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NON-SYMMETRIC TRANSLATION INVARIANT DIRICHLET FORMS ON HYPERGROUPS

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In this note translation-invariant Dirichlet forms on a commutative hypergroup are studied. The main theorem gives a characterisation of an invariant Dirichlet form in terms of the negative definite function associated with it. As an illustration constructions of potentials arising from invariant Dirichlet forms are given. The examples of one- and two-dimensional Jacobi hypergroups yield specifications of invariant Dirichlet forms, particularly in the case of Gelfand pairs of compact type.

The study of translation invariant Dirichlet forms on locally compact abelian groups is covered in the symmetric case by the Beurling-Deny theory (see [5]) and in the nonsymmetric case by Berg and Forst [3]. In this paper we consider the latter for commutative hypergroups, which is of importance for the study of transient convolution semigroups and their potential theory. We then present examples to illustrate these results.

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The analysis throughout will be carried out on a (locally compact) commutative hypergroup K . For a definition and properties, refer to Heyer [6] and Jewett [7]. We make use of the notation $K^\wedge$ for the dual hypergroup of K , and π for the Plancherel measure on $K^\wedge$ corresponding to the Haar measure ω_K on K , guaranteed by Spector [9, Theorem III.4] together with Levitan's theorem (Jewett [7, Theorem 7.3I]). For $\mu \in M(K)$ (the space of bounded Radon measures on K) and $\sigma \in M(K^\wedge)$ we have

$$\hat{\mu}(\gamma) = \int_K \bar{\gamma}(x) d\mu(x)$$

for all $\gamma \in K^\wedge$, and

$$\check{\sigma}(y) = \int_{K^\wedge} \chi(y) d\sigma(\chi) ,$$

for all $y \in K$. $\hat{\mu}$ and $\check{\sigma}$ define the Fourier transform and inverse Fourier transform respectively, and have obvious extensions to the respective subspaces of integrable functions.

Jewett ([7], Theorem 12.2C) has proved an inversion formula for Fourier transforms. We require the dual formulation of this:

LEMMA 1. *Suppose* $f \in L^1 \cap C(K^\wedge)$ *has* $\check{f} \in L^1(K)$. *Then* $(f^\vee)^\wedge(\gamma) = f(\gamma)$ *for all* $\gamma \in \operatorname{supp}\pi$.

Proof. We know from Jewett [7, Lemma 12.2B] that for $\mu \in M(K)$, $\sigma \in M(K^\wedge)$

$$\mu = \check{\sigma}\omega_K \text{ if and only if } \sigma = \hat{\mu}\pi .$$

Writing $\sigma = f\pi$ this gives

$$\mu = \check{f}\omega_K \text{ if and only if } f\pi = \hat{\mu}\pi ,$$

which yields $(f^\vee)^\wedge\pi = f\pi$, as required. □

Throughout the paper we assume $K^\wedge$ to be a hypergroup under pointwise multiplication, in which case $\pi = \omega_{K^\wedge}$, a Haar measure for $K^\wedge$, and $\operatorname{supp}\omega_{K^\wedge} = K^\wedge$ (Jewett [7, Theorem 12.4 A]). We denote the distinguished unit element of $K^\wedge$ by $\underline{1}$.

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Following Bloom and Heyer [4], we call a continuous function ψ on $K^\wedge$ strongly negative definite if $\psi(\underline{1})\geq 0$ and $\exp(-t\psi)$ is strongly positive definite on $K^\wedge$ for every $t>0$, in the sense that there exists a measure $\mu_t \in M_+(K)$ (which is necessarily unique) satisfying $\hat{\mu}_t(\chi) = \exp(-t\psi(\chi))$ for $\chi \in K^\wedge$, $t>0$. The set of strongly negative definite functions on $K^\wedge$ will be denoted by $SN(K^\wedge)$. It is known (Lasser [8, Proposition 1]) that every strongly negative definite function is negative definite. Various properties of strongly negative definite functions are to be found in Bloom and Heyer [4, Section 2].

The following result is needed in the sequel, and is given here for convenience.

LEMMA 2. *If* $\psi \in SN(K^\wedge)$ *then* $Re\psi \in SN(K^\wedge)$.

Proof. If $\hat{\mu}_t(\chi) = \exp(-t\psi(\chi))$ then $\hat{\mu}_t^-(\chi) = \exp(-t\psi^-(\chi))$ (note that $\chi^- = \bar{\chi}$) and hence

$$(\mu_{t/2} * \mu^-_{t/2})^\wedge(\chi) = \exp(-t/_2(\psi(\chi)+\psi^-(\chi)))$$

$$= \exp(-t/_2(\psi(\chi)+\overline{\psi(\chi)}))$$

$$= \exp(-tRe\psi(\chi)),$$

where we have made use of Bloom and Heyer [4, Proposition 2.3(c)]. We now just observe that $Re\psi$ is continuous and $Re\psi(\underline{1}) \geq 0$, which shows that $Re\psi \in SN(K^\wedge)$. □

Dirichlet forms for the non-symmetric case were introduced in Berg and Forst [3] (see Definition 2.3). The results proved there for general locally compact spaces hold, in particular, for hypergroups. We now consider the rôle of translation invariance in the context of the latter structure.

Let $(\mu_t)_{t>0}$ be a (vaguely) continuous convolution semigroup of subprobability measures on K. The Schoenberg correspondence (Bloom and Heyer [4, Theorem 3.7]) guarantees the existence of a continuous negative definite function ψ on $K^\wedge$ satisfying

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$$\hat{\mu}_t(\chi) = \exp(-t\psi(\chi))$$

for all $\chi \in K^\wedge$, $t>0$. The strongly continuous contraction semigroup $(P_t)_{t>0}$ of operators on $L^2(K)$ defined by $P_t f = \mu_t * f$ can be related through its infinitesimal generator (A, D_A) to ψ as follows:

THEOREM 1. *The domain* D_A *of* A *is given by*

$$D_A = \{f \in L^2(K) : \hat{f}\psi \in L^2(K^\wedge)\}$$

and

$$(Af)^\wedge = - \hat{f}\psi$$

for all $f \in D_A$.

The proof, which uses Levitan's theorem, follows that of the group case (see Berg and Forst [3, Proposition, p.205]).

Let $(R_\lambda)_{\lambda>0}$ denote the resolvent associated with the semigroup $(P_t)_{t>0}$. We call a positive closed form (β, V) on $L^2(K)$ *translation invariant* if each R_λ commutes with convolution by point measures, that is,

$$R_\lambda(f*\varepsilon_x) = (R_\lambda f)*\varepsilon_x$$

for all $\lambda>0$, $f \in L^2(K)$ and $x \in K$. With some minor changes the proof of Berg and Forst [3, Theorem 3.5] carries over to give:

THEOREM 2. *Let* (β, V) *be a translation invariant positive closed form on* $L^2(K)$ *and suppose that for the unit contraction* T_I *(the projection of the complex plane onto the unit interval)* $T_I f \in V$ *for all* $f \in V$. *The following are equivalent:*

(i) T_I *operates with respect to* β .

(ii) λR_λ *is sub-Markovian for all* $\lambda>0$.

(iii) λR_λ^* *is sub-Markovian for all* $\lambda>0$.

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(iv) T_I *operates with respect to* β^* *(the adjoint of* β *).*

One observes that an invariant positive closed form (β, V) on $L^2(K)$ such that $T_I f \in V$ for all $f \in V$ is a Dirichlet form if and only if any of the statements (i) to (iv) holds. Theorem 2 leads to a characterisation of translation invariant Dirichlet forms on $L^2(K)$.

THEOREM 3. *Let* (β, V) *be a translation invariant Dirichlet form on* $L^2(K)$. *There is a strongly negative definite function* $\psi : K^\wedge \to C$ *satisfying* $|Im\psi| \leq C\, Re\psi$ *for a suitable constant* $C>0$, *such that*

$$V = \{f \in L^2(K) \; : \; \int_{K^\wedge} |\hat{f}|^2 \, Re\psi d\omega_{K^\wedge} < \infty\}$$

and
$$\beta(f,g) = \int_{K^\wedge} \hat{f}\bar{\hat{g}} d\omega_{K^\wedge} \quad \text{for all } f, g \in V .$$

Conversely if ψ is any strongly negative definite function satisfying $|Im\psi| \leq C\, Re\psi$ for some constant $C>0$ then (β, V) with β, V as above constitutes a translation invariant Dirichlet form on $L^2(K)$.

Proof. We sketch the proof, indicating where it departs from the locally compact abelian group case.

The translation invariance of β tells us that

$$R_\lambda(f * \varepsilon_x) = (R_\lambda f) * \varepsilon_x, \; \lambda > 0, \; f \in L^2(K), \; x \in K .$$

Now

$$P_t f = \lim_{\lambda \to 0} exp(tA\lambda R_\lambda) \; f,$$

where $A = \lambda I - R_\lambda^{-1}$. It is clear that R_λ^{-1} , and hence A , is also translation invariant, whence follows the translation invariance of P_t . Similar comments hold for the hermitian part (α, V) of (β, V) $(\alpha = 1/_2(\beta + \beta^*))$ so that, following the Deny theory, there exists a strongly negative definite function $\psi^{(\alpha)} : K^\wedge \to R$ such that

$$V = \{f\epsilon L^2(K) : \int_{K^\wedge} |\hat{f}|^2 \psi^{(\alpha)} d\omega_{K^\wedge} < \infty\}$$

and

$$\alpha(f,g) = \int_{K^\wedge} \hat{f}\bar{\hat{g}}\psi^{(\alpha)} d\omega_{K^\wedge} \quad \text{for all} \quad f,g\epsilon V .$$

Since the P_t commute with translations we have $P_t f = \mu_t * f$, where $(\mu_t)_{t>0}$ is a vaguely continuous convolution semigroup of subprobability measures. Let ψ be the associated strongly negative definite function. Then, using Berg and Forst [3, Section 1.4],

$$\beta(f,g) = \int_{K^\wedge} \hat{f}\bar{\hat{g}}\psi d\omega_{K^\wedge}$$

and, in particular,

$$\alpha(f,f) = Re\beta(f,f) = \int_{K^\wedge} |\hat{f}|^2 Re\psi d\omega_{K^\wedge}$$

for all $f\epsilon D_A$. By Theorem 1, $\check{\phi}\epsilon D_A \subset V$ for all $\phi\epsilon C_c(K^\wedge)$ (the space of continuous functions on $K^\wedge$ having compact support) and, appealing to Lemma 1,

$$\int_{K^\wedge} |\phi|^2 Re\psi d\omega_{K^\wedge} = \int_{K^\wedge} |\phi|^2 \psi^{(\alpha)} d\omega_{K^\wedge}$$

whence it follows that $Re\psi = \psi^{(\alpha)}$ on $\operatorname{supp}\omega_{K^\wedge} = K^\wedge$. Similarly (see the proof of Berg and Forst [3, Theorem 3.7])

$$|Im\ \psi| \leqq C\ Re\ \psi \quad \text{on} \quad K^\wedge$$

and the condition on β follows.

Conversely suppose that ψ is a strongly negative definite function satisfying $|Im\psi| \leqq CRe\psi$ for some positive constant C. By Lemma 2, $Re\psi$ is a real strongly negative definite function on $K^\wedge$ which, by the Deny theory, defines a translation invariant Dirichlet form (α,V) on $L^2(K)$ where

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$$V = \{f \in L^2(K) : \int_{K^\wedge} |\hat{f}|^2 Re\psi d\omega_{K^\wedge} < \infty\}$$

and, for all $f, g \in V$,

$$\alpha(f,g) = \int_{K^\wedge} \hat{f}\bar{\hat{g}} Re\psi d\omega_{K^\wedge} .$$

As in Berg and Forst [3, Theorem 3.7],

$$\beta(f,g) = \int_{K^\wedge} \hat{f}\bar{\hat{g}} \psi d\omega_{K^\wedge}$$

defines a positive closed form on $L^2(K)$.

Now using Bloom and Heyer [4, Proposition 2.3(a)], $\lambda+\psi \in SN(K^\wedge)$ for every $\lambda>0$ and, for such λ , it is the case that $(\lambda+\psi)(\underline{1})>0$. Thus we can apply Bloom and Heyer [4, Proposition 2.5] to deduce the existence of $\rho_\lambda \in M^+(K)$ satisfying $\hat{\rho}_\lambda = (\lambda+\psi)^{-1}$ for all $\lambda>0$, and the resolvent $(R_\lambda)_{\lambda>0}$ associated with β is seen to be given by

$$R_\lambda f = \rho_\lambda * f, \quad f \in L^2(K) .$$

It remains to observe that each R_λ , being a convolution operator, commutes with translations, which means that β is translation invariant. □

For the purpose of illustration we draw a consequence from Theorem 3 which is known in the group case (see Berg and Forst [3, p.211]).

Let K be a strong hypergroup (that is, a hypergroup satisfying $K^{\wedge\wedge} \cong K$) , in which case it suffices to apply the usual (pointwise) notion of negative definiteness. In general the Deny theory yields that the symmetric Dirichlet form (α, V) is positive definite if and only if

$$(*) \qquad \frac{1}{Re\psi} \in L^1_{loc}(K^\wedge)$$

and, in this case, (α, V) can be extended to a regular Dirichlet form $(\beta, \tilde{V})$, where $\tilde{V}$ denotes the completion of V with respect to the norm $f \to ||f||_\alpha := \sqrt{\alpha(f,f)}$ and the unit contraction T_I operates on $\tilde{V}$ with respect to β and β^* .

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Suppose now that (*) holds. Then $\frac{1}{\psi}\in L^1_{loc}(K^\wedge)$ and $\frac{1}{\psi}\omega_{K^\wedge}$ is a translation bounded positive definite measure on $K^\wedge$. Let

$$\kappa := [(\frac{1}{\psi}\omega_{K^\wedge})^-]^\vee$$

defined via Godement's approach (applied to strong hypergroups in Bloom and Heyer [4]) by

$$\int_{K^\wedge} f^* \tilde{g} \frac{1}{\psi} d\omega_{K^\wedge} = \int_K \check{f} \bar{\check{g}} \, d\kappa \, .$$

whenever $f, g\in C_c(K^\wedge)$. Then $\kappa = \int_0^\infty \mu_t dt$ and $\rho_\lambda \to \kappa$ when $\lambda \to 0$. Moreover (as in Berg and Forst [3, Remark 3.8 (4)]) ,

$$\beta(\kappa * f, v) = \int_{K^\wedge} \hat{f}\bar{\hat{v}} d\omega_{K^\wedge}$$

$$= \int_K f\bar{v} d\omega_K$$

for all $v\in V$, whence $\kappa * f$ is the β-potential generated by $f\in C_c(K)$.

In the following examples we want to specify invariant Dirichlet forms (β, V) on hypergroups by making precise the Fourier transform $\hat{f}$ of $f\in V$ and the negative definite function ψ associated with (β, V) via the representation

$$\beta(f,f) = \int_{K^\wedge} |\hat{f}|^2 \psi d\omega_{K^\wedge}$$

for all $f\in V$. In the case of symmetric Dirichlet forms Lasser's Lévy-Khintchine representation ([8, Theorem 5]) is available at least for strong hypergroups satisfying his hypothesis *(F)* (thus in particular for discrete hypergroups and for hypergroups of type G_B) .

1. <u>One-dimensional Jacobi hypergroups</u>. These are hypergroups $K := [-1,1]$ with hypergroup dual $K^\wedge = \{R_n^{\alpha,\beta} : n\geq 0\} \cong Z_+$, where $R_n^{\alpha,\beta}$ denotes the normed Jacobi polynomial of degree $n\geq 0$ $(\alpha\geq\beta>-1,\ \alpha+\beta+1\geq 0)$. The polynomials $R_n^{\alpha,\beta}$ are orthogonal with respect to the Haar measure $\omega := A\,\lambda_{[-1,1]}$ with

$$A(x) := (1-x)^{\alpha}(1+x)^{\beta},\ x\in[-1,1].$$

$K^\wedge$ has 0 as unit and the identity map as involution.

(i) For any $f\in L^2(K)$ we have

$$\hat{f}(n) = \int_{-1}^{1} f(x)R_n^{\alpha,\beta}(x)\,d\omega(x)\ ,\ n\geq 0\ .$$

(ii) Any negative definite function ψ on $K^\wedge$ is of the form

$$\psi(n)=a+bn(n+\alpha+\beta+1)+\int_{-1}^{1}(1-R_n^{\alpha,\beta}(x))\,d\eta(x)\ ,\ n\geq 0$$

where $a,b\geq 0$, and $\eta\in M_+([-1,1))$ is a Lévy measure satisfying

$$\int_{-1}^{1}(1-x)\,d\eta(x)<\infty\ .$$

For certain pairs (α,β) the hypergroup duals $K^\wedge$ are duals of Gelfand pairs (G,H) of compact type, that is, with a semisimple connected compact Lie group G of finite centre and a maximal compact subgroup H of G such that G/H is of rank 1 .

1.1 <u>Subexample</u>. Let $G := SO(d,\mathbb{R})$ and $H := SO(d-1,\mathbb{R})\ (d\geq 3)$. Then $K = G//H \cong [-1,1]$ and $K^\wedge = \{R_n^{\alpha} : n\geq 0\}\cong\mathbb{Z}_+$, where R_n^{α} denotes the ultraspherical (Gegenbauer) polynomial of degree n and order $\alpha := \frac{d-3}{2}$.

Starting (as in Berg [2]) with an invariant Dirichlet form (β,V) on the Gelfand pair (G,H) we obtain

$$\beta(f,f) = \sum_{n\geq 0}||f_n||^2\psi(n)$$

for all f on the sphere $S^d(=G/H)$ having spherical representation $\sum_{n\geq 0} f_n$. Applying the canonical form of the negative definite function ψ on $\mathbb{Z}_+$ associated with (β,V) we obtain

$$\beta(f,f)=a||f||^2+b\langle f,\varepsilon_1'*f\rangle+\int_{-1}^{1}\langle f,f-\varepsilon_x*f\rangle\,d\eta(x)$$

for all

$$f \in V := \{g \in L^2(S^d, \omega_d) : \beta(g,g) < \infty\} .$$

Here ε_1' denotes the distribution $\phi \to \phi'(1)$ on R .

2. Two-dimensional Jacobi hypergroups. These are of the form $K := D := \{z \in C : |z| \leq 1\}$ with hypergroup dual $K^\wedge := \{R^\alpha_{m,n} : (m,n) \in Z^2_+\} \cong Z^2_+$, where

$$R^\alpha_{m,n}(z) := |z|^{|m-n|} R^{\alpha,|m-n|}_{m \wedge n}(2|z|^2-1),\ z \in D, (m,n) \in Z^2_+$$

$(\alpha \geq 0)$. The polynomials $R^\alpha_{m,n}$ are orthogonal with respect to the Haar measure $\omega := A\, \lambda_D$ with

$$A(x,y) := \frac{\alpha+1}{\pi}(1-x^2-y^2)^\alpha,\ (x,y) \in R^2,\ x^2+y^2 \leq 1 .$$

$K^\wedge$ has 0 as unit and $(m,n) \to (n,m)$ as involution.
These hypergroup structures in K and $K^\wedge$ have been studied by Annabi and Trimèche in [1].

(i) For any $f \in L^2(K)$ we have

$$\hat{f}(m,n) = \int_D f(z) R^\alpha_{n,m}(\bar{z})\, d\omega(z) ,\ (m,n) \in Z^2_+$$

(ii) Any negative definite function ψ on $K^\wedge$ is of the form

$$\psi(m,n) = d+a(m-n)^2-ic(m-n)+b(m+n+\frac{2mn}{\alpha+1})$$

$$+ \int_{D^*} [1-R^\alpha_{m,n}(x,y)+i(m-n)]\, d\eta(x,y)$$

for all $(m,n) \in Z^2_+$, where $a,b,d \in R_+$, $c \in R$, and $\eta \in M_+(D^*)$ is a Lévy measure satisfying

$$\int_{D^*} (1-x)\, d\eta(x,y) < \infty$$

$(D^* := D \backslash \{0\})$.

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For certain $\alpha \geqq 0$ these hypergroup duals are duals of Gelfand pairs (G,H) of compact type arising from symmetric spaces of rank 2.

2.1 <u>Subexample</u>. Let $G := U(d)$ and $H := U(d-1)(d \geqq 3)$. Then $K := G//H \cong D$ and $K^\wedge = Z_+^2 \cong \{R_{m,n}^\alpha : (m,n) \in Z_+^2\}$, where $\alpha := d-2$.

In both subexamples a given invariant Dirichlet form (β,V) with associated negative definite function ψ is positive definite with a regular Dirichlet space V under the norm $\sqrt{\beta}$ if and only if $\frac{1}{\psi} \in L_{loc}^1(K^\wedge)$.

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