Canad. Math. Bull. Vol. **25** (4), 1982

BOUNDED SOLUTIONS OF A FUNCTIONAL INEQUALITY

BY
MICHAEL ALBERT AND JOHN A. BAKER[(1)]

ABSTRACT. It is known that if f is a real valued function on a rational vector space V, $\delta > 0$,

(1) $$|f(x+y)-f(x)f(y)| \le \delta \quad \text{for all} \quad x, y \in V$$

and if f is unbounded then $f(x+y)=f(x)f(y)$ for all $x, y \in V$. In response to a problem of E. Lukacs, in this paper we study the bounded solutions of (1). For example, it is shown that if f is a bounded solution of (1) then $-\delta \le f(x) \le (1+(1+4\delta)^{1/2})/2$ for all $x \in V$ and these bounds are optimal.

Let V be a rational vector space and let R denote the set of real numbers. In [1] it was shown that if $\delta > 0$ and $f: V \to R$ such that

(1) $$|f(x+y)-f(x)f(y)| \le \delta \quad \text{for all} \quad x, y \in V$$

then either f is bounded or $f(x+y)=f(x)f(y)$ for all $x, y \in V$. A short proof of a more general result appears in [2]. In this paper we study the bounded solutions of (1).

Note that any function $f: V \to R$ which is sufficiently uniformly close to either 0 or 1 is a solution of (1). In fact if $\varepsilon > 0$ and $\varepsilon + \varepsilon^2 = \delta$ then (1) holds provided $|f(x)| \le \varepsilon$ for all $x \in V$. If $\varepsilon > 0$, $3\varepsilon + \varepsilon^2 = \delta$ and $|f(x)-1| \le \varepsilon$ for all $x \in V$ then (1) holds.

Observe that (1) has many constant solutions. Indeed, any $c \in R$ with $|c-c^2| \le \delta$ determines a constant solution of (1). If $\delta \ge \frac{1}{4}$, then $|c-c^2| \le \delta$ if and only if $(1-(1+4\delta)^{1/2})/2 \le c \le (1+(1+4\delta)^{1/2})/2$. If $0 < \delta < \frac{1}{4}$, then $|c-c^2| \le \delta$ if and only if

$$(1-(1+4\delta)^{1/2})/2 \le c \le (1-(1-4\delta)^{1/2})/2$$

or

$$(1+(1-4\delta)^{1/2})/2 \le c \le (1+(1+4\delta)^{1/2})/2.$$

We shall see that these bounds for the constant solutions are, except for one, the optimal bounds for the bounded solutions.

Received by the editors January 6, 1981 and, in revised form, September 4, 1981.
AMS Subject Classification Primary 39C05 Secondary 26D99, 39B20
(1) Research supported by NSERC Grant A7153

Inequality (1) arose in the work of E. Lukacs in probability theory (personal communication). He was interested in real functions f satisfying (1) and the conditions $f(0)=1$ and $|f(x)|\le 1$ for all real x.

In this paper it is shown that if $0<\delta<\frac{3}{4}$, $f: V\to R$ satisfies (1), $f(0)=1$ and $|f(x)|\le 1$ for all $x\in V$ then $1-\delta\le f(x)\le 1$ for all $x\in V$; moreover the bounds are optimal.

THEOREM 1. *Suppose* $\delta>0$, $f: V\to R$ *satisfies* (1) *and* f *is bounded. Then* $-\delta\le f(x)\le(1+(1+4\delta)^{1/2})/2$ *for all* $x\in V$. *Moreover these bounds are optimal.*

Proof. For any $x\in V$

$$-\delta\le f(x)-f(x/2)^2\le\delta$$

so that

$$f(x)\ge f(x/2)^2-\delta\ge-\delta.$$

To demonstrate the other inequality, suppose on the contrary that there exists $\varepsilon>0$ and $a\in V$ such that, $|f(a)|=\rho+\varepsilon$ where $\rho=(1+(1+4\delta)^{1/2})/2$. (Notice $\rho^2-\rho=\delta$.) Then

$$|f(2a)|\ge|f(a)^2|-\delta=(\rho+\varepsilon)^2-\delta=(\rho^2-\delta)+2\varepsilon\rho+\varepsilon^2=\rho+2\varepsilon\rho+\varepsilon^2>\rho+2\varepsilon$$

since $\rho>1$. It follows by induction that $|f(2^n a)|\to+\infty$ as $n\to+\infty$ contradicting the boundedness of f.

We have observed that the upper bound actually determines a constant solution to (1) and hence this bound is optimal. To see that the lower bound is optimal, let $\delta>0$, $0\ne x_0\in V$ and define $f: V\to R$ by letting $f(x)=-\delta$ if $x=x_0$ and $f(x)=0$ if $x_0\ne x\in V$. It is easy to check that this f satisfies (1).

THEOREM 2. *Suppose* $0<\delta<\frac{1}{4}$ *and* $f: V\to R$ *is a bounded solution of* (1). *Then either*

(a) $$-\delta\le f(x)<(1-(1-4\delta)^{1/2})/2,\qquad x\in V$$

or

(b) $$(1+(1-4\delta)^{1/2})/2\le f(x)\le(1+(1+4\delta)^{1/2})/2,\qquad x\in V.$$

Moreover these bounds are optimal.

Proof. Since $|f(0)-f(0)^2|\le\delta<\frac{1}{4}$ either

(i) $$(1-(1+4\delta)^{1/2})/2\le f(0)\le(1+(1+4\delta)^{1/2})/2.$$

or

(ii) $$(1+(1-4\delta)^{1/2})/2\le f(0)\le(1+(1+4\delta)^{1/2})/2.$$

If (i) holds then

$$|f(x)(1-f(0))|=|f(x+0)-f(x)f(0)|\le\delta$$

https://doi.org/10.4153/CMB-1982-071-9 Published online by Cambridge University Press

and

$$1-f(0)\geqslant 1-\{(1-(1-4\delta)^{1/2})/2\}=(1+(1-4\delta)^{1/2})/2$$

so

$$|f(x)|\leq \delta/(1-f(0))\leq 2\delta/(1+(1-4\delta)^{1/2})=(1-(1-4\delta)^{1/2})/2$$

for all $x\in V$. But $f(x)\geq -\delta$ for all $x\in V$ by Theorem 1. Since $\delta<(1-(1-4\delta)^{1/2})/2$, it follows that (a) holds.

Now suppose (ii) holds but $f(x_0)\leq 0$ for some $x_0\in V$. Now $|f(x)f(-x)|\geq f(x)f(-x)\geq f(0)-\delta>\frac{1}{2}-\frac{1}{4}=\frac{1}{4}>0$ for all $x\in V$ so $f(-x_0)<0$ as well. But $|f(x_0)|\leq\frac{1}{4}$ since $-\frac{1}{4}\leq-\delta\leq f(x_0)\leq 0$ so $|f(-x_0)|\geq\frac{1}{4}|f(x_0)|\geq 1$, a contradiction. Thus, if (ii) holds then $f(x)>0$ for all $x\in V$.

Now suppose (ii) holds and let $M=\sup\{f(x):x\in V\}>0$ and choose $\{x_i\}_{i=1}^{\infty}$ in V such that $f(x_i)\to M$ as $i\to\infty$. Now

$$\left|f(x_i)-f\left(\frac{x_i}{2}+x\right)f\left(\frac{x_i}{2}-x\right)\right|\leq\delta$$

so

(2) $$f\left(\frac{x_i}{2}+x\right)\geq (f(x_i)-\delta)/f\left(\frac{x_i}{2}-x\right)\geq (f(x_i)-\delta)/M$$

for all $x\in V$ and all $i=1,2,\ldots$. Replacing x by $y-(x_i/2)$ in (2) we find

$$f(y)\geq (f(x_i)-\delta)/M \quad \text{for all} \quad y\in V \quad \text{and all} \quad i=1,2,\ldots.$$

Hence $f(y)\geq(M-\delta)/M$ for all $y\in V$ so $f(x_i)\geq(M-\delta)/M$ for all $i=1,2,\ldots$ and thus $M^2\geq M-\delta$. Thus $M\geq(1+(1-4\delta)^{1/2})/2$ or $M\leq(1-(1-4\delta)^{1/2})/2<\frac{1}{2}$. But $M\geq f(0)>\frac{1}{2}$ so $M\geq(1+(1-4\delta)^{1/2})/2$. Thus $f(y)\geq(M-\delta)/M=1-(\delta/M)\geq 1-(2\delta/1+(1-4\delta)^{1/2})=(1+(1-4\delta)^{1/2})/2$ for all $y\in V$ and so, by Theorem 1, (b) holds.

The bounds in Theorem 2 are optimal as has been observed.

THEOREM 3. *Suppose* $0<\delta<1$ *and* $f:V\to R$ *satisfies* (1), $f(0)=1$ *and* $|f(x)|\leq 1$ *for all* $x\in V$.

If $\delta<\frac{3}{4}$ *then* $1-\delta\leq f(x)\leq 1$ *for all* $x\in V$.

If $\delta\geq\frac{3}{4}$ *and* $f(x)\geq 0$ *for some* $x\in V$ *then* $1-\delta\leq f(x)\leq 1$.

If $\delta\geq\frac{3}{4}$ *and* $f(x)<0$ *for some* $x\in V$ *then*

$$-\tfrac{1}{2}-(\delta-\tfrac{3}{4})^{1/2}\leq f(x)\leq -\tfrac{1}{2}+(\delta-\tfrac{3}{4})^{1/2}.$$

Proof. Since $|1-f(x)f(-x)|=|f(0)-f(x)f(-x)|\leq\delta$ it follows that

(*) $$f(x)f(-x)\geq 1-\delta>0 \quad \text{for all} \quad x\in V.$$

Hence $f(x)\neq 0$ and

(3) $$|f(x)|\geq(1-\delta)/|f(-x)|\geq 1-\delta>0 \quad \text{for all} \quad x\in V$$

because $0<|f(-x)|\leq 1$.

https://doi.org/10.4153/CMB-1982-071-9 Published online by Cambridge University Press

Now suppose there exists $x \in V$ such that $f(x)<0$. Then, according to (*), $f(-x)<0$. Assume $f(x/2)>0$ (and hence $f(-x/2)>0$). Then

$$|f(-x/2)-f(-x)f(x/2)| \leq \delta$$

and hence, by (*),

$$f(x/2)f(-x) \geq f(-x/2)-\delta \geq \{(1-\delta)/f(x/2)\}-\delta.$$

Hence

$$f(-x) \geq (1-\delta-\delta f(x/2))/(f(x/2))^2. \tag{4}$$

It follows that $1-\delta-\delta f(x/2)<0$ since $f(-x)<0$. From (*), since $f(-x)<0$, we deduce that

$$f(x) \leq (1-\delta)/f(-x)$$

and so, from (4),

$$f(x) \leq (1-\delta)f(x/2)^2/\{1-\delta-\delta f(x/2)\}<0.$$

Thus

$$\begin{aligned}\delta &\geq f(x/2)^2-f(x) \geq f(x/2)^2-[(1-\delta)f(x/2)^2/\{1-\delta-\delta f(x/2)\}]\\ &=-\delta f(x/2)^3/\{1-\delta-\delta f(x/2)\}.\end{aligned}$$

Since $1-\delta-\delta f(x/2)<0$ we have

$$1-\delta-\delta f(x/2) \leq -f(x/2)^3$$

or

$$\delta(1+f(x/2)) \geq 1+f(x/2)^3.$$

Since $f(x/2) \geq 0$ we find from the last inequality that

$$\delta \geq 1-f(x/2)+f(x/2)^2$$

of

$$(f(x/2)-\tfrac{1}{2})^2+(\tfrac{3}{4}-\delta) \leq 0 \tag{**}$$

from which it is clear that $\delta \geq \frac{3}{4}$.

Similarly, if $f(x/2)<0$ we deduce by a similar argument that

$$(f(x/2)+\tfrac{1}{2})^2+(\tfrac{3}{4}-\delta) \leq 0$$

and again conclude that $\delta \geq \frac{3}{4}$.

We have shown that if $f(x)<0$ for some $x \in V$ then $\delta \geq \frac{3}{4}$. Thus the first assertion of the Theorem follows from (3) as does the second.

To check the third assertion, suppose $f(x)<0$ for some $x \in V$ and $f(x/2)>0$.

https://doi.org/10.4153/CMB-1982-071-9 Published online by Cambridge University Press

Then (**) holds and so

$$|f(x/2)-\tfrac{1}{2}|\leq(\delta-\tfrac{3}{4})^{1/2}.$$

Hence

$$f(x/2)\geq\tfrac{1}{2}-(\delta-\tfrac{3}{4})^{1/2}>0\quad(\text{since } \tfrac{3}{4}\leq\delta<1).$$

But $f(x)\geq f(x/2)^2-\delta$ so

$$f(x)\geq(\tfrac{1}{2}-(\delta-\tfrac{3}{4})^{1/2})^2-\delta=-\tfrac{1}{2}-(\delta-\tfrac{3}{4})^{1/2}.$$

Similarly

$$f(-x)\geq-\tfrac{1}{2}-(\delta-\tfrac{3}{4})^{1/2}.$$

But $f(-x)<0$ and so

$$f(x)\leq(1-\delta)/f(-x)\leq(1-\delta)/(-\tfrac{1}{2}-(\delta-\tfrac{3}{4})^{1/2})=-\tfrac{1}{2}+(\delta-\tfrac{3}{4})^{1/2}.$$

A similar argument applies in case $f(x)<0$ and $f(x/2)<0$ to complete the proof.

To see that the estimates in the first assertion are optimal consider the function $f:R\to R$ defined by letting $f(x)=1$ for $x\geq0$ and $f(x)=1-\delta$ for $x<0$.

A continuous, monotonic example can be constructed by letting $f(x)=1$ for $x\geq0$ and $f(x)=\delta\exp(x)+1-\delta$ for $x<0$ where $0<\delta<1$. This f satisfies (1) $f(0)=1$, $1-\delta<f(x)\leq1$ for all $x\in R$ and $\inf\{f(x):x\in R\}=1-\delta$.

We doubt, but haven't been able to prove, that the bounds in the last assertion of Theorem 3 are optimal. However, if $\tfrac{3}{4}\leq\delta<1$ and $f:V\to R$ is defined by letting $f(0)=1$ and $f(x)=-\tfrac{1}{2}$ for $0\neq x\in V$ then the assumptions of Theorem 3 are satisfied.

This shows that such functions may assume negative values.

References

1. John Baker, J. Lawrence and F. Zorzitto, *The Stability of the Equation* $f(x+y)=f(x)f(y)$, Proc. Amer. Math. Soc. **74** (1979), 242–246.

2. John A. Baker, *The Stability of the Cosine Equation*, Proc. Amer. Math. Soc. **80** (1980), 411–416.

Department of Pure Mathematics
University of Waterloo
Waterloo, Ontario
Canada, N2L 3G1