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FROM SURFACES IN THE 5-SPHERE TO 3-MANIFOLDS IN COMPLEX PROJECTIVE 3-SPACE

J. BOLTON, C. SCHARLACH AND L. VRANCKEN

In a previous paper it was shown how to associate with a Lagrangian submanifold satisfying Chen's equality in 3-dimensional complex projective space, a minimal surface in the 5-sphere with ellipse of curvature a circle. In this paper we focus on the reverse construction.

1. INTRODUCTION

It was proved in [7] that at each point p of a totally real submanifold M^n of a holomorphic space form $\widetilde{M}(4c)$ of constant holomorphic sectional curvature $4c$ we have

$$\delta_M(p) \leqslant \frac{n^2(n-2)}{2(n-1)} H^2(p) + \frac{1}{2}(n+1)(n-2)c, \tag{1}$$

where H denotes the length of the mean curvature vector and δ_M is the Riemannian invariant introduced by Chen in [6], defined by

$$\delta_M(p) = \tau(p) - (\inf K)(p).$$

Here

$$(\inf K)(p) = \inf \{K(\pi) \mid \pi \text{ is a 2-dimensional subspace of } T_pM\},$$

where $K(\pi)$ is the sectional curvature of π, and $\tau(p) = \sum_{i<j} K(e_i \wedge e_j)$ denotes the scalar curvature defined in terms of an orthonormal basis $\{e_1, \dots, e_n\}$ of the tangent space T_pM.

Then M^n is said to satisfy Chen's equality if equality is attained in (1) for each $p \in M$. In the case where $n = 3$ and the surrounding space is $\mathbb{C}^3$ this corresponds to one of the classes of Lagrangian submanifolds studied by Bryant in [5].

In a previous paper [2] we gave a local construction which associated to a Lagrangian submanifold satisfing Chen's equality but having no totally geodesic points in complex

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projective space $\mathbb{C}P^3(4)$, a minimal surface in $S^5(1)$ with ellipse of curvature a circle. In this paper, we focus on the reverse construction.

In Section 2 we consider the case in which a minimal surface with ellipse of curvature a circle is contained in a totally geodesic $S^4(1)$ of $S^5(1)$. The immersion is then superminimal [**4**], and our construction in this case is based on the well known correspondence [**4**] between superminimal surfaces in $S^4(1)$ and horizontal holomorphic curves in $\mathbb{C}P^3(4)$.

In Section 3, which is the main part of the paper, we consider the case of a linearly full minimal surface in $S^5(1)$ whose ellipse of curvature is a circle. Here we use the theory of harmonic sequences to show how to construct locally a submanifold M^3 of $SO(6)$ whose Maurer-Cartan equations coincide with equations (9) to (14) of Section 4 of [**2**]. Then, since $SU(4)$ is a double cover of $SO(6)$, we obtain a local lift into $SU(4)$ for which projection onto the first column defines a Lagrangian immersion of M^3 into $\mathbb{C}P^3(4)$ satisfying Chen's equality. It will be apparent that the constructions described in this paper provide a local inverse of the construction described in [**2**].

2. Superminimal surfaces in $S^4(1)$

In this section we assume that N^2 is an oriented surface superminimally immersed in $S^4(1)$. The orientation, together with the metric induced on N^2, enables us to give N^2 the structure of a Riemann surface in such a way that the immersion is conformal.

We first recall the following result of Bryant [**4**] relating superminimal immersions of N^2 into $S^4(1)$ to holomorphic horizontal immersions of N^2 into $\mathbb{C}P^3(4)$.

Theorem 1. (Bryant) *Let $T : \mathbb{C}P^3(4) \to S^4(1)$ be the twistor fibration and let $\phi : N^2 \to S^4(1)$ be a superminimal immersion of a simply connected Riemann surface. Then there exists a unique horizontal holomorphic immersion $\widetilde{\phi} : N^2 \to \mathbb{C}P^3(4)$ such that $T \circ \widetilde{\phi} = \pm\phi$.*

Conversely if $\widetilde{\phi} : N^2 \to \mathbb{C}P^3(4)$ is a horizontal holomorphic curve, then $T \circ \widetilde{\phi} : N^2 \to S^4(1)$ is a (possibly branched) superminimal immersion.

Now, let $\widetilde{\phi} : N^2 \to \mathbb{C}P^3(4)$ be a horizontal holomorphic curve defined on a simply connected Riemann surface N^2 and let $p_i : S^7(1) \to \mathbb{C}P^3(4)$ denote the Hopf fibration determined by the complex structure on $\mathbb{R}^8 = \mathbb{C}^4$ given by multiplication by i. It is clear that the natural immersion ψ of the pullback bundle $M^3 = \widetilde{\phi}^*(S^7(1))$, defined so that the following diagram commutes, is invariant (and hence minimal) in the Sasakian space form $(S^7(1), I, \langle .,. \rangle)$. Here, I is the Sasakian structure determined on $S^7(1)$ by multiplication by i on $\mathbb{R}^8 = \mathbb{C}^4$.

$$\begin{array}{ccc} M^3 & \xrightarrow{\psi} & S^7(1) \subset \mathbb{C}^4 = \mathbb{H}^2 \\ \downarrow & & \downarrow p_i \\ N^2 & \xrightarrow{\widetilde{\phi}} & \mathbb{C}P^3(4). \end{array}$$

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In fact, we may use multiplication by i, j, k on $\mathbb{R}^8 = \mathbb{H}^2$ to define corresponding Hopf fibrations of $S^7(1)$ over $\mathbb{C}P^3(4)$, and we let $p_j : S^7(1) \to \mathbb{C}P^3$ be the one determined by multiplication by j. Since $\widetilde{\phi}$ is horizontal and holomorphic, the immersion ψ is horizontal with respect to p_j [**1**] and so we may apply the following special case of a theorem of Reckziegel [**9**].

THEOREM 2. (Reckziegel) *Let $\psi : M^3 \to S^7(1) \subset \mathbb{C}^4$ be an immersion which is horizontal with respect to the Hopf fibration $p_j : S^7(1) \to \mathbb{C}P^3$. Then $p_j\psi : M^3 \to \mathbb{C}P^3(4)$ is a Lagrangian immersion which is minimal if and only if ψ is minimal.*

Conversely, let $\widetilde{\psi} : M^3 \to \mathbb{C}P^3(4)$ be a Lagrangian immersion of a connected, simply connected manifold M^3. Then there exists a map $\psi : M^3 \to S^7(1)$, which is horizontal with respect to p_j, such that $p_j\psi = \widetilde{\psi}$. Moreover, any two such lifts ψ_1 and ψ_2 are related by $\psi_2 = e^{i\theta}\psi_1$ where θ is a constant.

Hence, combining the above two theorems, we see that starting from a superminimal immersion $\phi : N^2 \to S^4(1)$, we obtain a minimal Lagrangian immersion $p_j\psi : M^3 \to \mathbb{C}P^3(4)$. Note that $i\psi$ is tangential to the immersion ψ of M^3 into $S^7(1)$, and if D denotes the standard flat connection on $\mathbb{R}^8$ then for X tangential to M,

$$D_X(i\psi) = iD_X\psi = iX.$$

Hence if h denotes the second fundamental form of ψ in $S^7(1)$, we see that $h(.,i\psi) = 0$. It then follows from [**7**] and [**8**] that $p_j\psi : M^3 \to \mathbb{C}P^3(4)$ satisfies Chen's equality. Moreover, it is clear that if we apply the construction of [**2**] to $p_j\psi$ we recover the immersion ϕ.

3. Linearly full minimal surfaces in $S^5(1)$

Let $f : N^2 \to S^5(1)$ be a minimal immersion of an oriented surface. As in Section 2, we use the orientation and induced metric to give N^2 the structure of a Riemann surface in such a way that f is a conformal immersion. If II denotes the second fundamental form of f in S^5 we recall that the image under II of the unit circle in a tangent space of N^2 is a (possibly degenerate) ellipse called the *ellipse of curvature*.

From now on, we assume that $f : N^2 \to S^5(1)$ is a linearly full minimal immersion of an oriented surface with ellipse of curvature a non-degenerate circle at each point. We now show how to locally associate to such an immersion a unitary moving frame. The approach we follow here is based on the theory of harmonic sequences, which we describe briefly below for the special case of minimal surfaces in $S^5(1)$ with ellipse of curvature a circle. The reader is referred to [**3**] for more details in the general situation of minimal surfaces in $S^m(1)$ or $\mathbb{C}P^m(4)$.

Let $z = x + iy$ be a local complex coordinate on N^2, and denote $\dfrac{\partial}{\partial z}$ by ∂ and $\dfrac{\partial}{\partial \bar{z}}$

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by $\overline{\partial}$. We introduce $\mathbb{C}^6$-valued functions f_0, f_1, f_2 by

$$f_0 = f, \tag{2}$$

$$f_1 = \partial f, \tag{3}$$

$$f_2 = II(\partial, \partial), \tag{4}$$

where II now denotes the complex bilinear extension of the second fundamental form of f in $S^5(1)$. If $(\quad,\quad)$ is the complex bilinear extension of the standard inner product on $\mathbb{R}^6$, it follows that $(f_0, f_1) = 0$ while conformality of f is equivalent to

$$(f_1, f_1) = 0. \tag{5}$$

Thus f_0, f_1, $\overline{f}_1$ are mutually unitarily orthogonal and f_2 is the component of ∂f_1 unitarily orthogonal to f_0, f_1, $\overline{f}_1$.

If $f_2 = a - ib$ where a, b are $\mathbb{R}^7$ valued functions then, using minimality of f,

$$II\Big(\cos\phi\frac{\partial}{\partial x} + \sin\phi\frac{\partial}{\partial y}, \cos\phi\frac{\partial}{\partial x} + \sin\phi\frac{\partial}{\partial y}\Big) = 2(a\cos 2\phi + b\sin 2\phi),$$

so that the ellipse of curvature is a circle if and only if

$$f_2 \neq 0 \quad \text{and} \quad (f_2, f_2) = 0, \tag{6}$$

so that in this case f_2 and $\overline{f}_2$ are unitarily orthogonal. Hence, $f_0, f_1, \overline{f}_1, f_2, \overline{f}_2$ are mutually unitarily orthogonal non-zero vectors.

Finally, we define f_3 to be the component of ∂f_2 which is unitarily orthogonal to $\{f_0, f_1, \overline{f}_1, f_2, \overline{f}_2\}$. As the immersion is contained in $S^5(1)$, we deduce that f_3 and $\overline{f}_3$ are linearly dependent.

By Takahashi's Lemma, the minimality condition for f may be written as $\partial\overline{\partial} f_0 = \lambda f_0$ for some $\lambda \in \mathbb{R}$, and an inductive argument readily shows that if we put $w_p = \log|f_p|$, $\quad p = 1, 2, 3$, then

$$\partial f_0 = f_1, \tag{7}$$

$$\partial f_1 = f_2 + 2\partial w_1 f_1, \tag{8}$$

$$\partial f_2 = f_3 + 2\partial w_2 f_2, \tag{9}$$

while

$$\overline{\partial} f_1 = -e^{2w_1} f_0, \tag{10}$$

$$\overline{\partial} f_2 = -e^{2(w_2 - w_1)} f_1, \tag{11}$$

$$\overline{\partial} f_3 = -e^{2(w_3 - w_2)} f_2. \tag{12}$$

So far, everything is valid for an arbitrary choice of local complex coordinate but we now pick a special coordinate to facilitate calculations. It follows from (12) that

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$(\overline{\partial} f_3, f_3) = 0$, so that $(f_3, f_3)dz^6$ is a holomorphic differential on N^2. Hence, away from the isolated points at which $f_3 = 0$, we can choose a local complex coordinate z for which

$$(f_3, f_3) = 1, \tag{13}$$

so that

$$f_3 \text{ is real and } w_3 = 0. \tag{14}$$

We now introduce a unitary moving frame $\{F_0, \ldots, F_5\}$ by setting $F_0 = f_0$, $F_1 = e^{-w_1} f_1$, $F_2 = e^{-w_2} f_2$, $F_3 = f_3$, $F_{-1} = -\overline{F}_1$ and $F_{-2} = \overline{F}_2$ (the minus sign in the definition of F_{-1} is there for reasons connected with the theory of harmonic sequences, and makes no essential difference in the present paper). A straightforward computation shows that

$$dF_0 = e^{w_1} dz F_1 - e^{w_1} d\overline{z} F_{-1}, \tag{15}$$

$$dF_1 = -e^{w_1} d\overline{z} F_0 + (\partial w_1 dz - \overline{\partial} w_1 d\overline{z}) F_1 + e^{w_2 - w_1} dz F_2, \tag{16}$$

$$dF_{-1} = e^{w_1} dz F_0 + (-\partial w_1 dz + \overline{\partial} w_1 d\overline{z}) F_{-1} - e^{w_2 - w_1} d\overline{z} F_{-2}, \tag{17}$$

$$dF_2 = -e^{w_2 - w_1} d\overline{z} F_1 + (\partial w_2 dz - \overline{\partial} w_2 d\overline{z}) F_2 + e^{-w_2} dz F_3, \tag{18}$$

$$dF_{-2} = e^{w_2 - w_1} dz F_{-1} + (-\partial w_2 dz + \overline{\partial} w_2 d\overline{z}) F_{-2} + e^{-w_2} d\overline{z} F_3, \tag{19}$$

$$dF_3 = -e^{-w_2} d\overline{z} F_2 - e^{-w_2} dz F_{-2}. \tag{20}$$

We now consider the manifold W of unitary frames $\{V_0, V_1, V_{-1}, V_2, V_{-2}, V_3\}$ of the form

$$\{V_0, V_1, V_{-1}, V_2, V_{-2}, V_3\} = \{F_0, e^{i\alpha} F_1, e^{-i\alpha} F_{-1}, e^{i\beta} F_2, e^{-i\beta} F_{-2}, F_3\}, \quad \alpha,\ \beta \in \mathbb{R}.$$

Thus, we may regard W as the bundle of strongly adapted unitary frames over N^2, in that V_1 (*respectively* V_2) spans the (1,0) component of the complexified tangent space (*respectively* first normal space) of N^2. If we use $z = x + iy$, α and β as local coordinates on W, it follows easily from (15)-(20) that

$$dV_0 = e^{w_1 - i\alpha} dz V_1 - e^{w_1 + i\alpha} d\overline{z} V_{-1}, \tag{21}$$

$$dV_1 = -e^{w_1 + i\alpha} d\overline{z} V_0 + (\partial w_1 dz - \overline{\partial} w_1 d\overline{z} + i d\alpha) V_1 + e^{w_2 - w_1 - i(\beta - \alpha)} dz V_2, \tag{22}$$

$$dV_{-1} = e^{w_1 - i\alpha} dz V_0 + (-\partial w_1 dz + \overline{\partial} w_1 d\overline{z} - i d\alpha) V_{-1} - e^{w_2 - w_1 + i(\beta - \alpha)} d\overline{z} V_{-2}, \tag{23}$$

$$dV_2 = -e^{w_2 - w_1 - i(\alpha - \beta)} d\overline{z} V_1 + (\partial w_2 dz - \overline{\partial} w_2 d\overline{z} + i d\beta) V_2 + e^{-w_2 + i\beta} dz V_3, \tag{24}$$

$$dV_{-2} = e^{w_2 - w_1 + i(\alpha - \beta)} dz V_{-1} + (-\partial w_2 dz + \overline{\partial} w_2 d\overline{z} - i d\beta) V_{-2} + e^{-w_2 - i\beta} d\overline{z} V_3, \tag{25}$$

$$dV_3 = -e^{-w_2 - i\beta} d\overline{z} V_2 - e^{-w_2 + i\beta} dz V_{-2}. \tag{26}$$

We now wish to compare the above formulae to those obtained in Section 4 of [**2**]. We recall that there, with a Lagrangian submanifold M^3 of $\mathbb{C}P^3$ satisfying Chen's equality but having no totally geodesic points, we locally associated a smooth map $\{U_0, \ldots, U_5\} : M^3 \to SO(6)$ such that

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(i) the image of U_0 is a minimal surface in $S^5(1)$,
(ii) U_1 and U_2 span the tangent space to this surface,
(iii) U_3 and U_4 span the first normal space to this surface,
(iv) U_5 is the remaining orthogonal vector such that $\det(U_0, \dots, U_5) = 1$.

We now write

$$\begin{aligned}
\widetilde{U}_0 &= U_0, \\
\widetilde{U}_1 &= \frac{1}{\sqrt{2}}(U_1 - i\varepsilon_1 U_2), \\
\widetilde{U}_{-1} &= -\frac{1}{\sqrt{2}}(U_1 + i\varepsilon_1 U_2), \\
\widetilde{U}_2 &= \frac{1}{\sqrt{2}}(U_3 - i\varepsilon_2 U_4), \\
\widetilde{U}_{-2} &= \frac{1}{\sqrt{2}}(U_3 + i\varepsilon_2 U_4), \\
\widetilde{U}_3 &= U_5,
\end{aligned}$$

where $\varepsilon_1, \varepsilon_2 = \pm 1$ will be chosen later. If we now rewrite equations (9)-(14) of Section 4 of [**2**] with respect to this frame, we find that for suitably chosen functions a, b, c, d and orthonormal basis $\{\theta_1, \theta_2, \theta_3\}$ of local 1-forms on M we have

$$d\widetilde{U}_0 = b_{10}\widetilde{U}_1 + b_{-10}\widetilde{U}_{-1}, \tag{27}$$

$$d\widetilde{U}_1 = -\bar{b}_{10}\widetilde{U}_0 + i\varepsilon_1\Big(c\theta_1 + d\theta_2 + \Big(1 - \frac{1}{3}b\Big)\theta_3\Big)\widetilde{U}_1 + b_{21}\widetilde{U}_2 + b_{-21}\widetilde{U}_{-2}, \tag{28}$$

$$d\widetilde{U}_{-1} = -\bar{b}_{-10}\widetilde{U}_0 - i\varepsilon_1\Big(c\theta_1 + d\theta_2 - \Big(1 - \frac{1}{3}b\Big)\theta_3\Big)\widetilde{U}_{-1} + b_{2-1}\widetilde{U}_2 + b_{-2-1}\widetilde{U}_{-2}, \tag{29}$$

$$d\widetilde{U}_2 = -\bar{b}_{21}\widetilde{U}_1 - \bar{b}_{2-1}\widetilde{U}_{-1} + i\varepsilon_2\Big(c\theta_1 + d\theta_2 - \Big(1 + \frac{1}{3}b\Big)\theta_3\Big)\widetilde{U}_2 + b_{32}\widetilde{U}_3, \tag{30}$$

$$d\widetilde{U}_{-2} = -\bar{b}_{-21}\widetilde{U}_1 - \bar{b}_{-2-1}\widetilde{U}_{-1} - i\varepsilon_2\Big(c\theta_1 + d\theta_2 - \Big(1 + \frac{1}{3}b\Big)\theta_3\Big)\widetilde{U}_{-2} + b_{3-2}\widetilde{U}_3, \tag{31}$$

$$d\widetilde{U}_3 = -\bar{b}_{32}\widetilde{U}_2 - \bar{b}_{3-2}\widetilde{U}_{-2}, \tag{32}$$

where there exists a positive function λ such that

$$\begin{aligned}
b_{10} = -\bar{b}_{-10} &= \frac{1}{\sqrt{2}}\big(-a\theta_1 + (1+b)\theta_2\big) - i\frac{\varepsilon_1}{\sqrt{2}}\big((1+b)\theta_1 + a\theta_2\big), \\
b_{21} = -\bar{b}_{-2-1} &= \frac{1}{2}\lambda\big((1 - \varepsilon_1\varepsilon_2)\theta_1 + i(\varepsilon_1 - \varepsilon_2)\theta_2\big), \\
b_{-21} = -\bar{b}_{2-1} &= \frac{1}{2}\lambda\big((1 + \varepsilon_1\varepsilon_2)\theta_1 + i(\varepsilon_1 + \varepsilon_2)\theta_2\big), \\
b_{32} = \bar{b}_{3-2} &= \frac{1}{\sqrt{2}}\big(a\theta_1 + (1-b)\theta_2\big) + i\frac{\varepsilon_2}{\sqrt{2}}\big((1-b)\theta_1 - a\theta_2\big).
\end{aligned}$$

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We now find a hypersurface $\widehat{M^3}$ of the manifold W of strongly adapted unitary frames over N^2 described above, together with linearly independent local 1-forms θ_1, θ_2, θ_3, and local functions $\lambda > 0, a, b, c, d$ defined on $\widehat{M^3}$, such that the systems (21)-(26) and (27)-(32) of differential equations coincide.

So, assume that $a_{\ell k}$ (respectively $b_{\ell k}$) are the components of dV_k (respectively $d\widetilde{U}_k$) in the direction of V_ℓ (respectively $\widetilde{U}_\ell$). As $a_{-21} = 0$, it follows that we need $b_{-21} = 0$ and thus

$$\varepsilon_1 \varepsilon_2 = -1. \tag{33}$$

Next, we find that if we require that

$$b_{21} = a_{21},$$
$$b_{10} + b_{32} = a_{10} + a_{32},$$

then we need that

$$\lambda(\theta_1 + i\varepsilon_1\theta_2) = e^{w_2 - w_1 + i(\alpha-\beta)} dz, \tag{34}$$

$$\sqrt{2}(\theta_2 - i\varepsilon_1\theta_1) = (e^{w_1 - i\alpha} + e^{-w_2 + i\beta}) dz. \tag{35}$$

Hence, we see that the positive function λ must satisfy

$$\lambda(e^{w_1 - i\alpha} + e^{-w_2 + i\beta}) + \sqrt{2} i \varepsilon_1 e^{w_2 - w_1 + i(\alpha-\beta)} = 0$$

which, as λ is real, implies that

$$\lambda(e^{w_1 + i\alpha} + e^{-w_2 - i\beta}) - \sqrt{2} i \varepsilon_1 e^{w_2 - w_1 - i(\alpha-\beta)} = 0.$$

It follows from the two previous equations that the following conditions need to be satisfied:

$$0 \neq (e^{w_1 - i\alpha} + e^{-w_2 + i\beta}), \tag{36}$$

$$\lambda = -\sqrt{2} i \varepsilon_1 \frac{e^{w_2 - w_1 + i(\alpha-\beta)}}{(e^{w_1 - i\alpha} + e^{-w_2 + i\beta})}, \tag{37}$$

$$e^{w_2} \cos(2\alpha - \beta) + e^{-w_1} \cos(2\beta - \alpha) = 0, \tag{38}$$

where $\varepsilon_1 = \pm 1$ is determined by the requirement that λ be positive.

Lemma 1. *The conditions* (36) *and* (38) *determine a hypersurface* $\widehat{M^3}$ *of* W, *which may be parametrized by* z *and* $t = \alpha + \beta$.

Proof: We first introduce new coordinates s and t on W by

$$s = \alpha - \beta,$$
$$t = \alpha + \beta.$$

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Then (38) becomes

$$e^{w_2}\cos\Big(\frac{1}{2}t+\frac{3}{2}s\Big)+e^{-w_1}\cos\Big(\frac{1}{2}t-\frac{3}{2}s\Big)=0,$$

which we can rewrite as

$$(e^{w_2}+e^{-w_1})\cos\Big(\frac{1}{2}t\Big)\cos\Big(\frac{3}{2}s\Big)=(e^{w_2}-e^{-w_1})\sin\Big(\frac{1}{2}t\Big)\sin\Big(\frac{3}{2}s\Big).$$

It then follows that

$$\cot\Big(\frac{3}{2}s\Big)=\frac{e^{w_1+w_2}-1}{e^{w_1+w_2}+1}\tan\frac{1}{2}t. \tag{39}$$

To determine s explicitly in terms of t (up to an initial condition), we differentiate (39) with respect to t and find that

$$s'(t)=-\frac{1}{3}\frac{e^{2(w_1+w_2)}-1}{e^{2(w_1+w_2)}+1+2e^{(w_1+w_2)}\cos t}. \tag{40}$$

The denominator of the right hand side vanishes only if $w_1+w_2=0$ and $t=(2k+1)\pi$, $k\in\mathbb{Z}$, which is excluded by (36). The function $s(t)$ is now determined (up to a addition of an integer multiple of $(2\pi)/3$) by the condition that $\cos\big((3/2)s\big)=0$ when t is an integer multiple of 2π. □

We now compute the 1-forms θ_1, θ_2, θ_3 and the function λ on $\widehat{M}^3$. As λ is real valued, we see using (37) that

$$\begin{aligned}\lambda^2&=\lambda\overline{\lambda}\\&=\frac{2e^{2(w_2-w_1)}}{(e^{w_1-i\alpha}+e^{-w_2+i\beta})(e^{w_1+i\alpha}+e^{-w_2-i\beta})}\\&=\frac{2e^{2(w_2-w_1)}}{e^{2w_1}+e^{-2w_2}+2e^{w_1-w_2}\cos t}\\&=\frac{e^{3(w_2-w_1)}}{\cosh(w_1+w_2)+\cos t}.\end{aligned}$$

Hence, as λ is positive, it follows that

$$\lambda=\frac{e^{3(w_2-w_1)/2}}{\sqrt{\cosh(w_1+w_2)+\cos t}}. \tag{41}$$

From (35), we obtain

$$\sqrt{2}(\theta_2-i\varepsilon_1\theta_1)=\big(e^{w_1-(i(s+t))/2}+e^{-w_2+(i(t-s))/2}\big)dz, \tag{42}$$

which determines the 1-forms θ_1 and θ_2. The 1-form θ_3 is determined by the condition that

$$a_{11}+a_{22}=b_{11}+b_{22}.$$

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Indeed, taking into (33) into account, it follows that

$$\theta_3 = -i\varepsilon_1(\partial(w_1+w_2)dz - \overline{\partial}(w_1+w_2)d\overline{z}) + \frac{1}{2}\varepsilon_1\,dt. \tag{43}$$

We may proceed in two different ways in order to obtain a Lagrangian immersion of $\widehat{M}^3$ into $\mathbb{C}P^3(4)$. The first possibility is to use the following existence and uniqueness result of [**8**].

Theorem 3. *Let $(M^n, \langle .,. \rangle)$ be an n-dimensional simply connected Riemannian manifold. Let σ be a symmetric bilinear vector-valued form on M^n satisfying*

(i) *$\langle \sigma(X,Y), Z\rangle$ is totally symmetric,*

(ii) *$(\nabla\sigma)(X,Y,Z) = \nabla_X\sigma(Y,Z) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z)$ is totally symmetric,*

(iii) *$R(X,Y)Z = \langle Y,Z\rangle X - \langle X,Z\rangle Y + \sigma(\sigma(Y,Z),X) - \sigma(\sigma(X,Z),Y)$.*

Then there exists a Lagrangian isometric immersion $x : (M^n, \langle .,. \rangle) \longrightarrow \mathbb{C}P^n(4)$ such that the second fundamental form h satisfies $h(X,Y) = J\sigma(X,Y)$. Moreover, x is determined uniquely modulo holomorphic isometries of $\mathbb{C}P^n(4)$.

The above result may be applied in the following way. We start with the minimal surface N^2 equipped with the special local complex coordinate z chosen so that (13) holds. We consider the 3-dimensional manifold $\widehat{M}^3$ of W constructed in Lemma 1, but excluding those points where $w_1(z) + w_2(z) = 0$ and $\cos t = -1$. We define 1-forms θ_1, θ_2 and θ_3 on $\widehat{M}^3$ using (42) and (43), where ε_1 is determined by the two equations (37) and (41) for λ and the initial condition chosen for s. We denote the dual vector fields corresponding to these 1-forms by E_1, E_2 and E_3 and define a metric on $\widehat{M}^3$ by requiring that E_1, E_2 and E_3 form an orthonormal moving frame on $\widehat{M}^3$. We define a positive function λ on $\widehat{M}^3$ by (41) and introduce a symmetric bilinear vector valued form σ on $\widehat{M}^3$ by

$$\begin{aligned} \sigma(E_1,E_1) &= \lambda E_1, & \sigma(E_1,E_3) &= 0,\\ \sigma(E_1,E_2) &= -\lambda E_2, & \sigma(E_2,E_3) &= 0,\\ \sigma(E_2,E_2) &= -\lambda E_1, & \sigma(E_3,E_3) &= 0. \end{aligned}$$

It is then straighforward to compute that all the conditions of Theorem 3 are satisfied and hence there exists a Lagrangian immersion with the desired properties of $\widehat{M}^3$ into $\mathbb{C}P^3(4)$.

The second way to proceed is to continue with the comparison of the systems (21)–(26) and (27)–(32) in order to determine the functions a, b, c and d explicitly. The requirement that

$$a_{10} - a_{32} = b_{10} - b_{32},$$

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necessitates that

$$\begin{aligned}(e^{w_1-i\alpha}-e^{-w_2+i\beta})dz &= -\sqrt{2}(a+i\varepsilon_1 b)(\theta_1+i\varepsilon_1\theta_2)\\ &= -i\varepsilon_1\sqrt{2}(\theta_2-i\varepsilon_1\theta_1)(a+i\varepsilon_1 b)\\ &= -i\varepsilon_1(a+i\varepsilon_1 b)\big(e^{w_1-(i(s+t))/2}+e^{-w_2+\frac{1}{2}i(t-s)}\big)dz,\end{aligned}$$

where we have used (42) for the final equality. Hence,

$$\begin{aligned}(a+i\varepsilon_1 b) &= i\varepsilon_1\frac{\big(e^{w_1-(i(t+s))/2}-e^{-w_2+(i(t-s))/2}\big)}{\big(e^{w_1-(i(s+t))/2}+e^{-w_2+(i(t-s))/2}\big)}\\ &= i\varepsilon_1\frac{\big(e^{w_1+w_2}-e^{it}\big)}{\big(e^{w_1+w_2}+e^{it}\big)},\end{aligned}$$

which determines a and b. Specifically, we have that

$$\begin{aligned}a &= 2\varepsilon_1\frac{e^{w_1+w_2}\sin t}{e^{2(w_1+w_2)}+1+2e^{w_1+w_2}\cos t},\\ b &= -\frac{1-e^{2(w_1+w_2)}}{e^{2(w_1+w_2)}+1+2e^{w_1+w_2}\cos t}.\end{aligned}$$

Finally, in order to obtain c and d, we consider the condition that

$$a_{11}-a_{22}=b_{11}-b_{22}.$$

This yields

$$\partial(w_1-w_2)dz-\overline{\partial}(w_1-w_2)d\overline{z}+ids=2\varepsilon_1\Big(c\theta_1+d\theta_2-\frac{1}{3}b\theta_3\Big)$$

or, equivalently,

$$\begin{aligned}\partial(w_1-w_2)dz-\overline{\partial}(w_1-w_2)d\overline{z}-\frac{1}{3}i\frac{e^{2(w_1+w_2)}-1}{e^{2(w_1+w_2)}+1+2e^{(w_1+w_2)}\cos t}\,dt\\ = 2\varepsilon_1\Big(c\theta_1+d\theta_2+\frac{1}{3}\frac{1-e^{2(w_1+w_2)}}{e^{2(w_1+w_2)}+1+2e^{w_1+w_2}\cos t}\theta_3\Big).\end{aligned}$$

Using (43) the above equation gives

$$\begin{aligned}&\partial(w_1-w_2)dz-\overline{\partial}(w_1-w_2)d\overline{z}\\ &= 2\varepsilon_1\Big(c\theta_1+d\theta_2-\frac{1}{3}i\varepsilon_1\frac{1-e^{2(w_1+w_2)}}{e^{2(w_1+w_2)}+1+2e^{w_1+w_2}\cos t}\big(\partial(w_1+w_2)dz-\overline{\partial}(w_1+w_2)d\overline{z}\big)\Big).\end{aligned}$$

However, it follows from (35) that dz and $d\overline{z}$ may be expressed as linear combinations of θ_1 and θ_2, so that c and d are uniquely determined by the above equation. It is now straightforward to check that the systems (21)-(26) and (27)-(32) coincide. Therefore, using the double cover of $SO(6)$ by $SU(4)$ as described in Section 4 of [**2**], we obtain a Lagrangian immersion satisfying Chen's equality.

Again, it is clear that if we apply the construction of [**2**] to this Lagrangian immersion, we obtain the linearly full minimal immersion $f : N^2 \to S^5(1)$ from which we started.

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Departmentof Mathematical Sciences
University of Durham
Durham DH1 3LE
United Kingdom
e-mail: john.bolton@dur.ac.uk

Fachbereich Mathematik, Sekr. MA 8-3
Technische Universität Berlin
Strasse des 17 Juni 136
10623 Berlin
Germany
e-mail: cs@sfb288.math.tu-berlin.de

Laboratoire de Mathématiques
LAMATH, ISTV2
Université de Valenciennes
59313 Valenciennes Cedex 9
France
e-mail: luc.vrancken@univ-valenciennes.fr

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