On the other hand, for $t \in \mathcal{I}_k$ we have

$$\|v-a\|_{0,\mathcal{I}_k} \leq \|v_0 - a\|_{0,\mathcal{I}_k} + \|v - v_0\|_{0,\mathcal{I}_k} \leq \|v_0 - a\|_{0,\mathcal{I}_k} + C\,\|R_0\|_{0,\mathcal{I}_k}^{1/2} \leq C\,2^{-\alpha k},$$

where the constant changes after each inequality and it is allowed to depend on $\|a\|_{L^\infty}$ but not on k. Hence,

$$\|v-a\|_{0,\mathcal{I}_k}\,|\Omega_k| \leq C\,2^{-(3+4\alpha)k}, \qquad \|v-a\|_{0,\mathcal{I}_k}^2\,|\Omega_k| \leq C\,2^{-(3+5\alpha)k}. \tag{11.4}$$

Our choice of α ensures that both quantities go to 0 when $k \to \infty$. We conclude that the maps $t \mapsto \|v(\cdot,t) - a\|_{L^2(\mathbb{R}^3)}^2$ and $t \mapsto \int a \cdot (v_\varepsilon - a)dx$ can be extended by 0 to a continuous function in the whole $\mathbb{R}$.

Next, we show that we may extend our weak solution to $\mathbb{R}^3 \times \mathbb{R}$ by setting $(v,p) = (a,0)$ for $t \geq 1$. For simplicity, we still denote this extension as (v,p). It will be a weak solution in $\mathbb{R}^3 \times \mathbb{R}$ if and only if for any solenoidal test-function $\phi \in C_c^\infty(\mathbb{R}^3 \times \mathbb{R})$ we have

$$\int_{\mathbb{R}} \int_{\mathbb{R}^3} [\partial_t \phi \cdot v + \nabla \phi : (v \otimes v)]\,dxdt = 0.$$

We split the integral:

$$\begin{aligned} \int_1^\infty \int_{\mathbb{R}^3} [\partial_t \phi \cdot v + \nabla \phi : (v \otimes v)]\,dxdt &= \int_1^\infty \int_{\mathbb{R}^3} [\partial_t \phi \cdot a + \nabla \phi : (a \otimes a)]\,dxdt \\ &= \int_1^\infty \int_{\mathbb{R}^3} [\partial_t \phi(x,t) \cdot a(t)]\,dxdt = 0, \end{aligned}$$

where we have used that, for a fixed time t, $a(t)$ is just a constant vector and $\partial_t\phi(\cdot,t)$ is a compactly supported divergence-free. Thus, the spatial integral vanishes for each time t. We conclude that (v,p) will be a weak solution in $\mathbb{R}^3\times\mathbb{R}$ if and only if

$$\int_{-\infty}^{1}\int_{\mathbb{R}^3}[\partial_t\phi\cdot v+\nabla\phi:(v\otimes v)]dxdt=0. \tag{11.5}$$

We choose a cutoff function $\theta\in C^\infty(\mathbb{R})$ that equals 0 if $t\le 0$ and equals 1 if $t\ge 1/2$. For $j\ge 1$ consider $\theta_j(t):=\theta(1+2^j(t-1))$. Since $(1-\theta_j)\phi\in C_c^\infty(\mathbb{R}^3\times(-\infty,1))$ is solenoidal and (v,p) is a weak solution in $\mathbb{R}^3\times(-\infty,1)$, we see that

$$\int_{-\infty}^{1}\int_{\mathbb{R}^3}[\partial_t\phi\cdot v+\nabla\phi:(v\otimes v)]dxdt=\int_{1-2^{-j}}^{1}\int_{\mathbb{R}^3}[\partial_t(\theta_j\phi)\cdot v+\nabla(\theta_j\phi):(v\otimes v)]dxdt. \tag{11.6}$$

Let us study the second term on the right-hand side. We fix $t\in\mathcal{I}_k$ and we write

$$\int_{\mathbb{R}^3}\nabla(\theta_j\phi):(v\otimes v)\,dx=\int_{\mathbb{R}^3}\nabla(\theta_j\phi):(v\otimes v-a\otimes a)\,dx.$$

Taking into account that

$$v\otimes v-a\otimes a=(v-a)\otimes(v-a)+a\otimes(v-a)+(v-a)\otimes a$$

and (11.4), we surmise that for $t\in\mathcal{I}_k$

$$\left|\int_{\mathbb{R}^3}\nabla(\theta_j\phi):(v\otimes v)\,dx\right|\le C\,\|\phi\|_1\,2^{-(3+5\alpha)k}.$$

Therefore,

$$\left|\int_{1-2^{-j}}^{1}\int_{\mathbb{R}^3}\nabla(\theta_j\phi):(v\otimes v)\,dxdt\right|\le C\,\|\phi\|_1\sum_{k=j}^{\infty}2^{-(3+5\alpha)k}\,|\mathcal{I}_k|\le C\,\|\phi\|_1\,2^{-(4+5\alpha)j}$$

because $3+5\alpha>0$ by our choice of α. Regarding the first term on the right-hand side of (11.6), we split the integral into

$$\int_{1-2^{-j}}^{1}\int_{\mathbb{R}^3}\partial_t(\theta_j\phi)\cdot v\,dxdt=\int_{1-2^{-j}}^{1}\int_{\mathbb{R}^3}\partial_t(\theta_j\phi)\cdot a\,dxdt+\int_{1-2^{-j}}^{1}\int_{\mathbb{R}^3}\partial_t(\theta_j\phi)\cdot(v-a)\,dxdt.$$

Again, the first term vanishes because if we keep t fixed $a(t)$ is just a constant vector and $\partial_t(\theta_j\phi)(\cdot,t)$ is a compactly supported divergence-free field. Thus, the spatial integral vanishes for each time t. Concerning the second term, we estimate

$$\begin{aligned}\left|\int_{1-2^{-j}}^{1}\int_{\mathbb{R}^3}\partial_t(\theta_j\phi)\cdot(v-a)\,dxdt\right|&\le\sum_{k=j}^{\infty}\left\|\partial_t(\theta_j\phi)\right\|_0\|v-a\|_{0,\mathcal{I}_k}\,|\Omega_k|\,|\mathcal{I}_k|\\&\le C(\|\phi\|_0+\|\partial_t\phi\|_0)2^j\sum_{k=j}^{\infty}2^{-(3+4\alpha)k}2^{-k}\\&\le C(\|\phi\|_0+\|\partial_t\phi\|_0)2^{-(3+4\alpha)j}.\end{aligned}$$

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Hence,

$$\left|\int_{-\infty}^{1}\int_{\mathbb{R}^3}[\partial_t\phi\cdot v+\nabla\phi:(v\otimes v)]dxdt\right|\le C(\|\phi\|_1+\|\partial_t\phi\|_0)2^{-(3+4\alpha)j},$$

where the constant may depend on a but it is independent of j. Since $3+4\alpha>0$ and $j\ge 1$ is arbitrary, we conclude that (11.5) holds, so (v,p) is a weak solution in $\mathbb{R}^3\times\mathbb{R}$.

Regarding the q-singular set, fix a spatial ball $B(0,r)$ and let $t_k := 1-2^{-k}$ for $k\ge 1$. For sufficiently large k we have $\chi(t_k)=1$ and, since the velocity is unchanged at the endpoints of the intervals $\mathcal{I}_k$, we have

$$v(x,t_k)=a+2^{-\alpha k}U\left(2^{(1+\alpha)k}x\right).$$

We compute

$$\begin{aligned}\|v(\cdot,t_k)\|_{L^q(B(0,r))}&\ge\|v(\cdot,t_k)-a\|_{L^q(B(0,r))}-\|a\|_{L^q(B(0,r))}\\&\ge-\|a\|_{L^\infty}\left(\frac{4}{3}\pi r^3\right)^{1/q}+2^{-[\alpha+3(1+\alpha)/q]k}\|U\|_{L^q(\mathbb{R}^3)},\end{aligned}$$

where we have used that for sufficiently large k the ball $B(0,2^{(1+\alpha)k}r)$ contains the support of U. By our choice of α we have $\alpha+3(1+\alpha)/q<0$, so

$$\lim_{k\to\infty}\|v(\cdot,t_k)\|_{L^q(B(0,r))}=+\infty.$$

Since the ball $B(0,r)$ is arbitrary, we see that $(0,1)\in\mathcal{S}_v^q$.

Finally, since $v\in C^\beta_{\mathrm{loc}}(\mathbb{R}^3\times(-\infty,1))$ and $v=a$ in $(\mathbb{R}^3\backslash B(0,2^{-k}))\times(1-2^{-k},1)$ for any $k\ge 0$, we conclude that $v\in C^\beta_{\mathrm{loc}}(\mathbb{R}^3\times\mathbb{R}\backslash\{(0,1)\})$. In particular, the q-singular set reduces to $\{(0,1)\}$. This completes the proof of the lemma.

11.3. Proof of Theorem 1.5

After a temporal rescaling, we may assume that $T=1$. After a translation, we may assume that $\overline{B}(0,4\rho)$ is contained in U for sufficiently small $\rho>0$. By subtracting a time dependent constant, we may assume that $p_0(0,t)=0$. Let $a(t)=v_0(0,t)$. We glue (v_0,p_0) and $(a,-\partial_t a\cdot x)$ using Lemma 2.11, obtaining a subsolution $(v_1,p_1,\mathring{R}_1)$ such that

$$(v_1,p_1,\mathring{R}_1)(x,t)=\begin{cases}(v_0,p_0,0) & \text{if } x\notin B(0,4\rho),\\(a(t),-\partial_t a(t)\cdot x,0) & \text{if } x\in\overline{B}(0,3\rho).\end{cases}$$

It is not difficult to deduce from Lemma 2.14 that by reducing ρ we can obtain $\|v_1-v_0\|_0$ and $\rho^3\|\mathring{R}_1\|_0^{1/2}$ arbitrarily small.

We apply Theorem 1.7 to obtain a weak solution of the Euler equations (v_2,p_2) that equals (v_0,p_0) outside $B(0,4\rho)\times[0,1]$ and $(a,-\partial_t a\cdot x)$ in $\overline{B}(0,3\rho)\times[0,1]$. Note that we may choose a nonincreasing energy profile that is arbitrarily close to the original one but still satisfies (1.3) because $\rho^3\|\mathring{R}_1\|_0^{1/2}$ is arbitrarily small. Since $\|v_1-v_0\|_0$ is arbitrarily small, we conclude that $\|v_2-v_0\|_0$ may be chosen to be arbitrarily small.

Next, we construct the blowup in $B(0,\rho)\times(0,1]$. By [4], for any $k\ge 1$ there exists a function $f^k\in C^{2^{-k}}(B(0,\rho))$ taking values in $[1-2\cdot 4^{-k},1-4^{-k}]$ and whose graph G^k has Hausdorff dimension $4-2^{-k}$. We want the q-singular set $\mathcal{S}^q$ of the final solution to contain all of these G^k so that its Hausdorff dimension is 4. To do that, we choose a sequence $\{x_i\}_{i=1}^\infty$ dense in $B(0,\rho)$ and we denote

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$\tau_i^k := f^k(x_i)$. We see that $\{(x_i, \tau_i^k)\}_{i=1}^\infty$ is dense in G^k. Using Lemma 11.1 we will construct a sequence of blowups converging to each of the points (x_i, τ_i^k). Thus, these blowups would accumulate at every point in $\bigcup_{k\geq 1} G^k$, which means that this set will be contained in the singular set, as we wanted.

In order to glue the blowups given by Lemma 11.1 using Lemma 11.2, they must have disjoint supports. Hence, we have study the geometry of the situation. Let

$$t_{ij}^k := \tau_i^k - 4^{-(k+j)}$$

so that the sequence $\{t_{ij}^k\}_{j=1}^\infty$ is contained in $(1-4^{-(k-1)}, 1-4^{-k})$ and converges to τ_i^k. We want to apply Lemma 11.1 to construct a blowup in

$$U_{ij}^k := B(x_i, \varepsilon_{ij}^k) \times (t_{ij}^k, t_{ij}^k + \varepsilon_{ij}^k).$$

It is clear that choosing ε_{ij}^k sufficiently small ensures that the sets $\overline{U}_{ij}^k$ are disjoint for a fixed i and k, but it is not so clear if i is not kept fixed.

We will try to isolate the sequence corresponding to a fixed i. Let $L^k := \|f^k\|_{C^{2^{-k}}}$ and consider the sets

$$\mathcal{C}_i^k := \left\{(x,t) \in \mathbb{R}^3 \times [1-4^{-(k-1)}, \tau_i^k] : \left|t - \tau_i^k\right| \geq (L^k+1)\,|x - x_i|^{\,2^{-k}}\right\}.$$

By the definition of L^k, for any $i \neq i' \geq 1$ we have $(x_i, \tau_i^k) \notin \mathcal{C}_{i'}^k$. We define $j_0(1) = 1$ and for $i > 1$ we define $j_0(i)$ to be the minimum $j_0 \geq 1$ such that

$$\left[\mathcal{C}_i^k \cap [\mathbb{R}^3 \times (t_{ij_0}^k, \tau_i^k)]\right] \cap \mathcal{C}_{i'}^k = \varnothing \qquad \forall i' < i.$$

As we have mentioned, $(x_i, \tau_i^k) \notin \mathcal{C}_{i'}^k$ for $i' < i$, so there exists a neighborhood of (x_i, τ_i^k) disjoint from the union of these sets. Since $\mathcal{C}_i^k \cap [\mathbb{R}^3 \times (t_{ij_0}^k, \tau_i^k)]$ will be contained in this neighborhood of (x_i, τ_i^k) for sufficiently large j_0, $j_0(i)$ is well defined. The point is that we will only add blowups for $j \geq j_0(i)$.

Next, let us define

$$\varepsilon_{ij}^k := 4^{-2^k[i+j+k+\log_4(L^k+1)]}\delta,$$

where $0 < \delta \leq 1$ will be chosen later. Since $\varepsilon_{ij}^k \leq 4^{-(k+j+1)}$, we see that the $\{U_{ij}^k\}_{j=1}^\infty$ are pairwise disjoint for fixed k, i. Furthermore, this definition ensures that

$$4^{-(k+j)} - \varepsilon_{ij}^k \geq (L^k+1)(\varepsilon_{ij}^k)^{2^{-k}},$$

which means that $U_{ij}^k \subset \mathcal{C}_i^k$. Therefore, it follows from the definition of $j_0(i)$ that the sets

$$\{U_{ij}^k : i, k \geq 1, j \geq j_0(i)\}$$

are pairwise disjoint, as claimed.

Let (v_{ij}^k, p_{ij}^k) be the weak solution of the Euler equations given by Lemma 11.1 using the parameter ε_{ij}^k. After a translation, we may assume that the set where $(v_{ij}^k, p_{ij}^k) \neq (a, -\partial_t a \cdot x)$ is contained in U_{ij}^k. We define

$$v_3 := a + \sum_{k,i\geq 1} \sum_{j\geq j_0(i)} (v_{ij}^k - a),$$

$$p_3 := -\partial_t a \cdot x + \sum_{k,i\geq 1} \sum_{j\geq j_0(i)} (p_{ij}^k + \partial_t a \cdot x).$$

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By Lemma 11.2, the pair (v_3, p_3) is a weak solution of the Euler equations in $\mathbb{R}^3 \times \mathbb{R}$. Note that any point (x, t) not in the closure of $\bigcup_{k\geq 1} G^k$ has a neighborhood V that intersects only a finite number of the U^k_{ij}. We conclude that the q-singular set of the weak solution is the closure of the union of the q-singular sets of the v^k_{ij}, that is,

$$\mathcal{S}^q = \{(x_i, t^k_{ij} + \varepsilon^k_{ij}) : k, i \geq 1, j \geq j_0(i)\} \cup \bigcup_{k\geq 1} G^k \cup \left(\overline{B}(0,\rho) \times \{1\}\right)$$

and $v \in C^\beta_{\mathrm{loc}}(\mathbb{R}^4 \backslash \mathcal{S}^q)$. Regarding the energy, let us consider the partial sums

$$\widetilde{v}^k_{ij} := a + \sum_{k'=1}^{k} \sum_{i'=1}^{i} \sum_{j'=j_0(i')}^{j} (v^{k'}_{i'j'} - a),$$
$$e^k_{ij}(t) := \int_{B(0,2\rho)} \left|\widetilde{v}^k_{ij}(x,t)\right|^2 dx.$$

Taking into account the identity $v^2 = (v-a)^2 + a^2 + 2a\cdot(v-a)$, we may write

$$e^k_{ij}(t) = \frac{32}{3}\pi\rho^3 a(t)^2 + \sum_{k'=1}^{k} \sum_{i'=1}^{i} \sum_{j'=j_0(i')}^{j} \left(\int |v^{k'}_{i'j'} - a|^2 dx + 2\int a\cdot(v^{k'}_{i'j'} - a)\, dx\right)$$

because the support of the $(v^{k'}_{i'j'} - a)$ are pairwise disjoint. We see that e^k_{ij} is continuous by Lemma 11.1. Furthermore, by Equation (11.1) and our choice of ε^k_{ij} it converges uniformly to $\int_{B(0,2\rho)} |v_4(x,t)|^2 dx$ which is, therefore, continuous. In addition, it can get arbitrarily close to $\frac{32}{3}\pi\rho^3 a(t)^2$ by reducing $\delta > 0$.

Let us glue this blowup to (v_2, p_2). Since $x_i \in B(0,\rho)$ and we may assume that $\varepsilon^k_{ij} \leq \rho$ by further reducing δ, we see that $U^k_{ij} \subset B(0,2\rho)\times(0,1)$, so $(v_4, p_4) = (a, -\partial_t a\cdot x)$ outside $B(0,2\rho)\times(0,1)$. Hence, it glues well with (v_2, p_2). We conclude that there exists a weak solution of the Euler equations (v_4, p_4) that equals (v_0, p_0) outside $B(0,4\rho)\times(0,1)$ and has a q-singular set $\mathcal{S}^q \subset \overline{B}(0,\rho)\times(0,1]$ with Hausdorff dimension 4. In addition, $v \in C^\beta_{\mathrm{loc}}((\mathbb{R}^3\times[0,1])\backslash\mathcal{S}^q)$. Furthermore, $\|v_4(\cdot,0) - v_0(\cdot,0)\|_{C^0}$ is arbitrarily small and $t \mapsto \int |v_4(x,t)|^2 dx$ is continuous and arbitrarily close to $t \mapsto \int |v_0(x,t)|^2 dx$.

To complete the proof it suffices to modify (v_4, p_4) in $(B(0,3\rho)\backslash\overline{B}(0,2\rho))\times[0,1]$ to ensure that the energy profile is nonincreasing. Let $\widetilde{e}(t) := \int |v_4(x,t)|^2 dx$ and fix a nonincreasing function $e(t) > e_4(t)$. It is easy to obtain a sequence of strictly positive smooth functions $\{\delta_k\}_{k=1}^\infty$ whose sum is $e(t) - \widetilde{e}(t)$ and such that $\|\delta_k\|_{L^\infty} \lesssim 2^{-k}$. Let $r_k := r_0\rho 2^{-k/3}$. By reducing r_0 if necessary, we can find a sequence of pairwise disjoint balls $\overline{B}(x_k, r_k) \subset B(0,3\rho)\backslash\overline{B}(0,2\rho)$.

Fix $k \geq 1$ and let $e_k(t) := a(2^{-k/3}t)^2\,|B(0,r_0\rho)| + 2^k\delta_k(2^{-k/3}t)$. We use Theorem 1.7 to construct a weak solution of the Euler equations (u_k, π_k) that equals $(a(2^{-k/3}t), -(\partial_t a)(2^{-k/3}t)\cdot x)$ for $x \notin B(0,r_0\rho)$ and such that $\int_{B(0,r_0\rho)} |u_k|^2 dx = e_k(t)$. In addition, $v \in C^\beta$.

Finally, we consider

$$v_5(x,t) := a(t) + \sum_{k=1}^{\infty} [u_k(2^{k/3}(x - x_k), 2^{k/3}t) - a(t)],$$
$$p_5(x,t) := -\partial_t a(t)\cdot x + \sum_{k=1}^{\infty} [\pi_k(2^{k/3}x, 2^{k/3}t) + \partial_t a(t)\cdot x].$$

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The pair (v_5, p_5) is a weak solution of the Euler equations that equals $(a, -\partial_t a \cdot x)$ for $x \notin B(0, 3\rho)\backslash\overline{B}(0, 2\rho)$ and $v_5 \in C^\beta$. Regarding the energy

$$\begin{aligned}\int_{B(0,3\rho)\backslash\overline{B}(0,2\rho)} |v_5(x,t)|^2 dx &= a(t)^2 \left|B(0,3\rho)\backslash\overline{B}(0,2\rho)\right| \\ &\quad + \sum_{k=1}^{\infty} \int_{B(x_k,r_k)} \left[|u_5(2^{k/3}(x-x_k),t)|^2 - a(t)^2\right] dx \\ &= a(t)^2 \left|B(0,3\rho)\backslash\overline{B}(0,2\rho)\right| + \sum_{k=1}^{\infty} \delta_k(t) \\ &= \int_{B(0,3\rho)\backslash\overline{B}(0,2\rho)} |v_4(x,t)|^2 dx + e(t) - \widetilde{e}(t).\end{aligned}$$

We glue (v_5, p_5) to (v_4, p_4), obtaining the desired weak solution of the Euler equations (v_6, p_6). Note that $\|(v_6 - v_0)(\cdot, 0)\|_{C^0}$ can be taken to be arbitrarily small because we can do so with $\|v_2 - v_0\|_0$ and $e - \widetilde{e}$. This completes the proof.

A. Hölder and Besov spaces

Let Ω be an open subset of Euclidean space. We denote by $C^0(\Omega)$ the set of bounded continuous functions on Ω, which we equip with the supremum norm, denoted by $\|f\|_0 := \sup_{x\in\Omega} |f(x)|$. More generally, for any $N \geq 0$ we define the space $C^N(\Omega)$ as the set of functions that have bounded continuous derivatives of any order $k \leq N$. On this space, we define the following seminorms and norms, respectively:

$$[f]_N := \max_{|\beta|=N} \left\|D^\beta f\right\|_0, \quad \|f\|_N := \sum_{j=0}^{N} [f]_j.$$

Here β denotes a multi-index and $|\beta|$ denotes its length. Given $N \geq 0$ and $\alpha \in (0, 1)$, we define the Hölder space $C^{N+\alpha}(\Omega)$ as the set of functions $f \in C^N(\Omega)$ such that the following quantity is finite:

$$[f]_{N+\alpha} := \max_{|\beta|=N} \sup_{x\neq y} \frac{\left|D^\beta f(x) - D^\beta f(y)\right|}{|x-y|^{\alpha}}.$$

This set becomes a Banach space when equipped with the following norm:

$$\|f\|_{N+\alpha} := \|f\|_N + [f]_{N+\alpha}.$$

When we work in a subset $E \subset \Omega$ instead of the whole Ω, we will write $\|\cdot\|_{N;E}$. When the functions also depend on time, we also take the supremum in $t \in [0, T]$.

The Hölder norms satisfy the following inequalities:

$$[f]_s \leq C\left(\varepsilon^{t-s}[f]_t + \varepsilon^{-s}\|f\|_0\right) \tag{A.1}$$

for $0 \leq s \leq t$ and all $\varepsilon > 0$, and

$$[fg]_s \leq [f]_s \|g\|_0 + \|f\|_0 [g]_s \tag{A.2}$$

for $0 \leq s \leq 1$. The constant C only depends on the Hölder exponents involved and on the domain Ω. For the applications in this article, since f will be a compactly supported function defined on the whole $\mathbb{R}^n$, C will just depend on the Hölder exponents. From (A.1) with $\varepsilon = \|f\|_0^{1/r} [f]_r^{-1/r}$ we obtain the

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following interpolation inequality:

$$[f]_s \leq C \|f\|_0^{1-s/r} [f]_r^{s/r}. \tag{A.3}$$

Let β by a multi-index. By induction on $|\beta|$ and the rule for the first derivative of a product, one easily deduces

$$D^\beta(fg) = \sum_{|\gamma|+|\delta|=|\beta|} C_{|\beta|,\gamma,\delta} D^\gamma f \, D^\delta g,$$

from which it immediately follows that

$$[fg]_N \leq C_N \sum_{j=0}^N [f]_j \, [g]_{N-j}. \tag{A.4}$$

The following proposition is standard:

Proposition A.1. *Let $N \in \mathbb{N}$ and $\alpha \in (0,1)$. Let $f \in C_c^{N,\alpha}(\mathbb{R}^m, \mathbb{R})$ and $F \in C_c^{N,\alpha}(\mathbb{R}^m, \mathbb{R}^m)$. There exists a constant $C = C(N,\alpha)$ such that the potential-theoretic solutions of*

$$\Delta\phi = f, \qquad \Delta\psi = \operatorname{div} F$$

satisfy

$$\|\phi\|_{N+2+\alpha} \leq C \|f\|_{N+\alpha}, \qquad \|\psi\|_{N+1+\alpha} \leq C \|F\|_{N+\alpha}.$$

Note that in the previous proposition, one does not get any information about the C^α norm of the solution, aside from estimating it by higher-order norms. For this, we will need to introduce negative regularity spaces. Let us consider a Littlewood–Payley decomposition, for example, as in [2, Section 2.2]. For this, we take smooth radial functions $\chi, \varphi : \mathbb{R}^3 \to [0,1]$, whose supports are contained in the ball $B(0, \frac{4}{3})$ and in the annulus $\{\frac{3}{4} < |\xi| < \frac{8}{3}\}$ respectively, with the property that

$$\chi(\xi) + \sum_{N=0}^\infty \varphi(2^{-N}\xi) = 1$$

for all $\xi \in \mathbb{R}^3$. In terms of the Fourier multipliers $P_< := \chi(D)$ and $P_N := \varphi(2^{-N}D)$, the Besov norm $B^s_{\infty,\infty}$ (which is equivalent to the Hölder norm C^s if $s \in \mathbb{R}^+ \backslash \mathbb{N}$, and strictly weaker if $s \in \mathbb{N}$) can be written as

$$\|f\|_{B^s_{\infty,\infty}} := \|P_< f\|_0 + \sup_{N \geq 0} 2^{Ns} \|P_N f\|_0. \tag{A.5}$$

Here $s \in \mathbb{R}$ is any real number. Again, when dealing with time-dependent functions, we consider the supremum in time of $\|f(t)\|_{B^s_{\infty,\infty}}$.

B. Some auxiliary estimates

The first lemma of this appendix shows that we can find a cutoff function with well-behaved bounds on its derivatives:

Lemma B.1. *Let $A \subset \mathbb{R}^n$ be a measurable set and let $r > 0$. There exists a cutoff function $\varphi_r \in C^\infty(\mathbb{R}^n, [0,1])$ whose support is contained in $A + B(0,r)$ and such that $\varphi_r \equiv 1$ in a neighborhood of A.*

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Furthermore, for any $N \geq 0$ we have

$$\|\varphi_r\|_N \leq C(N,n)\, r^{-N}.$$

Proof. We choose a nonnegative function $\psi \in C_c^\infty(B(0,1))$ such that $\int \psi = 1$. For $\varepsilon > 0$ we denote

$$A_\varepsilon := A + B(0,\varepsilon), \qquad \psi_\varepsilon(x) = \varepsilon^{-n}\,\psi(x/\varepsilon).$$

Note that $\int \psi_\varepsilon = 1$. The desired cutoff function is:

$$\varphi_r(x) := \left(\mathbb{1}_{A_{r/2}} * \psi_{r/2}\right)(x) = \int_{A_{r/2}} \psi_{r/2}(x-y)\ dy.$$

It is easy to see that its support is contained in $A + B(0,r)$ and that $\varphi_r \equiv 1$ in a neighborhood of A. Furthermore, it is smooth and

$$\begin{aligned}\partial^\alpha \varphi_r(x) &= \left(\mathbb{1}_{A_{r/2}} * \partial^\alpha \psi_{r/2}\right)(x) = \int_{A_{r/2}} \partial_x^\alpha \left[(r/2)^{-n}\ \psi\left(\frac{x-y}{r/2}\right)\right]\ dy \\ &= (r/2)^{-|\alpha|} \int_{A_{r/2}} (r/2)^{-n} (\partial^\alpha \psi)\left(\frac{x-y}{r/2}\right)\ dy.\end{aligned}$$

Hence,

$$\begin{aligned}|\partial^\alpha \varphi_r(x)| &\leq (r/2)^{-|\alpha|} \int_{A_{r/2}} (r/2)^{-n} \left|(\partial^\alpha \psi)\left(\frac{x-y}{r/2}\right)\right|\ dy \\ &\leq (r/2)^{-|\alpha|} \int_{\mathbb{R}^n} (r/2)^{-n} \left|(\partial^\alpha \psi)\left(\frac{x-y}{r/2}\right)\right|\ dy \\ &\leq (r/2)^{-|\alpha|} \int_{\mathbb{R}^n} |\partial^\alpha \psi(y)|\ dy,\end{aligned}$$

from which the result follows. □

The second instrumental lemma provides a bound for the solution to a transport equation. The proof is standard, see, for example, [6].

Lemma B.2. *Let $f \in C^\infty(\mathbb{R}^3 \times \mathbb{R})$ be the solution of the transport equation*

$$\begin{cases}\partial_t f + v \cdot \nabla f = g, \\ f(\cdot, 0) = f_0\end{cases}$$

for some vector field $v \in C^\infty(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R}^3)$ and $g \in C^\infty(\mathbb{R}^3 \times \mathbb{R})$. Then, for $0 \leq \alpha \leq 1$ and $|t|\,\|v\|_1 \leq 1$ we have

$$\|f(\cdot,t)\|_\alpha \leq e^\alpha \left(\|f_0\|_\alpha + \int_0^t \|g(\cdot,\tau)\|_\alpha\, d\tau\right).$$

In this article we also need and extension theorem for Hölder continuous functions defined on a domain Ω:

Theorem B.3. *Let $N \geq 0$ and $\alpha \in (0,1)$. Let $\Omega \subset \mathbb{R}^n$ be a domain with smooth boundary. Then, there exists a linear map $T : C^{N+\alpha}(\Omega) \to C^{N+\alpha}(\mathbb{R}^n)$ such that*

- *$Tf = f$ on Ω for each $f \in C^{N+\alpha}(\Omega)$ and*
- *the norm of T is bounded by a constant depending only on Ω and N.*

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Proof. The proof is essentially [33, Lemma 6.37] and is based on a rectification of the boundary and a reflection. We must, however, make some remarks. In [33] they assume $N \geq 1$ because they are considering sets with less regular boundaries. In the case of a smooth boundary, the result also holds for C^{α} functions. We must warn the reader that they are using the notation $C^{N+\alpha}\left(\overline{\Omega}\right)$ to denote Hölder continuous functions because they use $C^{N+\alpha}(\Omega)$ to denote locally Hölder continuous functions. Finally, it is also interesting to note that the form of operator T itself does not depend on N and α, although its norm as an operator $C^{N+\alpha}(\Omega) \to C^{N+\alpha}(\mathbb{R}^n)$ does, of course. This holds provided that we follow the construction of [33] but use smooth parametrizations of Ω, instead of less regular ones. □

The last result of this appendix is a lemma that establishes a bound for a Besov norm of functions that are compactly supported in a collared neighborhood of an $(n-1)$-dimensional surface. This can be seen as a Poincaré inequality with negative regularity.

Lemma B.4. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain with smooth boundary. Let $\alpha \in (0,1)$ and let $r > 0$ be sufficiently small (depending on Ω). Consider a function $f \in C_c^{\infty}(\mathbb{R}^n)$ supported in $\{x \in \mathbb{R}^n : 0 < \operatorname{dist}(x,\Omega) < r\}$. We have*

$$\|f\|_{B^{-1+\alpha}_{\infty,\infty}} \leq C(\Omega,\alpha)\, r^{1-\alpha}\, \|f\|_0 .$$

Proof. Let us denote $U := \{x \in \mathbb{R}^n : 0 < \operatorname{dist}(x,\Omega) < r\}$. We begin by computing

$$|P_N f(x)| \leq 2^{nN} \int_U \left|h(2^N(x-y))\right| |f(y)|\, dy \lesssim \|f\|_0\, 2^{nN} \int_U \langle 2^N(x-y)\rangle^{-8n}\, dy,$$

where we have used that $|h(x)| \lesssim \langle x\rangle^{-8n}$ because h is in the Schwartz class. Here $P_N f$ stands for the nonhomogeneous dyadic blocks in the Littlewood-Paley decomposition of f, and h is the corresponding convolution kernel.

We claim that the following estimate holds:

$$2^{nN} \int_U \langle 2^N(x-y)\rangle^{-8n}\, dy \lesssim \min\{1, 2^N r\}. \tag{B.1}$$

Using this, the bound for f readily follows:

$$\begin{aligned} \|f\|_{B^{-1+\alpha}_{\infty,\infty}} &= \sup_N 2^{N(-1+\alpha)}\, \|P_N f\|_{L^\infty} \\ &\lesssim \sup_{2^N < r^{-1}} 2^{N(-1+\alpha)} (\|f\|_0\, 2^N r) + \sup_{2^N \geq r^{-1}} 2^{N(-1+\alpha)}\, \|f\|_0 \\ &\lesssim r^{1-\alpha}\, \|f\|_0 . \end{aligned}$$

Therefore, the problem is reduced to estimating the integral above.

Let

$$U_R := \{x \in \mathbb{R}^n : 0 < \operatorname{dist}(x,\Omega) < R\}.$$

For each point $x \in \partial\Omega$ and for each small enough $R > 0$, there is a boundary normal chart

$$X_{x,R} : Q_R \to U_{2R}\,,$$

where $Q_R := [0,R) \times (-R,R)^{n-1}$, such that $\Psi_{x,R}(0) = x$ and

$$X_{x,R}(Q_R) \supset \{y \in \overline{U_R} : |x-y| < R/2\}\,.$$

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Recall that, by the definition of boundary normal coordinates, the pullback $X_{x,R}^* g_0$ of the Euclidean metric $g_0 := (\delta_{ij})$ to this chart satisfies $X_{x,R}^* g_0(0) = g_0$. One can also assume that $\nabla X_{x,R}(0)$ is the identity matrix, and that

$$|x - y| > \frac{R}{4} \tag{B.2}$$

for all $y \in X_{x,R}(Q_{R/2})$ and all $x \notin X_{x,R}(Q_{R/2}) \cup \Omega$.

Since $\partial\Omega$ is a smooth compact hypersurface of $\mathbb{R}^n$, it is standard that there is some small enough $R > 0$ and a finite collection of charts $\{X^j := X_{x_j,R}\}_{j=1}^J$ as above such that $\{U^j := X^j(Q_{R/2})\}_{j=1}^J$ is a cover of the set $\overline{U_{R/2}}$ and which satisfy

$$\|(X^j)^* g_0 - g_0\|_{C^2(Q_R)} < \frac{1}{100} \tag{B.3}$$

and

$$|X^j(z) - X^j(\tilde{z})| \geq \frac{1}{2}|z - \tilde{z}| \tag{B.4}$$

for all $z, \tilde{z} \in Q_R$. Moreover, the distance between the point $X^j(z)$ and the boundary is comparable to its first coordinate, in the sense that

$$\frac{z_1}{2} \leq \operatorname{dist}(X^j(z), \Omega) \leq 2z_1$$

for all $z \in Q_R$.

Let us now estimate the integral (B.1) over the subset $U^j \cap U$, with $x \in U$. If $x \notin X^j(Q_R)$, by (B.2), one immediately has

$$2^{nN} \int_{U^j \cap U} \langle 2^N (x - y) \rangle^{-8n} \, dy \lesssim 2^{nN} \langle 2^N R \rangle^{-8n} |U^j \cap U| \lesssim r \leq \{1, 2^N r\},$$

where we have also used that $|U| \lesssim r$. If $x \in X^j(Q_R)$, one can write $x = X^j(\tilde{z})$ for some $\tilde{z} \in Q_R$. By (B.4), one then has

$$\begin{aligned} 2^{nN} \int_{U^j \cap U} \langle 2^N (x - y) \rangle^{-8n} \, dy &\lesssim 2^{nN} \int_0^{2r} \int_{(-R,R)^{n-1}} \langle 2^N (z - \tilde{z}) \rangle^{-8n} J_{X^j}(z) \, dz_1 \, dz' \\ &\lesssim 2^{nN} \int_0^{2r} \int_{(-R,R)^{n-1}} \langle 2^N (z_1 - \tilde{z}_1) \rangle^{-4n} \langle 2^N (z' - \tilde{z}') \rangle^{-4n} \, dz_1 \, dz' . \end{aligned}$$

Here we have used the notation $z = (z_1, z') \in [0, R) \times (-R, R)^{n-1}$ and the fact that the Jacobian J_{X^j} is bounded by (B.3).

We can now carry out the integrations in z_1 and z' separately. Since

$$2^{(n-1)N} \int_{(-R,R)^{n-1}} \langle 2^N (z' - \tilde{z}') \rangle^{-4n} \, dz_1 \, dz' \leq 2^{(n-1)N} \int_{\mathbb{R}^{n-1}} \langle 2^N (z' - \tilde{z}') \rangle^{-4n} \, dz_1 \, dz' \lesssim 1,$$

the estimate follows from the fact that

$$2^N \int_0^{2r} \langle 2^N (z_1 - \tilde{z}_1) \rangle^{-4n} \, dz_1 \leq 2^N \int_{-4r}^{4r} \langle 2^N s \rangle^{-4n} \, ds = \int_{-2^{N+2}r}^{2^{N+2}r} \langle s \rangle^{-4n} \, ds \lesssim \min\{1, 2^N r\}.$$

The estimate (B.1) then follows by summing over j. □

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