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MULTIVARIATE REGULARLY VARYING INSURANCE AND FINANCIAL RISKS IN MULTIDIMENSIONAL RISK MODELS

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Abstract

Multivariate regular variation is a key concept that has been applied in finance, insurance, and risk management. This paper proposes a new dependence assumption via a framework of multivariate regular variation. Under the condition that financial and insurance risks satisfy our assumption, we conduct asymptotic analyses for multidimensional ruin probabilities in the discrete-time and continuous-time cases. Also, we present a two-dimensional numerical example satisfying our assumption, through which we show the accuracy of the asymptotic result for the discrete-time multidimensional insurance risk model.

Keywords: Ruin probability; dependence; asymptotics; multivariate regular variation

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1. Introduction

Consider an insurance company that invests an asset in a financial market. Such a situation carries two kinds of risk, as summarized by [24] and [32]. One kind, called insurance risk in the literature, stems from insurance claims, and the other, called financial risk, stems from financial investments. These two types of risk should be dealt with carefully in conducting solvency assessments of insurance companies. In this paper, we use multidimensional risk models to accommodate the two types of risks in the discrete-time and continuous-time cases.

Major events, such as the COVID-19 crisis and natural catastrophes, significantly affect almost all economic sectors. This applies in particular to financial markets and the insurance industry, implying that it makes sense to consider interdependencies between financial and insurance risks. In this paper, we propose to describe the dependence structure using Assumption 1 within a framework of general multivariate regular variation (MRV). Under certain conditions, multiple dependence structures satisfy Assumption 1, such as those of Proposition 2 and the relation (14) in this paper. Moreover, Assumption 1 includes scale mixtures, which have a variety of interpretations in different applications. For instance, they can

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reflect both individual and common factors of a risk class, and they can describe loss variables. However, the implications of Assumption 1 are not confined to this, and one can find available information after carefully studying Assumption 1.

Since the pioneering work of [24] and [32], various studies have been performed on discrete-time insurance risk models with financial and insurance risks. Some recent works include [6], [9], [21], [29], and [33]. All of these works restrict attention to an insurance company with one business line. However, special attention is also given to those insurance companies with multiple business lines. In Section 4 of this paper, we introduce the multidimensional discrete-time risk model (16) with a dependence structure between financial and insurance risks satisfying Assumption 1. We then obtain a precise asymptotic expansion for the ruin probability that holds uniformly for the whole time horizon. Furthermore, we present a two-dimensional numerical example. The contribution of this numerical example is twofold. First, it provides a specific dependence structure that satisfies Assumption 1. Second, it shows the accuracy of the asymptotic expansions for the ruin probability.

For the continuous-time risk model with financial and insurance risks, there is abundant literature on the asymptotic estimation of ruin probabilities; see [3], [10], [12], [15], [19], [26], and [28]. However, in almost all of these papers, it is assumed that financial and insurance risks are independent. There is no doubt that, in a similar or the same macroeconomic environment, it is unrealistic to assume that financial and insurance risks over the same period are independent. In Section 5 of this paper, under Assumption 1, we construct a multidimensional continuous-time risk model with a dependence structure between insurance and financial risks. Three ruin probabilities in the infinite-time horizon are then investigated.

The rest of the paper consists of four parts. Section 2 describes regular variation and MRV and presents some properties of MRV. Section 3 introduces Assumption 1. Under Assumption 1, in Section 4 we study the d-dimensional discrete-time risk model (16) (for $d \geq 1$) and construct a two-dimensional numerical example satisfying Assumption 1. In Section 5 we consider a d-dimensional continuous-time risk model under Assumption 1.

2. Preliminaries

If there exist two positive functions $g(\cdot, \cdot)$ and $f(\cdot, \cdot)$ satisfying

$$l_1 = \liminf_{x\to\infty} \frac{f(x, t)}{g(x, t)} \leq \limsup_{x\to\infty} \frac{f(x, t)}{g(x, t)} = l_2,$$

then we say that $f(x, t) \lesssim g(x, t)$ holds if $l_2 \leq 1$; $f(x, t) \gtrsim g(x, t)$ holds if $l_1 \geq 1$; $f(x, t) \sim g(x, t)$ holds if $l_1 = l_2 = 1$; and $f(x, t) \asymp g(x, t)$ holds if $0 < l_1 \leq l_2 < \infty$. All limit relations, unless otherwise stated, hold as $x \to \infty$. Throughout this paper, C denotes a generic positive constant without relation to x, which may vary with the context. Moreover, for any $x, y \in \mathbb{R}$, we write $x \wedge y = \min\{x, y\}$, $x \vee y = \max\{x, y\}$, and $x^+ = x \vee 0$.

In this paper, random and real vectors, supposed to be d-dimensional, are expressed by bold English letters. For two real vectors $\boldsymbol{a} = (a_1, \ldots, a_d)$ and $\boldsymbol{b} = (b_1, \ldots, b_d)$, we assume that operations between vectors such as $\boldsymbol{a} > \boldsymbol{b}$, $\boldsymbol{a} \pm \boldsymbol{b}$, $\boldsymbol{ab}$, and $\boldsymbol{a}^{-1}$ should be interpreted componentwise, and we let $[\boldsymbol{a}, \boldsymbol{b}] = [a_1, b_1] \times \cdots \times [a_d, b_d]$, $[\boldsymbol{a}, \infty) = [a_1, \infty) \times \cdots \times [a_d, \infty)$. Additionally, for $y \in \mathbb{R}$, write $y\boldsymbol{a} = (ya_1, \ldots, ya_d)$ as usual. We also write $\mathbf{0} = (0, \ldots, 0)$, $\mathbf{1} = (1, \ldots, 1)$, and $\mathbf{e}_k = (0, \ldots, 1, \ldots, 0)$, where the value 1 occurs in the kth spot.

Definition 1. A distribution F on $[0, \infty)$ satisfying $\bar{F}(x) > 0$ for all $x \geq 0$ is said to be of regular variation, and we write $F \in \mathcal{R}_{-\alpha}$, if

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$$\lim_{x\to\infty} \frac{\bar{F}(xy)}{\bar{F}(x)} = y^{-\alpha}, \qquad y > 0, \tag{1}$$

for some $0 < \alpha < \infty$.

For some distribution function with $F \in \mathcal{R}_{-\alpha}$, $0 < \alpha < \infty$, we get by [5, Proposition 2.2.3] that there exist positive constants $C_F > 1$ and D_F such that, for all $x, xy \geq D_F$ and any $0 < \varepsilon < \alpha$,

$$\frac{1}{C_F}(y^{-\alpha+\varepsilon} \wedge y^{-\alpha-\varepsilon}) \leq \frac{\bar{F}(xy)}{\bar{F}(x)} \leq C_F(y^{-\alpha+\varepsilon} \vee y^{-\alpha-\varepsilon}). \tag{2}$$

From the relation (2), it follows that for $p > \alpha$,

$$x^{-p} = o(\bar{F}(x)), \qquad x \to \infty. \tag{3}$$

Definition 2. An $\mathbb{R}_+^d$-valued random vector $\mathbf{Z}$ is said to follow a multivariate regularly varying distribution if there exist a Radon measure ν not identically 0 and a positive normalizing function $a(\cdot)$ monotonically increasing to ∞, such that when $t \to \infty$,

$$t\,\mathbb{P}\left(\frac{\mathbf{Z}}{a(t)} \in \cdot\right) \xrightarrow{v} \nu(\cdot) \quad \text{on } [0, \infty]^d \backslash \{\mathbf{0}\}, \tag{4}$$

where $\xrightarrow{v}$ denotes vague convergence.

From the above definition, one can show that the Radon measure ν is homogeneous—namely, that there exists some $\alpha > 0$, denoting the MRV index, such that the equality

$$\nu(sB) = s^{-\alpha}\nu(B) \tag{5}$$

holds for every Borel set $B \subset [0, \infty]^d \backslash \{\mathbf{0}\}$. For details on this, see [27, p. 178]. From [17, p. 459], we obtain $a(t) \in \mathcal{R}_{1/\alpha}$, and hence there exists some distribution $F \in \mathcal{R}_{-\alpha}$ such that

$$\bar{F}(t) \sim \frac{1}{a^{\leftarrow}(t)}, \quad \text{as } t \to \infty,$$

where $a^{\leftarrow}(t)$ is the generalized inverse function of $a(t)$. Consequently, the relation (4) can be expressed as follows:

$$\frac{1}{\bar{F}(x)}\mathbb{P}\left(\frac{\mathbf{Z}}{x} \in \cdot\right) \xrightarrow{v} \nu(\cdot), \quad \text{on } [0, \infty]^d \backslash \{\mathbf{0}\}. \tag{6}$$

Hence, we write $\mathbf{Z} \in \mathrm{MRV}(\alpha, F, \nu)$. For more details on MRV, the reader is referred to [27, Chapter 6] or [17, Chapter 13].

Next we present a lemma that is important in the following proofs.

Lemma 1. *Let random vector $\mathbf{Z} \in MRV(\alpha, F, \nu)$ for some $\alpha > 0$, let $\boldsymbol{\xi}$ be a positive random vector with arbitrarily dependent components satisfying $\mathbb{E}\boldsymbol{\xi}^{\beta} < \infty$ for some $\beta > \alpha$, and let $\{\Delta_t, t \in \mathcal{T}\}$ be a set of random events such that $\lim_{t\to t_0} \mathbb{P}(\Delta_t) = 0$ for some t_0 in the closure of the index set $\mathcal{T}$. Furthermore, assume that $\{\boldsymbol{\xi}, \{\Delta_t, t \in \mathcal{T}\}\}$ and $\mathbf{Z}$ are independent. Then*

$$\lim_{t\to t_0} \limsup_{x\to\infty} \frac{\mathbb{P}(\mathbf{Z}\boldsymbol{\xi} > x\mathbf{1}, \Delta_t)}{\bar{F}(x)} = 0.$$

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Proof. Since the random vector $\mathbf{Z} \in \mathrm{MRV}(\alpha, F, \nu)$, the marginal tail of $\mathbf{Z}$ is regularly varying from [17, p. 458]. Therefore, we have $Z_k \in \mathcal{R}_{-\alpha}$, $1 \le k \le d$. Then

$$\lim_{t\to t_0} \limsup_{x\to\infty} \frac{\mathbb{P}\left(\mathbf{Z}\boldsymbol{\xi} > x\mathbf{1},\, \Delta_t\right)}{\bar{F}(x)} \le \lim_{t\to t_0} \limsup_{x\to\infty} \frac{\mathbb{P}\left(Z_k\xi_k > x,\, \Delta_t\right)}{\bar{F}(x)} = 0,$$

where we used [30, Lemma 6.2]. This completes the proof. □

3. Main assumption

This paper proposes the following assumption to describe certain dependence structures between the random vectors $\mathbf{U}$ and $\mathbf{V}$.

Assumption 1. *Let* $\{\boldsymbol{U}_i\boldsymbol{V}_i = (U_{1i}V_{1i}, \ldots, U_{di}V_{di}), i \in \mathbb{N}\}$, $d \ge 1$, *be a sequence of independent and identically distributed (i.i.d.) nonnegative random vectors with generic vector* $\boldsymbol{UV} = (U_1V_1, \ldots, U_dV_d) \in MRV(\alpha, F, \nu)$ *such that* $\nu\left((\mathbf{1}, \boldsymbol{\infty})\right) > 0$.

Remark 1. From Assumption 1, we can derive that

$$\lim_{x\to\infty} \frac{\mathbb{P}\left(\bigcap_{k=1}^{d}\{U_kV_k > x\}\right)}{\bar{F}(x)} = \nu\left((\mathbf{1}, \boldsymbol{\infty})\right) > 0 \tag{7}$$

and

$$\lim_{x\to\infty} \frac{\mathbb{P}\left(U_kV_k > x\right)}{\bar{F}(x)} = \nu\left((\mathbf{e}_k, \boldsymbol{\infty}]\right) := a_k > 0, \qquad 1 \le k \le d. \tag{8}$$

The relation (7) indicates that $U_1V_1, \ldots, U_dV_d$ reveal large joint movements and accordingly are asymptotically dependent. The relation (8) indicates that the tails of the products $U_1V_1, \ldots, U_dV_d$ are regularly varying and that U_iV_i and U_jV_j have comparable tails. Furthermore, note that by (5) and (7), for any $(b_1, \ldots, b_d) \in [0, \infty]^d \backslash \{\mathbf{0}\}$,

$$\lim_{x\to\infty} \frac{\mathbb{P}\left(\bigcap_{k=1}^{d}\{U_kV_k > b_kx\}\right)}{\bar{F}(x)} = \nu\left((b_1, \infty] \times \cdots \times (b_d, \infty]\right) > 0.$$

Remark 2. Set $\mathbf{U} = U\mathbf{1}$; then the product $\mathbf{UV}$ can be rewritten as

$$\mathbf{UV} = (UV_1, \ldots, UV_d), \tag{9}$$

which is called the scale mixture. This structure can be applied in various areas, including investments or insurance portfolios, and it has various interpretations in different applications. For example, [18] developed a method to estimate the tail dependence of heavy-tailed scale mixtures of multivariate distributions. The paper [34] studied the asymptotic relations between the value-at-Risk and multivariate tail conditional expectation for heavy-tailed scale mixtures of multivariate distributions. Additionally, the class (9) of loss distributions is a subset of all multivariate regularly varying distributions considered in [14], but it includes various loss distributions. In addition, the scale mixture can reflect both individual factors (via the V_i) and the common factor (via U) of a risk class. For instance, U is a common systemic risk factor associated with macroeconomic conditions, including regulatory bodies and economic conditions, while the quantities V_k, $1 \le k \le d$, are individual risks explained as individual business risks and assets.

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Remark 3. Since the structure of $\mathbf{UV} \in \mathrm{MRV}(\alpha, F, \nu)$ in Assumption 1 makes no dependence assumption between random vectors **U** and **V**, it allows for a wide range of dependence structures between **U** and **V**, and many dependence structures satisfy Assumption 1 under certain conditions (such as those of Proposition 2 and the relation (14)). This further enhances the practical and theoretical interest of Assumption 1.

For Assumption 1, the following question arises naturally: what are the conditions under which the distribution of **UV** is MRV? This question has received an increasing amount of attention. The paper [4] demonstrated the MRV of the product $\boldsymbol{UV}$, when **U** is MRV and independent of **V**, as in Proposition 1 (see also [4, Appendix], [11, Theorem 1], and [16, Lemma 3.1]).

Proposition 1. *Let the random vector $\boldsymbol{U} \in MRV(\alpha, F, \nu)$ for some $\alpha > 0$, and let $\boldsymbol{V}$ be a nonnegative random vector with dependent components satisfying $\mathbb{E}\boldsymbol{V}^p < \infty$ for some $p > \alpha$. Assume that $\boldsymbol{V}$ and $\boldsymbol{U}$ are independent. Then the following relation holds for every Borel set $K \subset [0, \infty]^d \backslash \{\mathbf{0}\}$:*

$$\lim_{x\to\infty} \frac{1}{\bar{F}(x)} \mathbb{P}\left(\frac{\boldsymbol{UV}}{x} \in K\right) = \mathbb{E}\left[\nu(\boldsymbol{V}^{-1}K)\right],$$

where

$$\begin{aligned}\boldsymbol{V}^{-1}K &= \{(b_1, \ldots, b_d) \mid (V_1 b_1, \ldots, V_d b_d) \in K\} \\ &= \left\{\left(V_1^{-1}a_1, \ldots, V_d^{-1}a_d\right),\ (a_1, \ldots, a_d) \in K\right\}.\end{aligned}$$

The paper [7] studied the tail asymptotics of the nonnegative random loss $\sum_{i=1}^d R_i S$, where the stand-alone risk vector $\mathbf{R} = (R_1, \ldots, R_d)$ follows a multivariate regularly varying distributions with index $\alpha > 0$ and is independent of S, representing the background risk factor. Furthermore, [7] also assumed that $\mathbb{E}S^{\alpha+\delta} < \infty$ for some $\delta > 0$. Essentially, the conditions in [7] imply that the random vector $(R_1 S, \ldots, R_d S)$ is MRV by Proposition 1.

The hypothesis of independence between **U** and **V** in Proposition 1 may be too strong in certain settings. If this condition can be weakened, Assumption 1 will become broader and more meaningful. The paper [11] further improved Proposition 1 to the following result, Proposition 2, which seems meaningful in the context of actuarial risk theory.

Proposition 2. *Let us assume that*

$$t\,\mathbb{P}\left(\left(\frac{\mathbf{U}}{a(t)}, \mathbf{V}\right) \in \cdot\right) \xrightarrow{v} (\nu \times L)(\cdot) \tag{10}$$

on the Borel sets of $([0, \infty]^d \backslash \{\mathbf{0}\}) \times [0, \infty]^d$, where L represents a probability measure on $[0, \infty]^d$ and ν denotes a Radon measure on $[0, \infty]^d \backslash \{\mathbf{0}\}$ not concentrated at ∞. Suppose that, for some $\delta > 0$ and any $i = 1, 2, \ldots, d$,

$$\lim_{\varepsilon\to 0} \limsup_{t\to\infty} t\mathbb{E}\left[\left(\frac{|\mathbf{U}|\,|V_i|}{a(t)}\right)^{\delta} \mathbb{I}_{[|\mathbf{U}|/a(t)\le\varepsilon]}\right] = 0, \tag{11}$$

and also that

$$\int_{[\mathbf{0},\,\infty]} ||\mathbf{v}||^{\alpha} L(\mathrm{d}\mathbf{v}) < \infty, \tag{12}$$

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where $\mathbb{I}_{[E]}$ *is the indicator of the Borel set E and* $||\cdot||$ *denotes some norm on* $\mathbb{R}^d$. *Then the random vector* $\mathbf{UV}$ *follows a multivariate regularly varying distribution under the relations (10), (11), and (12). More precisely,*

$$t\mathbb{P}(\mathbf{UV}\in a(t)\cdot\,)\xrightarrow{v}\nu_L(\cdot)$$

as $t\to\infty$, *where* ν_L *is the measure defined on* $[0,\infty]^d\backslash\{\mathbf{0}\}$ *by*

$$\nu_L(A)=(\nu\times L)(\{(\mathbf{x},\mathbf{y}),\mathbf{x}\in\mathbf{y}^{-1}A\})=\int_{[0,\infty]^d}\nu(\mathbf{y}^{-1}A)L(\mathrm{d}y).$$

Assume that $(U_1,\ldots,U_d,V)$, $d\geq 1$, is a positive random vector. Li [19] introduced the following dependence structure among the components of $(U_1,\ldots,U_d,V)$:

- There are some d-variate function $f:[0,\infty]^d\backslash\{\mathbf{0}\}\mapsto(0,\infty)$, some univariate function $h:(0,\infty)\mapsto(0,\infty)$ satisfying

$$0<\inf_{0<t<\infty}h(t)\leq\sup_{0<t<\infty}h(t)<\infty,\tag{13}$$

  and some distribution F supported on $(0,\infty)$ with an infinite upper endpoint, such that for every $\mathbf{b}\in[0,\infty]^d\backslash\{\mathbf{0}\}$ the relation

$$\mathbb{P}(U_1>b_1u,\ldots,U_d>b_du|V=v)\sim f(\mathbf{b})h(v)\bar{F}(u)\tag{14}$$

  holds uniformly for $v\in(0,\infty)$ as $u\to\infty$.

According to the relation (2.10) in [19], taking $\mathbf{b}=\mathbf{e}_k$, $1\leq k\leq d$, in (14) yields that the relation

$$\mathbb{P}(U_k>u|V=v)\sim\frac{1}{\mu}h(v)\bar{F}_k(u)\tag{15}$$

holds uniformly for $v\in(0,\infty)$ as $u\to\infty$, where $\bar{F}_k(u)$ is the distribution function of U_k, and $\mu=\mathbb{E}[h(V)]\in(0,\infty)$. Therefore, the dependence structure specified in (14) implies that the marginal vector (U_k,V) follows a parallel two-dimensional dependence structure as shown in (15). The structure (15) is defined in [2] and further developed in [1] and [22]; it includes a wide range of commonly used bivariate copulas.

Now define a new positive random variable V^* with distribution

$$\mathbb{P}(V^*\in\mathrm{d}v)=\frac{1}{\mu}h(v)\mathbb{P}(V\in\mathrm{d}v).$$

For simplicity, we introduce a new random variable ξ with distribution function $F\in\mathcal{R}_{-\alpha}$ for $0<\alpha<\infty$, and let ξ and V^* be independent of all other sources of randomness. If

$$\mathbb{E}(V^{\alpha+\varepsilon})<\infty$$

holds for $\varepsilon>0$, then by (13),

$$\mathbb{E}[(V^*)^{\alpha+\varepsilon}]=\frac{1}{\mu}\int_0^\infty v^{\alpha+\varepsilon}h(v)\mathbb{P}(V\in\mathrm{d}v)\leq C\mathbb{E}(V^{\alpha+\varepsilon})<\infty,$$

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and for $\frac{\alpha}{\alpha+\varepsilon} < p < 1$ and large $u > 0$,

$$\mathbb{P}(V > u^p) \le \mathbb{E}(V^{\alpha+\varepsilon})u^{-p(\alpha+\varepsilon)} = o(1)\bar{F}(u)$$

and

$$\mathbb{P}(V^* > u^p) \le \mathbb{E}[(V^*)^{\alpha+\varepsilon}]u^{-p(\alpha+\varepsilon)} = o(1)\bar{F}(u)$$

hold. Consequently, we have that by (14),

$$\begin{aligned}
\mathbb{P}(\mathbf{U}V > \mathbf{b}u) &= \mathbb{P}(\mathbf{U}V > \mathbf{b}u,\, 0 < V \le u^p) + \mathbb{P}(\mathbf{U}V > \mathbf{b}u,\, V > u^p) \\
&= \int_{0+}^{u^p} \mathbb{P}\left(\mathbf{U} > \mathbf{b}\frac{u}{v} | V = v\right) \mathbb{P}(V \in \mathrm{d}v) + o(1)\bar{F}(u) \\
&\sim \mu f(\mathbf{b}) \int_{0+}^{u^p} \bar{F}\left(\frac{u}{v}\right) \mathbb{P}(V^* \in \mathrm{d}v) \\
&= \mu f(\mathbf{b}) \left[\mathbb{P}(\xi V^* > u) - \mathbb{P}(\xi V^* > u,\, V^* > u^p)\right] \\
&\sim \mu f(\mathbf{b})\mathbb{E}[(V^*)^\alpha]\bar{F}(u),
\end{aligned}$$

where $\mathbb{P}(\xi V^* > u) \sim \bar{F}(u)\mathbb{E}[(V^*)^\alpha]$ is due to the relation (4.4) in [28]. Therefore, $(U_1 V, \ldots, U_d V)$ satisfies Assumption 1.

4. The study of a *d*-dimensional discrete-time risk model under Assumption 1

This section considers an insurer who runs multiple lines of business and makes risky assets along individual lines. For every $i \in \mathbb{N} = \{1, 2, \ldots\}$ and integer $d \ge 1$, the real-valued random variable X_{ki}, $1 \le k \le d$, denotes the net insurance loss (described by the aggregate claim amount minus the aggregate premium income) of the kth business of the insurer over the period i, and the positive random variable θ_{ki} denotes the discount factor, related to the return on the investment, of the kth business of the insurer over the same period. Let $\{(X_1, \ldots, X_d), (X_{1i}, \ldots, X_{di}), i \in \mathbb{N}\}$ and $\{(\theta_1, \ldots, \theta_d), (\theta_{1i}, \ldots, \theta_{di}), i \in \mathbb{N}\}$ be sequences of i.i.d. random vectors. The stochastic discounted value of total net insurance losses for the insurance company up to the time n can be described as

$$\begin{aligned}
\mathbf{S}_n = (S_{1n}, \ldots, S_{dn}) &= \left(\sum_{i=1}^n X_{1i} \prod_{j=1}^i \theta_{1j}, \ldots, \sum_{i=1}^n X_{di} \prod_{j=1}^i \theta_{dj}\right) \\
&= \left(\sum_{i=1}^n X_{1i} Y_{1i}, \ldots, \sum_{i=1}^n X_{di} Y_{di}\right), \quad n \in \mathbb{N}, \qquad (16)
\end{aligned}$$

where multiplication over the empty set is understood to be 1, and $Y_{ki} = \prod_{j=1}^i \theta_{kj}$, $1 \le k \le d$. The product $\boldsymbol{\rho} x = (\rho_1 x, \ldots, \rho_d x)$ denotes the vector of initial reserves assigned to different businesses, with positive $\rho_1, \ldots, \rho_d$ such that $\sum_{k=1}^d \rho_k = 1$. For $n \in \mathbb{N}$, the ruin probability

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can be defined as

$$\Psi(x, n) := \mathbb{P}\left(\bigcap_{k=1}^{d}\left\{\max_{1\le m\le n} S_{km} > \rho_k x\right\}\right) = \mathbb{P}\left(\max_{1\le m\le n} \mathbf{S}_m > \boldsymbol{\rho} x\right)$$

$$= \mathbb{P}\left(\max_{1\le m\le n} \sum_{i=1}^{m} \boldsymbol{X}_i \boldsymbol{Y}_i > \boldsymbol{\rho} x\right),$$

which denotes the probability of the insurance company's wealth process going below zero by time n. When $d = 1$, [11] considered the ruin probability of (16) if the random vector

$$(X_{11}Y_{11}, X_{12}Y_{12}, \ldots, X_{1n}Y_{1n})$$

is MRV, and [31] studied the ruin probability of (16) if the random vector (X_{1i}, θ_{1i}), $i \in \mathbb{N}$, follows a bivariate regular variation structure. When $d = 2$, according to [8], we can obtain that the random vector

$$\left(\sum_{i=1}^{m} X_{1i} \prod_{j=1}^{i} \theta_{1j}, \sum_{i=1}^{m} X_{2i} \prod_{j=1}^{i} \theta_{2j}\right)$$

still follows a bivariate-regular-variation structure if the pairs $(X_{1i}\theta_{1i}, X_{2i}\theta_{2i})$, $i \in \mathbb{N}$, follow some bivariate-regular-variation structure. In this section, we assume that $(X_{1i}^{+}\theta_{1i}, \ldots, X_{di}^{+}\theta_{di})$, $i \in \mathbb{N}$ and $d \ge 1$, is a sequence of i.i.d. random vectors with generic vector $(X_1^{+}\theta_1, \ldots, X_d^{+}\theta_d) \in \mathrm{MRV}(\alpha, F, \nu)$ such that $\nu((\mathbf{1}, \infty)) > 0$; then we study the asymptotic formula for the ruin probability for $n \in \mathbb{N}$.

Theorem 1. *Consider the d-dimensional discrete-time risk model (16). For each $i \in \mathbb{N}$, assume that $\boldsymbol{X}_i^{+}\boldsymbol{\theta}_i$ satisfies Assumption 1. If there is a constant $\beta > \alpha$ such that $\mathbb{E}\boldsymbol{\theta}^{\beta} < \mathbf{1}$, then it holds uniformly for $n \in \mathbb{N}$ that*

$$\Psi(x, n) = \mathbb{P}\left(\max_{1\le m\le n} \sum_{i=1}^{m} \boldsymbol{X}_i \boldsymbol{Y}_i > \boldsymbol{\rho} x\right)$$

$$\sim \mathbb{P}\left(\sum_{i=1}^{n} \boldsymbol{X}_i \boldsymbol{Y}_i > \boldsymbol{\rho} x\right) \sim \bar{F}(x) \sum_{i=1}^{n} \mathbb{E}\left[\nu(\boldsymbol{Y}_{i-1}^{-1}(\boldsymbol{\rho}, \infty])\right]. \tag{17}$$

4.1. Numerical results

This subsection presents a two-dimensional numerical example to examine the accuracy of Theorem 1. All computations are conducted in R. For simplicity, we assume that $\theta_1 = \theta_2 = \theta$, and let θ be exponential with parameter $\lambda = 2$.

We first construct a random pair (X_1, X_2). Let the distribution function F of the random variable Z follow a Pareto distribution, with scale parameter $\kappa = 4$ and shape parameter $\alpha = 2$ $(\alpha > 0)$; that is, $F(x) = 1 - \left(\frac{\kappa}{\kappa + x}\right)^{\alpha} \in \mathcal{R}_{-\alpha}$, $x > 0$. Suppose that the dependence structure between Z and θ is established by a Farlie–Gumbel–Morgenstern (FGM) copula,

$$C(u, v) = uv + \gamma uv(1 - u)(1 - v), \qquad \gamma \in [-1, 1],\ (u, v) \in [0, 1]^2. \tag{18}$$

Then, by [1, Example 2.2] or [22, Example 3.2], the random pair (Z, θ) satisfies

$$\mathbb{P}(Z > z \mid \theta = t) \sim \bar{F}(z) h(t), \qquad \text{uniformly for all } t \in [0, \infty),$$

with $h(t)=1+\gamma(1-2e^{-\lambda t})$. Let $(X_1, X_2)=(\zeta_1 Z, \zeta_2 Z)$, where ζ_k, $k=1, 2$, are uniform on [0,1] and independent of (Z, θ), and $\mathbb{E}\zeta_k^q<\infty$ for some $q>\alpha$.

We next verify that $(X_1\theta, X_2\theta)$ satisfies Assumption 1. Using [19, Example 3.1], we obtain

$$\mathbb{P}(X_1>b_1 x, X_2>b_2 x|\theta=t)\sim V(b_1, b_2)h(t)\bar{F}(x), \quad \text{uniformly for all } t\in[0, \infty),$$

where $b_1,\ b_2>0$ are constants, and

$$V(b_1, b_2)=\mathbb{E}\left(\frac{\zeta_1^\alpha}{b_1^\alpha}\bigwedge\frac{\zeta_2^\alpha}{b_2^\alpha}\right)=\frac{b_1+b_2}{(\alpha+1)(b_1\vee b_2)^{\alpha+1}}-\frac{2b_1b_2}{(\alpha+2)(b_1\vee b_2)^{\alpha+2}}.$$

Moreover, for $\beta=3>\alpha$ satisfying $\mathbb{E}\theta^\beta=\frac{6}{\lambda^3}<1$ and $\alpha/\beta<p<1$, we obtain that

$$\begin{aligned}
&\mathbb{P}(X_1\theta>b_1x, X_2\theta>b_2x)\\
&=\mathbb{P}(X_1\theta>b_1x, X_2\theta>b_2x, 0<\theta\le x^p)+\mathbb{P}(X_1\theta>b_1x, X_2\theta>b_2x, \theta>x^p)\\
&=\int_{0+}^{x^p}\mathbb{P}\left(X_1>b_1x/t, X_2>b_2x/t|\theta=t\right)\mathbb{P}(\theta\in \mathrm{d}t)+o(1)\bar{F}(x)\\
&\sim V(b_1, b_2)\bar{F}(x)\int_{0+}^{x^p} t^\alpha h(t)\mathbb{P}(\theta\in\mathrm{d}t)\\
&\sim V(b_1, b_2)\bar{F}(x)\left[\mathbb{E}\theta^\alpha+\gamma\mathbb{E}\theta^\alpha-2\gamma\mathbb{E}\left(\theta^\alpha e^{-\lambda\theta}\right)\right] := \mu V(b_1, b_2)\bar{F}(x),
\end{aligned}$$

where $\mu=\mathbb{E}\theta^\alpha+\gamma\mathbb{E}\theta^\alpha-2\gamma\mathbb{E}\left(\theta^\alpha e^{-\lambda\theta}\right)$, and in the second and fourth steps we use the relation $\mathbb{P}(\theta>x^p)\le x^{-p\beta}\mathbb{E}\theta^\beta=o(1)\bar{F}(x)$ from (3). This implies that $\mathbb{P}(X_1\theta>\cdot, X_2\theta>\cdot)$ possesses a bivariate regularly varying tail. Set $\gamma=0.5$, so $\mu=11/16$ and $\nu((1, \infty]\times(1, \infty])=\mu V(1, 1)=11/96>0$; then $(X_1\theta, X_2\theta)$ satisfies Assumption 1.

Finally, we estimate and compare the numerical results of the asymptotic formula (17) and the simulation of $\Psi(x, n)$. Set $n=5$; then the asymptotic formula (17) becomes

$$\Psi(x, n)\sim\mu V(\rho_1, \rho_2)\bar{F}(x)\sum_{i=1}^n\left(\mathbb{E}\theta^\alpha\right)^{i-1} := \Psi_1(x, n).$$

Denote by $\Psi_2(x, n)$ the Monte Carlo simulation of $\Psi(x, n)$. From [25, Exercise 3.23] we can generate an FGM random pair (Z, θ), and then we get $(X_1, X_2, \theta)=(Z\zeta_1, Z\zeta_2, \theta)$ by generating two independent uniform (0,1) variates, ζ_1 and ζ_2. We simulate $N=10^7$ samples of $((X_{11}, X_{21}, \theta_1), \ldots, (X_{1n}, X_{2n}, \theta_n))$, and for each $k=1, \ldots, N$, we denote by $\left((X_{11}^{(k)}, X_{21}^{(k)}, \theta_1^{(k)}), \ldots, (X_{1n}^{(k)}, X_{2n}^{(k)}, \theta_n^{(k)})\right)$ the independent copy of $((X_{11}, X_{21}, \theta_1), \ldots, (X_{1n}, X_{2n}, \theta_n))$. Hence, $\Psi_2(x, n)$ is estimated by

$$\frac{1}{N}\sum_{k=1}^N\mathbb{I}_{\left\{\max_{1\le m\le n}\sum_{i=1}^m X_{1i}^{(k)}\prod_{j=1}^i\theta_j^{(k)}>\rho_1x,\quad \max_{1\le m\le n}\sum_{i=1}^m X_{2i}^{(k)}\prod_{j=1}^i\theta_j^{(k)}>\rho_2x\right\}}.$$

From Table 1, we observe that the ratio $\Psi_1(x, n)/\Psi_2(x, n)$ approaches 1 as x becomes large. Fixing x, we notice that the ruin probabilities decay when ρ_1 decreases from 0.5 to 0.1, meaning that a larger value of $(1-\rho_1)$ leads to a smaller ruin probability of the corresponding business line from the insurance company.

TABLE 1. Asymptotic versus simulated values

x	250	300	350	400	450
$\rho_1 = 0.5$	(125, 125)	(150, 150)	(175, 175)	(200, 200)	(225, 225)
$\Psi_1(x, n)$	2.2×10^{-4}	1.54×10^{-4}	1.13×10^{-4}	8.71×10^{-5}	6.89×10^{-5}
$\Psi_2(x, n)$	2.35×10^{-4}	1.61×10^{-4}	1.15×10^{-4}	8.68×10^{-5}	6.9×10^{-5}
Ψ_1/Ψ_2	0.936	0.957	0.983	1.003	0.999
$\rho_1 = 0.3$	(75, 175)	(90, 210)	(105, 245)	(120, 280)	(135, 315)
$\Psi_1(x, n)$	1.77×10^{-4}	1.23×10^{-4}	9.09×10^{-5}	6.98×10^{-5}	5.53×10^{-5}
$\Psi_2(x, n)$	1.86×10^{-4}	1.28×10^{-4}	9.26×10^{-5}	7.06×10^{-5}	5.49×10^{-5}
Ψ_1/Ψ_2	0.952	0.961	0.982	0.989	1.007
$\rho_1 = 0.1$	(25, 225)	(30, 270)	(35, 315)	(40, 360)	(45, 405)
$\Psi_1(x, n)$	1.28×10^{-4}	8.96×10^{-5}	6.61×10^{-5}	5.08×10^{-5}	4.02×10^{-5}
$\Psi_2(x, n)$	1.35×10^{-4}	9.1×10^{-5}	6.68×10^{-5}	5.11×10^{-5}	4.03×10^{-5}
Ψ_1/Ψ_2	0.948	0.985	0.990	0.994	0.998

4.2. Some lemmas before the proof of Theorem 1

Clearly, one can derive the following relation for all $x > 0$:

$$\mathbb{P}\left(\boldsymbol{X}_i^+\boldsymbol{Y}_i > \boldsymbol{b}x\right) = \mathbb{P}\left(\boldsymbol{X}_i\boldsymbol{Y}_i > \boldsymbol{b}x\right), \qquad \text{for any } \boldsymbol{b} \in [0, \infty]^d \backslash \{\boldsymbol{0}\} \text{ and } i \in \mathbb{N}. \tag{19}$$

Lemma 2. *If the conditions of Theorem 1 hold, then for every fixed* $\boldsymbol{b} \in [0, \infty]^d \backslash \{\boldsymbol{0}\}$ *and* n *we have*

$$\mathbb{P}\left(\sum_{i=1}^{n} \boldsymbol{X}_i\boldsymbol{Y}_i > \boldsymbol{b}x\right) \sim \sum_{i=1}^{n} \mathbb{P}\left(\boldsymbol{X}_i^+\boldsymbol{Y}_i > \boldsymbol{b}x\right) \sim \sum_{i=1}^{n} \bar{F}(x)\mathbb{E}\left[\nu\left(\boldsymbol{Y}_{i-1}^{-1}(\boldsymbol{b}, \boldsymbol{\infty}]\right)\right].$$

Proof. For any $1 \le i \le n$, we have by the conditions of Theorem 1 that $\boldsymbol{X}_i^+\boldsymbol{\theta}_i \in \mathrm{MRV}(\alpha, F, \nu)$ and

$$\mathbb{E}\boldsymbol{Y}_i^\beta = \left(\mathbb{E}Y_{1i}^\beta, \ldots, \mathbb{E}Y_{di}^\beta\right) = \left(\prod_{j=1}^{i} \mathbb{E}\theta_{1j}^\beta, \ldots, \prod_{j=1}^{i} \mathbb{E}\theta_{dj}^\beta\right) < \boldsymbol{\infty}. \tag{20}$$

Since $\boldsymbol{X}_i^+\boldsymbol{\theta}_i$ and $\boldsymbol{Y}_{i-1} = \prod_{j=1}^{i-1} \boldsymbol{\theta}_j$ are independent, it follows from Proposition 1 that

$$\mathbb{P}\left(\boldsymbol{X}_i^+\boldsymbol{Y}_i > \boldsymbol{b}x\right) = \mathbb{P}\left(\frac{\boldsymbol{X}_i^+\boldsymbol{\theta}_i\boldsymbol{Y}_{i-1}}{x} \in (\boldsymbol{b}, \boldsymbol{\infty}]\right) \sim \bar{F}(x)\mathbb{E}\left[\nu\left(\boldsymbol{Y}_{i-1}^{-1}(\boldsymbol{b}, \boldsymbol{\infty}]\right)\right]. \tag{21}$$

Hence, it suffices to show that for every fixed $\boldsymbol{b} \in [0, \infty]^d \backslash \{\boldsymbol{0}\}$ and n, the following asymptotic formula holds:

$$\mathbb{P}\left(\sum_{i=1}^{n} \boldsymbol{X}_i\boldsymbol{Y}_i > \boldsymbol{b}x\right) \sim \sum_{i=1}^{n} \mathbb{P}\left(\boldsymbol{X}_i^+\boldsymbol{Y}_i > \boldsymbol{b}x\right). \tag{22}$$

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Now we aim to show the upper bound of $\mathbb{P}\left(\sum_{i=1}^{n} \boldsymbol{X}_i \boldsymbol{Y}_i > \boldsymbol{b}x\right)$. For an arbitrary $\varepsilon > 0$, choose small $v_1 > 0$ such that $(1-v_1)^{-\alpha} \le 1+\varepsilon$ and $(1+v_1)^{-\alpha} \ge 1-\varepsilon$ hold. Then

$$\begin{aligned}\mathbb{P}\left(\sum_{i=1}^{n} \boldsymbol{X}_i \boldsymbol{Y}_i > \boldsymbol{b}x\right) &\le \mathbb{P}\left(\sum_{i=1}^{n} \boldsymbol{X}_i^+ \boldsymbol{Y}_i > \boldsymbol{b}x\right) \\ &= \mathbb{P}\left(\sum_{i=1}^{n} \boldsymbol{X}_i^+ \boldsymbol{Y}_i > \boldsymbol{b}x, \bigcup_{m=1}^{n} \left(\boldsymbol{X}_m^+ \boldsymbol{Y}_m > \boldsymbol{b}(1-v_1)x\right)\right) \\ &\quad + \mathbb{P}\left(\sum_{i=1}^{n} \boldsymbol{X}_i^+ \boldsymbol{Y}_i > \boldsymbol{b}x, \bigcap_{m=1}^{n} \left(\boldsymbol{X}_m^+ \boldsymbol{Y}_m > \boldsymbol{b}(1-v_1)x\right)^c\right) \\ &:= K_1(x, n) + K_2(x, n).\end{aligned}$$

For $K_1(x, n)$, we have by the relation (21) that

$$\begin{aligned}K_1(x, n) &\le \sum_{m=1}^{n} \mathbb{P}\left(\boldsymbol{X}_m^+ \boldsymbol{Y}_m > \boldsymbol{b}(1-v_1)x\right) \sim \sum_{m=1}^{n} \bar{F}\left((1-v_1)x\right) \mathbb{E}\left\{\nu(\boldsymbol{Y}_{m-1}^{-1}\boldsymbol{b}, \boldsymbol{\infty}]\right\} \\ &\sim (1-v_1)^{-\alpha} \bar{F}(x) \sum_{m=1}^{n} \mathbb{E}\left\{\nu\left(\boldsymbol{Y}_{m-1}^{-1}\boldsymbol{b}, \boldsymbol{\infty}\right]\right\} \\ &\le (1+\varepsilon) \sum_{m=1}^{n} \bar{F}(x) \mathbb{E}\left\{\nu\left(\boldsymbol{Y}_{m-1}^{-1}\boldsymbol{b}, \boldsymbol{\infty}\right]\right\} \sim (1+\varepsilon) \sum_{m=1}^{n} \mathbb{P}\left(\boldsymbol{X}_m^+ \boldsymbol{Y}_m > \boldsymbol{b}x\right).\end{aligned}$$

For $K_2(x, n)$ we obtain, for $1 \le k \le d$ and some $b_k > 0$,

$$\begin{aligned}K_2(x, n) &= \mathbb{P}\left(\sum_{i=1}^{n} \boldsymbol{X}_i^+ \boldsymbol{Y}_i > \boldsymbol{b}x, \bigcap_{m=1}^{n} \bigcup_{l=1}^{d} \left(X_{lm}^+ Y_{lm} \le b_l(1-v_1)x\right)\right) \\ &= \mathbb{P}\left(\sum_{i=1}^{n} \boldsymbol{X}_i^+ \boldsymbol{Y}_i > \boldsymbol{b}x, \bigcup_{j=1}^{n} \left(X_{kj}^+ Y_{kj} > \frac{b_k x}{n}\right), \bigcap_{m=1}^{n} \bigcup_{l=1}^{d} \left(X_{lm}^+ Y_{lm} \le b_l(1-v_1)x\right)\right) \\ &\le \sum_{j=1}^{n} \mathbb{P}\left(\sum_{i=1}^{n} \boldsymbol{X}_i^+ \boldsymbol{Y}_i > \boldsymbol{b}x, X_{kj}^+ Y_{kj} > \frac{b_k x}{n}, \bigcup_{l=1}^{d} \left(X_{lj}^+ Y_{lj} \le b_l(1-v_1)x\right)\right) \\ &\le \sum_{j=1}^{n} \sum_{l=1}^{d} \mathbb{P}\left(\sum_{i=1}^{n} X_{li}^+ Y_{li} > b_l x, X_{kj}^+ Y_{kj} > \frac{b_k x}{n}, X_{lj}^+ Y_{lj} \le b_l(1-v_1)x\right) \\ &\le \sum_{j=1}^{n} \sum_{l=1}^{d} \sum_{1 \le i \le n, i \ne j} \mathbb{P}\left(X_{li}^+ Y_{li} > \frac{b_l v_1 x}{n-1}, X_{kj}^+ Y_{kj} > \frac{b_k x}{n}\right) \\ &= \left(\sum_{j=1}^{n} \sum_{l=1}^{d} \sum_{1 \le i < j \le n} + \sum_{j=1}^{n} \sum_{l=1}^{d} \sum_{1 \le j < i \le n}\right) \mathbb{P}\left(X_{li}^+ \theta_{li} Y_{l,i-1} > \frac{b_l v_1 x}{n-1}, X_{kj}^+ \theta_{kj} Y_{k,j-1} > \frac{b_k x}{n}\right) \\ &= o(1)\bar{F}(x),\end{aligned} \tag{23}$$

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where we applied (20), Lemma 1, and (8) in the last step, using the independence between

$$X_{li}^{+}\theta_{li} \quad \text{and} \quad (Y_{l,i-1}, X_{kj}^{+}\theta_{kj}Y_{k,j-1})$$

when $i > j$ and the independence between $X_{kj}^{+}\theta_{kj}$ and $(Y_{k,j-1}, X_{li}^{+}\theta_{li}Y_{l,i-1})$ when $i < j$. Hence, for large x, we get

$$\mathbb{P}\left(\sum_{i=1}^{n} \boldsymbol{X}_i\boldsymbol{Y}_i > \boldsymbol{b}x\right) \le (1 + C\varepsilon)\sum_{i=1}^{n} \mathbb{P}\left(\boldsymbol{X}_i^{+}\boldsymbol{Y}_i > \boldsymbol{b}x\right).$$

Finally, we construct the lower bound of $\mathbb{P}\left(\sum_{i=1}^{n} \boldsymbol{X}_i\boldsymbol{Y}_i > \boldsymbol{b}x\right)$. We have

$$\begin{aligned}
\mathbb{P}\left(\sum_{i=1}^{n} \boldsymbol{X}_i\boldsymbol{Y}_i > \boldsymbol{b}x\right) &\ge \mathbb{P}\left(\sum_{i=1}^{n} \boldsymbol{X}_i\boldsymbol{Y}_i > \boldsymbol{b}x, \bigcup_{m=1}^{n} (\boldsymbol{X}_m\boldsymbol{Y}_m > \boldsymbol{b}(1+v_1)x)\right) \\
&\ge \sum_{m=1}^{n} \mathbb{P}\left(\sum_{i=1}^{n} \boldsymbol{X}_i\boldsymbol{Y}_i > \boldsymbol{b}x, \boldsymbol{X}_m\boldsymbol{Y}_m > \boldsymbol{b}(1+v_1)x\right) \\
&\quad - \sum_{1\le m<k\le n} \mathbb{P}\left(\sum_{i=1}^{n} \boldsymbol{X}_i\boldsymbol{Y}_i > \boldsymbol{b}x, \boldsymbol{X}_m\boldsymbol{Y}_m > \boldsymbol{b}(1+v_1)x, \boldsymbol{X}_k\boldsymbol{Y}_k > \boldsymbol{b}(1+v_1)x\right),
\end{aligned}$$

so we find

$$\begin{aligned}
&\mathbb{P}\left(\sum_{i=1}^{n} \boldsymbol{X}_i\boldsymbol{Y}_i > \boldsymbol{b}x\right) \\
&\ge \sum_{m=1}^{n} \mathbb{P}\left(\boldsymbol{X}_m\boldsymbol{Y}_m > \boldsymbol{b}(1+v_1)x\right) \\
&\quad - \sum_{m=1}^{n} \mathbb{P}\left(\boldsymbol{X}_m\boldsymbol{Y}_m > \boldsymbol{b}(1+v_1)x, \bigcup_{k=1}^{d}\left(\sum_{i=1}^{n} X_{ki}Y_{ki} \le b_k x\right)\right) \\
&\quad - \sum_{1\le m<k\le n} \mathbb{P}\left(\sum_{i=1}^{n} \boldsymbol{X}_i\boldsymbol{Y}_i > \boldsymbol{b}x, \boldsymbol{X}_m\boldsymbol{Y}_m > \boldsymbol{b}(1+v_1)x, \boldsymbol{X}_k\boldsymbol{Y}_k > \boldsymbol{b}(1+v_1)x\right) \\
&:= K_1'(x,n) - K_2'(x,n) - K_3'(x,n).
\end{aligned}$$

For $K_1'(x,n)$, we have by the relations (19) and (21) that

$$\begin{aligned}
K_1'(x,n) &\sim (1+v_1)^{-\alpha}\bar{F}(x)\sum_{m=1}^{n} \mathbb{E}\left(\nu\left(\boldsymbol{Y}_{m-1}^{-1}\boldsymbol{b}, \infty\right]\right) \\
&\ge (1-\varepsilon)\bar{F}(x)\sum_{m=1}^{n} \mathbb{E}\left(\nu\left(\boldsymbol{Y}_{m-1}^{-1}\boldsymbol{b}, \infty\right]\right) \\
&\ge (1-C\varepsilon)\sum_{m=1}^{n} \mathbb{P}\left(\boldsymbol{X}_m^{+}\boldsymbol{Y}_m > \boldsymbol{b}x\right).
\end{aligned}$$

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For $K_2'(x, n)$, we obtain

$$K_2'(x, n) \le \sum_{m=1}^{n}\sum_{k=1}^{d} \mathbb{P}\left(\boldsymbol{X}_m\boldsymbol{Y}_m > \boldsymbol{b}(1+v_1)x, \sum_{i=1}^{n} X_{ki}Y_{ki} \le b_k x\right)$$

$$\le \sum_{m=1}^{n}\sum_{k=1}^{d} \mathbb{P}\left(X_{km}^{+}Y_{km} > b_k(1+v_1)x, \sum_{1\le i\ne m\le n} X_{ki}Y_{ki} \le -v_1 b_k x\right)$$

$$\le \sum_{m=1}^{n}\sum_{k=1}^{d}\sum_{1\le i<m\le n} \mathbb{P}\left(X_{km}^{+}\theta_{km}Y_{k,m-1} > b_k(1+v_1)x, |X_{ki}|Y_{ki} \ge \frac{v_1 b_k x}{n-1}\right)$$

$$+ \sum_{m=1}^{n}\sum_{k=1}^{d}\sum_{1\le m<i\le n} \mathbb{P}\left(X_{km}^{+}Y_{km} > b_k(1+v_1)x, |X_{ki}|Y_{ki} \ge \frac{v_1 b_k x}{n-1}\right)$$

$$= o(1)\bar{F}(x) + \sum_{m=1}^{n}\sum_{k=1}^{d}\sum_{1\le m<i\le n} \mathbb{P}\left(X_{km}^{+}Y_{km} > b_k(1+v_1)x, |X_{ki}|Y_{ki} \ge \frac{v_1 b_k x}{n-1}\right),$$

where the last equality follows from Lemma 1, (20), and (8) by the independence between $X_{km}^{+}\theta_{km}$ and $(Y_{k,m-1}, |X_{ki}|Y_{ki})$. For

$$\frac{\alpha}{\beta} < p < 1$$

we find that, by Chebyshev's inequality,

$$\sum_{m=1}^{n}\sum_{k=1}^{d}\sum_{1\le m<i\le n} \mathbb{P}\left(X_{km}^{+}Y_{km} > b_k(1+v_1)x, |X_{ki}|Y_{ki} \ge \frac{v_1 b_k x}{n-1}\right) \le \sum_{m=1}^{n}\sum_{k=1}^{d}$$

$$\sum_{1\le m<i\le n} \mathbb{P}\left(X_{km}^{+}Y_{km} > b_k(1+v_1)x, |X_{ki}|Y_{ki} \ge \frac{v_1 b_k x}{n-1}, \theta_{km} \le x^p\right) + dn^3\mathbb{P}\left(\theta_{km} > x^p\right)$$

$$\le \sum_{m=1}^{n}\sum_{k=1}^{d}\sum_{1\le m<i\le n} \mathbb{P}\left(X_{km}^{+}Y_{km} > b_k(1+v_1)x, |X_{ki}|\prod_{j=1,j\ne m}^{i} \theta_{kj} \ge \frac{v_1 b_k x^{1-p}}{n-1}\right)$$

$$+ dn^3 x^{-p\beta}\mathbb{E}\theta_{km}^{\beta} = o(1)\bar{F}(x),$$

where the last step is due to (3), Lemma 1, (20), and (8), using the independence between the product $X_{km}^{+}\theta_{km}$ and $\left(Y_{k,m-1}, |X_{ki}|\prod_{j=1,j\ne m}^{i}\theta_{kj}\right)$. For $K_3'(x, n)$, by Lemma 1 we have

$$K_3'(x, n) \le \sum_{1\le m<k\le n} \mathbb{P}\left(\boldsymbol{X}_m^{+}\boldsymbol{Y}_m > \boldsymbol{b}(1+v_1)x, \boldsymbol{X}_k^{+}\boldsymbol{Y}_k > \boldsymbol{b}(1+v_1)x\right) = o(1)\bar{F}(x).$$

Therefore, for large x, we obtain

$$\mathbb{P}\left(\sum_{i=1}^{n}\boldsymbol{X}_i\boldsymbol{Y}_i > \boldsymbol{b}x\right) \ge (1-C\varepsilon)\sum_{i=1}^{n}\mathbb{P}\left(\boldsymbol{X}_i^{+}\boldsymbol{Y}_i > \boldsymbol{b}x\right).$$

This completes the proof. □

Lemma 3. *If the conditions of Theorem 1 hold, then for every fixed* $\boldsymbol{b} \in [0, \infty]^d \backslash \{\boldsymbol{0}\}$ *we get*

$$\lim_{N\to\infty} \limsup_{x\to\infty} \frac{\mathbb{P}\left(\sum_{i=N+1}^{\infty} \boldsymbol{X}_i^+ \boldsymbol{Y}_i > \boldsymbol{b}x\right)}{\bar{F}(x)} = \lim_{N\to\infty} \limsup_{x\to\infty} \frac{\sum_{i=N+1}^{\infty} \mathbb{P}\left(\boldsymbol{X}_i^+ \boldsymbol{Y}_i > \boldsymbol{b}x\right)}{\bar{F}(x)} = 0.$$

Proof. For any fixed $0 < p < \beta$ and $0 < \varepsilon < \beta - \alpha$, we choose some $0 < q < 1$ satisfying

$$\left\{\left[\mathbb{E}\left(\frac{\theta_k}{q}\right)^{\alpha-\varepsilon}\right] \bigvee \left[\mathbb{E}\left(\frac{\theta_k}{q}\right)^{\alpha+\varepsilon}\right]\right\} < 1, \quad 1 \le k \le d.$$

Then there exists some $n_1 > 0$ such that the following relation holds:

$$\sum_{i=n_1+1}^{\infty} \left\{\left[\mathbb{E}\left(\frac{\theta_1}{q}\right)^{\alpha-\varepsilon}\right]^{i-1} \bigvee \left[\mathbb{E}\left(\frac{\theta_1}{q}\right)^{\alpha+\varepsilon}\right]^{i-1}\right\} \le C\varepsilon.$$

Since $X_{1i}^+\theta_{1i}$ is of regular variation from (8), applying [20, Lemma 1] for large x we obtain

$$\begin{aligned}
&\mathbb{P}\left(\sum_{i=n_1+1}^{\infty} \boldsymbol{X}_i^+ \boldsymbol{Y}_i > \boldsymbol{b}x\right) \le \mathbb{P}\left(\sum_{i=n_1+1}^{\infty} \boldsymbol{X}_i^+ \boldsymbol{Y}_i > \sum_{i=n_1+1}^{\infty} \boldsymbol{b}(1-q)q^{i-1}x\right) \\
&\le \sum_{i=n_1+1}^{\infty} \mathbb{P}\left(\boldsymbol{X}_i^+ \boldsymbol{\theta}_i \frac{\boldsymbol{Y}_{i-1}}{q^{i-1}} > \boldsymbol{b}(1-q)x\right) \le \sum_{i=n_1+1}^{\infty} \mathbb{P}\left(X_{1i}^+ \theta_{1i} \frac{Y_{1,i-1}}{q^{i-1}} > b_1(1-q)x\right) \\
&\le C\bar{F}(b_1(1-q)x) \sum_{i=n_1+1}^{\infty} \mathbb{E}\left[\left(\frac{Y_{1,i-1}}{q^{i-1}}\right)^{\alpha-\varepsilon} \bigvee \left(\frac{Y_{1,i-1}}{q^{i-1}}\right)^{\alpha+\varepsilon}\right] \\
&\le C\bar{F}(x) \sum_{i=n_1+1}^{\infty} \left\{\left[\mathbb{E}\left(\frac{\theta_1}{q}\right)^{\alpha-\varepsilon}\right]^{i-1} \bigvee \left[\mathbb{E}\left(\frac{\theta_1}{q}\right)^{\alpha+\varepsilon}\right]^{i-1}\right\} \\
&\le C\varepsilon\bar{F}(x).
\end{aligned}$$

This completes the proof. □

4.3. Proof of Theorem 1

Proof. First we show that for any n_2 sufficiently large, the relation (17) holds uniformly for $1 \le n \le n_2$. By Lemma 2, the following relations hold uniformly for $1 \le n \le n_2$:

$$\begin{aligned}
\Psi(x, n) = \mathbb{P}\left(\max_{1\le m\le n} \sum_{i=1}^{m} \boldsymbol{X}_i \boldsymbol{Y}_i > \boldsymbol{\rho}x\right) &\le \mathbb{P}\left(\sum_{i=1}^{n} \boldsymbol{X}_i^+ \boldsymbol{Y}_i > \boldsymbol{\rho}x\right) \\
&\sim \sum_{i=1}^{n} \mathbb{P}\left(\boldsymbol{X}_i^+ \boldsymbol{Y}_i > \boldsymbol{\rho}x\right)
\end{aligned}$$

and

$$\Psi(x, n) = \mathbb{P}\left(\max_{1\le m\le n}\sum_{i=1}^{m}\boldsymbol{X}_i\boldsymbol{Y}_i > \boldsymbol{\rho}x\right)$$
$$\ge \mathbb{P}\left(\sum_{i=1}^{n}\boldsymbol{X}_i\boldsymbol{Y}_i > \boldsymbol{\rho}x\right) \sim \sum_{i=1}^{n}\mathbb{P}\left(\boldsymbol{X}_i^{+}\boldsymbol{Y}_i > \boldsymbol{\rho}x\right).$$

Next, we study the uniformity of the relation (17) when $n > n_2$. Clearly,

$$\begin{aligned} 0 < \nu\left(\boldsymbol{\rho}, \boldsymbol{\infty}\right]\sum_{i=1}^{\infty}\mathbb{E}\left(\bigwedge_{k=1}^{d} Y_{k,i-1}^{\alpha}\right) &= \sum_{i=1}^{\infty}\mathbb{E}\left(\nu\left(\bigvee_{k=1}^{d} Y_{k,i-1}^{-1}\left(\boldsymbol{\rho}, \boldsymbol{\infty}\right]\right)\right) \\ \le \sum_{i=1}^{\infty}\mathbb{E}\left(\nu\left(\boldsymbol{Y}_{i-1}^{-1}\boldsymbol{\rho}, \boldsymbol{\infty}\right]\right) &\le \sum_{i=1}^{\infty}\mathbb{E}\left(\nu\left(\bigwedge_{k=1}^{d} Y_{k,i-1}^{-1}\left(\boldsymbol{\rho}, \boldsymbol{\infty}\right]\right)\right) \\ \le \nu\left(\boldsymbol{\rho}, \boldsymbol{\infty}\right]\sum_{i=1}^{\infty}\left(\mathbb{E}Y_{1,i-1}^{\alpha} + \cdots + \mathbb{E}Y_{d,i-1}^{\alpha}\right) &= \nu\left(\boldsymbol{\rho}, \boldsymbol{\infty}\right]\sum_{k=1}^{d}\sum_{i=1}^{\infty}\left(\mathbb{E}\theta_k^{\alpha}\right)^{i-1} \\ < \infty. \end{aligned} \tag{24}$$

On the one hand, it holds uniformly for $n > n_2$ that, by Lemmas 2 and 3 and the relation (24),

$$\begin{aligned} \mathbb{P}\left(\max_{1\le k\le n}\sum_{i=1}^{k}\boldsymbol{X}_i\boldsymbol{Y}_i > \boldsymbol{\rho}x\right) &\ge \mathbb{P}\left(\max_{1\le k\le n_2}\sum_{i=1}^{k}\boldsymbol{X}_i\boldsymbol{Y}_i > \boldsymbol{\rho}x\right) \\ &\ge \mathbb{P}\left(\sum_{i=1}^{n_2}\boldsymbol{X}_i\boldsymbol{Y}_i > \boldsymbol{\rho}x\right) \sim \sum_{i=1}^{n_2}\mathbb{P}\left(\boldsymbol{X}_i^{+}\boldsymbol{Y}_i > \boldsymbol{\rho}x\right) \\ &\ge \left(\sum_{i=1}^{n} - \sum_{i=n_2+1}^{\infty}\right)\mathbb{P}\left(\boldsymbol{X}_i^{+}\boldsymbol{Y}_i > \boldsymbol{\rho}x\right) \\ &\gtrsim \sum_{i=1}^{n}\mathbb{P}\left(\boldsymbol{X}_i^{+}\boldsymbol{Y}_i > \boldsymbol{\rho}x\right). \end{aligned}$$

On the other hand, it holds uniformly for $n > n_2$ that, for v_3 satisfying $(1 - v_3)^{-\alpha} \le 1 + \varepsilon$,

$$\begin{aligned} \mathbb{P}\left(\max_{1\le k\le n}\sum_{i=1}^{k}\boldsymbol{X}_i\boldsymbol{Y}_i > \boldsymbol{\rho}x\right) &\le \mathbb{P}\left(\sum_{i=1}^{n}\boldsymbol{X}_i^{+}\boldsymbol{Y}_i > \boldsymbol{\rho}x\right) \\ &\le \mathbb{P}\left(\sum_{i=1}^{n_2}\boldsymbol{X}_i^{+}\boldsymbol{Y}_i > \boldsymbol{\rho}(1 - v_3)x\right) + 2\sum_{k=1}^{d}\mathbb{P}\left(\sum_{i=n_2+1}^{n} X_{ki}^{+}Y_{ki} > \rho_k v_3 x\right) \\ &:= I_1(x, n_2) + I_2(x, n). \end{aligned}$$

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For $I_1(x, n_2)$, we have by Lemma 2 that

$$I_1(x, n_2) \sim (1 - v_3)^{-\alpha} \bar{F}(x) \sum_{i=1}^{n_2} \mathbb{E}\left\{ \nu(\boldsymbol{Y}_{i-1}^{-1}\boldsymbol{\rho}, \boldsymbol{\infty}] \right\} \sim (1 - v_3)^{-\alpha} \sum_{i=1}^{n_2} \mathbb{P}\left(\boldsymbol{X}_i^{+}\boldsymbol{Y}_i > \boldsymbol{\rho} x \right)$$

$$\leq (1 + \varepsilon) \sum_{i=1}^{n} \mathbb{P}\left(\boldsymbol{X}_i^{+}\boldsymbol{Y}_i > \boldsymbol{\rho} x \right).$$

For $I_2(x, n)$, we obtain by Lemmas 2 and 3 and the relation (24) that

$$I_2(x, n) \leq 2 \sum_{k=1}^{d} \mathbb{P}\left(\sum_{i=n_2+1}^{\infty} X_{ki}^{+} Y_{ki} > \rho_k v_3 x \right) = o(1)\bar{F}(x) = o(1) \sum_{i=1}^{n} \mathbb{P}\left(\boldsymbol{X}_i^{+}\boldsymbol{Y}_i > \boldsymbol{\rho} x \right).$$

This completes the proof. □

5. The study of *d*-dimensional continuous-time risk model under Assumption 1

In this section we consider an insurance company with d lines of business, for $d \geq 1$. Let x be the initial reserve and let $\boldsymbol{\rho}$ be the allocation vector. For each $1 \leq k \leq d$, the price process of the investment portfolio in the kth business is expressed by a geometric Lévy process $\{e^{L_k(t)},\ t \geq 0\}$; namely, $\{L_k(t),\ t \geq 0\}$ is a Lévy process that has stationary and independent increments, is stochastically continuous, and starts from 0. Then the insurer's discounted risk process, $\mathbf{R}(t) = (R_1(t),\ \ldots,\ R_d(t))$, is given by

$$\mathbf{R}(t) = \left(\sum_{i=1}^{N(t)} A_{1i} e^{-L_1(\tau_i)} - c_1 \int_0^t e^{-L_1(s)} \mathrm{d}s,\ \ldots,\ \sum_{i=1}^{N(t)} A_{di} e^{-L_d(\tau_i)} - c_d \int_0^t e^{-L_d(s)} \mathrm{d}s \right), \tag{25}$$

with $t \geq 0$, where A_{ki}, $1 \leq k \leq d$ and $i \in \mathbb{N}$, denotes the ith claim amount from the kth business, and $(c_1,\ \ldots,\ c_d)$ denotes the vector of constant premium rates. The successive claim arrival times are denoted by $0 < \tau_1 < \tau_2 < \cdots$, and the claim arrival process $\{N(t); t \geq 0\}$ is a renewal process with the following finite renewal function:

$$\lambda(t) = \mathbb{E}N(t) = \sum_{i=1}^{\infty} \mathbb{P}(\tau_i \leq t).$$

The vectors $(A_{1i},\ \ldots,\ A_{di})$, $i \in \mathbb{N}$, form a sequence of i.i.d. nonnegative random vectors with a generic random vector $(A_1,\ \ldots,\ A_d)$. The inter-arrival times are denoted by $\chi_1 = \tau_1$ and $\chi_i = \tau_i - \tau_{i-1}$ for $i = 2,\ 3,\ \ldots$. The sequence $\{\chi_i,\ i \geq 1\}$ is i.i.d. with generic random variable χ. We define the Laplace exponent for the Lévy process $\{L_k(t),\ t \geq 0\}$, $1 \leq k \leq d$, by the formula

$$\phi_k(s) = \log \mathbb{E} e^{-sL_k(1)}, \qquad s \in (-\infty, \infty). \tag{26}$$

If $\phi_k(s)$ is finite, then $\mathbb{E}e^{-sL_k(t)} = e^{t\phi_k(s)} < \infty$ for $t \geq 0$.

In the multivariate risk model case, ruin may appear in various situations. Thus, there are several versions of the probabilities of ruin based on various ruin sets. For the model (25) we adopt the following three ruin times:

$$T_{\max} = \inf\{t > 0 : \boldsymbol{\rho} x - R(t) < \mathbf{0}\}, \tag{27}$$

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which is the first instant when all line reserves become negative simultaneously,

$$T_{\min} = \inf\left\{t > 0\colon \min_{1\le k\le d}\{\rho_k x - R_k(t)\} < 0\right\}, \tag{28}$$

which is the first instant when at least one of the businesses is below zero, and

$$T_{\text{ult}} = \inf\left\{t > 0\colon \inf_{0\le s\le t}\{\rho_1 x - R_1(s)\} < 0, \ldots, \inf_{0\le s\le t}\{\rho_d x - R_d(s)\} < 0\right\}, \tag{29}$$

which is the first instant at which all lines have at some point run into deficit, though not necessarily simultaneously. Here, $\inf\emptyset$ is understood as ∞ by convention. In this section we study the following three types of ruin probabilities:

$$\Psi_{\#}(x, \infty) = \mathbb{P}\left(T_{\#} < \infty | \mathbf{R}(0) = \boldsymbol{\rho} x\right), \qquad 0 < t \le \infty, \tag{30}$$

where '#' denotes either 'max', 'min', or 'ult'. The ruin probability represents a significant indicator of the functional quality of the insurance company and has been studied extensively by [3], [13], [16], and [19], and others.

Theorem 2. *Consider the d-dimensional continuous-time risk model (25). Assume that the vector* $\mathbf{A}_i e^{\mathbf{L}(\tau_{i-1})-\mathbf{L}(\tau_i)} = \left(A_{1i}e^{L_1(\tau_{i-1})-L_1(\tau_i)}, \ldots, A_{di}e^{L_d(\tau_{i-1})-L_d(\tau_i)}\right)$, $i\in\mathbb{N}$, *satisfies Assumption 1. If* $\mathbb{E}L_k(1) < \infty$ *for* $1\le k\le d$ *and there is a constant* $\beta > \alpha$ *such that the Laplace exponent of the Lévy process* $\{L_k(t),\ t\ge 0\}$ *satisfies* $\phi_k(\beta) < 0$, *then the following asymptotic formulas hold:*

$$\Psi_{max}(x,\ \infty) \sim \Psi_{ult}(x,\ \infty) \sim \bar{F}(x)\sum_{i=1}^{\infty}\mathbb{E}\left(\nu\left\{\left(e^{\mathbf{L}(\tau_{i-1})}\boldsymbol{\rho}, \boldsymbol{\infty}\right]\right\}\right), \tag{31}$$

$$\Psi_{min}(x,\ \infty)$$
$$\sim \bar{F}(x)\sum_{k=1}^{d}(-1)^{k-1}\sum_{1\le m_1<\cdots<m_k\le d}\sum_{i=1}^{\infty}\mathbb{E}\left(\nu\left\{\left(e^{\mathbf{L}(\tau_{i-1})}\sum_{l=1}^{k}\rho_{m_l}e_{m_l}, \boldsymbol{\infty}\right]\right\}\right). \tag{32}$$

Remark 4. Suppose that $L_k(t)$, $1\le k\le d$, follow the jump-diffusion process

$$L_k(t) = r_k t + \sigma_k W_k(t) + \sum_{i=1}^{N(t)} B_{ki}, \tag{33}$$

where $r_k\in\mathbb{R}$ stands for the log return rate, $\sigma_k > 0$ denotes the volatility, and $N(t)$ represents the aforementioned claim arrival process. Here, $\{W_k(t),\ t\ge 0\}$ is a Wiener process and B_{ki} describes the jump sizes. Assume that for $1\le k\le d$ all the random sources $\{(A_{ki}, B_{ki}), i\ge 1\}$, $\{N(t), t\ge 0\}$, and $\{W_k(t), t\ge 0\}$ are mutually independent and $(A_{1i}e^{-B_{1i}}, \ldots, A_{di}e^{-B_{di}})$ satisfies Assumption 1. We observe that $(A_{1i}e^{-B_{1i}}, \ldots, A_{di}e^{-B_{di}})$ and $(e^{-r_1\chi_i-\sigma_1 W_1(\chi_i)}, \ldots, e^{-r_d\chi_i-\sigma_d W_d(\chi_i)})$ are independent and

$$\mathbb{E}[e^{-\beta r_1\chi_i-\beta\sigma_1 W_1(\chi_i)}] < \infty,$$

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for $\beta > \alpha$. Furthermore,

$$\begin{aligned}&\mathbb{P}\left(\mathbf{A}_i e^{\mathbf{L}(\tau_{i-1})-\mathbf{L}(\tau_i)} > \mathbf{b}x\right)\\ &= \mathbb{P}\left(A_{1i}e^{-B_{1i}}e^{-r_1\chi_i-\sigma_1 W_1(\chi_i)} > b_1 x, \ldots, A_{di}e^{-B_{di}}e^{-r_d\chi_i-\sigma_d W_d(\chi_i)} > b_d x\right)\end{aligned}$$

for any $\mathbf{b} \in [0, \infty]^d \setminus \{\mathbf{0}\}$, which implies that the random vector $\boldsymbol{A}_i e^{\mathbf{L}(\tau_{i-1})-\mathbf{L}(\tau_i)}$ is MRV by Proposition 1.

5.1. Some lemmas before the proof of Theorem 2

Lemma 4. *Let the conditions of Theorem 2 hold. Then for any* $\mathbf{b} \in [0, \infty]^d \setminus \{\mathbf{0}\}$ *and* n *we get*

$$\mathbb{P}\left\{\sum_{i=1}^{n} \boldsymbol{A}_i e^{-\mathbf{L}(\tau_i)} > \boldsymbol{b}x\right\} \sim \sum_{i=1}^{n} \bar{F}(x)\mathbb{E}\left(\nu\left\{\left(e^{\mathbf{L}(\tau_{i-1})}\boldsymbol{b}, \infty\right]\right\}\right).$$

Proof. From the conditions of Theorem 2, we obtain by Hölder's inequality that for any fixed $i \in \mathbb{N}$,

$$\mathbb{E}e^{-p\mathbf{L}(\tau_{i-1})} \leq \left(\mathbb{E}e^{-\beta\mathbf{L}(\tau_{i-1})}\right)^{p/\beta} = \left(\mathbb{E}e^{\tau_{i-1}\boldsymbol{\phi}(\beta)}\right)^{p/\beta} < \infty, \qquad p \leq \beta. \tag{34}$$

Hence by Proposition 1 we find

$$\begin{aligned}\mathbb{P}\left(\boldsymbol{A}_i e^{-\mathbf{L}(\tau_i)} > \boldsymbol{b}x\right) &= \mathbb{P}\left\{\frac{\boldsymbol{A}_i e^{[\mathbf{L}(\tau_{i-1})-\mathbf{L}(\tau_i)]} \cdot e^{-\mathbf{L}(\tau_{i-1})}}{x} \in (\boldsymbol{b}, \infty]\right\}\\ &\sim \bar{F}(x)\mathbb{E}\left(\nu\left\{\left(e^{\mathbf{L}(\tau_{i-1})}\boldsymbol{b}, \infty\right]\right\}\right).\end{aligned} \tag{35}$$

Indeed, in Lemma 2, for each $i = 1, \ldots, n$, take $\boldsymbol{X}_i = \boldsymbol{A}_i$,

$$\boldsymbol{\theta}_i = e^{[\mathbf{L}(\tau_{i-1})-\mathbf{L}(\tau_i)]}, \qquad \boldsymbol{Y}_{i-1} = e^{-\mathbf{L}(\tau_{i-1})};$$

then, applying Lemma 2 together with (35), we obtain

$$\mathbb{P}\left\{\sum_{i=1}^{n} \boldsymbol{A}_i e^{-\mathbf{L}(\tau_i)} > \boldsymbol{b}x\right\} \sim \sum_{i=1}^{n} \bar{F}(x)\mathbb{E}\left(\nu\left\{\left(e^{\mathbf{L}(\tau_{i-1})}\boldsymbol{b}, \infty\right]\right\}\right).$$

The proof is complete. □

Lemma 5. *Let the conditions of Theorem 2 hold. Then for any* $\mathbf{b} \in [0, \infty]^d \setminus \{\mathbf{0}\}$ *we get*

$$\mathbb{P}\left\{\sum_{i=1}^{\infty} \boldsymbol{A}_i e^{-\mathbf{L}(\tau_i)} > \mathbf{b}x\right\} \sim \bar{F}(x)\sum_{i=1}^{\infty} \mathbb{E}\left(\nu\left\{\left(e^{\mathbf{L}(\tau_{i-1})}\boldsymbol{b}, \infty\right]\right\}\right).$$

Proof. Take $\boldsymbol{X}_i = \boldsymbol{A}_i$ and

$$\boldsymbol{\theta}_i = e^{[\mathbf{L}(\tau_{i-1})-\mathbf{L}(\tau_i)]}, \qquad \boldsymbol{Y}_{i-1} = e^{-\mathbf{L}(\tau_{i-1})},$$

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for each $i = 1, \ldots, n$. Using Theorem 1 with $n = \infty$ and (35), we obtain

$$\mathbb{P}\left\{\sum_{i=1}^{\infty} \boldsymbol{A}_i e^{-\mathbf{L}(\tau_i)} > \mathbf{b}x\right\} \sim \bar{F}(x) \sum_{i=1}^{\infty} \mathbb{E}\left(\nu\left\{\left(e^{\mathbf{L}(\tau_{i-1})}\boldsymbol{b}, \infty\right]\right\}\right).$$

This completes the proof. □

The following lemma is significant in the literature on the uniform estimates for the ruin probability of insurance risk models with an exponential Lévy process investment return (see for example [19, Lemma 4.8] or [28, Lemma 4.6]).

Lemma 6. *Let Z_k, $1 \le k \le d$, be an exponential functional of the Lévy process $\{L_k(t),\ t \ge 0\}$ defined as*

$$Z_k = \int_0^\infty e^{-L_k(t)} \mathrm{d}t.$$

If $\mathbb{E}L_k(1) < \infty$, then for every $\beta > 0$ satisfying $\phi_k(\beta) < 0$, we have $\mathbb{E}Z_k^\beta < \infty$.

Proof. From [28], the condition $\phi_k(\beta) < 0$ implies that $\mathbb{E}L_k(1) > 0$. Since $0 < \mathbb{E}L_k(1) < \infty$, we can apply [23, Lemma 2.1] to complete the proof. □

5.2. Proof of Theorem 2

Proof. From the conditions in Theorem 2 we may conclude that $\phi_k(\alpha) < 0$ for $\alpha < \beta$ and $1 \le k \le d$. Then

$$\begin{aligned}
0 &< \nu\left(\boldsymbol{\rho}, \infty\right] \sum_{i=1}^{\infty} \mathbb{E}\left(\bigwedge_{k=1}^{d} e^{-\alpha L_k(\tau_{i-1})}\right) = \sum_{i=1}^{\infty} \mathbb{E}\left(\nu\left(\bigvee_{k=1}^{d} e^{L_k(\tau_{i-1})}\left(\boldsymbol{\rho}, \infty\right]\right)\right) \\
&\le \sum_{i=1}^{\infty} \mathbb{E}\left(\nu\left(e^{\mathbf{L}(\tau_{i-1})}\boldsymbol{\rho}, \infty\right]\right) \le \sum_{i=1}^{\infty} \mathbb{E}\left(\nu\left(\bigwedge_{k=1}^{d} e^{L_k(\tau_{i-1})}\left(\boldsymbol{\rho}, \infty\right]\right)\right) \\
&\le \nu\left(\boldsymbol{\rho}, \infty\right] \sum_{i=1}^{\infty} \mathbb{E}\left\{e^{-\alpha L_1(\tau_{i-1})} + \cdots + e^{-\alpha L_d(\tau_{i-1})}\right\} \\
&= \nu\left(\boldsymbol{\rho}, \infty\right] \sum_{k=1}^{d} \sum_{i=1}^{\infty} \left[\mathbb{E}e^{\chi\phi_k(\alpha)}\right]^{i-1} < \infty. \qquad (36)
\end{aligned}$$

Since

$$\sum_{i=1}^{n} \mathbb{E}\left(\nu\left(e^{\mathbf{L}(\tau_{i-1})}\boldsymbol{\rho}, \infty\right]\right) \uparrow \sum_{i=1}^{\infty} \mathbb{E}\left(\nu\left(e^{\mathbf{L}(\tau_{i-1})}\boldsymbol{\rho}, \infty\right]\right) \quad \text{as } n \to \infty,$$

there exists some large n_3 such that, for any $\varepsilon > 0$,

$$\sum_{i=1}^{\infty} \mathbb{E}\left(\nu\left(e^{\mathbf{L}(\tau_{i-1})}\boldsymbol{\rho}, \infty\right]\right) - \sum_{i=1}^{n_3} \mathbb{E}\left(\nu\left(e^{\mathbf{L}(\tau_{i-1})}\boldsymbol{\rho}, \infty\right]\right) < \varepsilon. \qquad (37)$$

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We first focus on the relation (31) for $\Psi_{\max}(x,\ \infty)$. By (25), (27), and (30), we have

$$\Psi_{\max}(x,\ \infty)=\mathbb{P}\left(\bigcup_{0<s<\infty}\left\{\boldsymbol{\rho}x+\boldsymbol{c}\int_0^s e^{-\mathbf{L}(u)}\mathrm{d}u-\sum_{i=1}^{\infty}\boldsymbol{A}_i e^{-\mathbf{L}(\tau_i)}\mathbb{I}_{[\tau_i\le s]}<\mathbf{0}\right\}\right).$$

Then, by Lemma 5, it holds that

$$\Psi_{\max}(x,\ \infty)\le\mathbb{P}\left(\sum_{i=1}^{\infty}\boldsymbol{A}_i e^{-\mathbf{L}(\tau_i)}>\boldsymbol{\rho}x\right)\sim\sum_{i=1}^{\infty}\bar{F}(x)\mathbb{E}\left(\nu\left\{\left(e^{\mathbf{L}(\tau_{i-1})}\boldsymbol{\rho},\infty\right]\right\}\right). \tag{38}$$

Let

$$\mathbf{Z}=(Z_1,\ldots,Z_d)=\left(\int_0^\infty e^{-L_1(s)}\mathrm{d}s,\ldots,\int_0^\infty e^{-L_d(s)}\mathrm{d}s\right).$$

By Lemma 1, we have

$$\begin{aligned}\Psi_{\max}(x,\ \infty)&\ge\mathbb{P}\left\{\sum_{i=1}^{n_3}\boldsymbol{A}_i e^{-\mathbf{L}(\tau_i)}>\boldsymbol{\rho}x+\boldsymbol{c}\int_0^\infty e^{-\mathbf{L}(s)}\mathrm{d}s\right\}\\
&\ge\mathbb{P}\left\{\bigcup_{i=1}^{n_3}\left(\boldsymbol{A}_i e^{-\mathbf{L}(\tau_i)}>\boldsymbol{\rho}x+\boldsymbol{c}\mathbf{Z}\right)\right\}\\
&\ge\sum_{i=1}^{n_3}\mathbb{P}\left(\boldsymbol{A}_i e^{-\mathbf{L}(\tau_i)}>\boldsymbol{\rho}x+\boldsymbol{c}\mathbf{Z}\right)-\sum_{1\le i<j\le n_3}\mathbb{P}\left(\boldsymbol{A}_i e^{-\mathbf{L}(\tau_i)}>\boldsymbol{\rho}x,\boldsymbol{A}_j e^{-\mathbf{L}(\tau_j)}>\boldsymbol{\rho}x\right)\\
&:=I_1(x)+o(1)\bar{F}(x).\end{aligned} \tag{39}$$

For $I_1(x)$, choose v_4 satisfying

$$(1+v_4)^{-\alpha}\ge(1-\varepsilon).$$

By Lemma 4, $F\in\mathcal{R}_{-\alpha}$, and (37), we have

$$\begin{aligned}&\sum_{i=1}^{n_3}\mathbb{P}\left(\boldsymbol{A}_i e^{-\mathbf{L}(\tau_i)}>\boldsymbol{\rho}(1+v_4)x\right)\sim(1+v_4)^{-\alpha}\bar{F}(x)\sum_{i=1}^{n_3}\mathbb{E}\left(\nu\left\{\left(e^{\mathbf{L}(\tau_{i-1})}\boldsymbol{\rho},\infty\right]\right\}\right)\\
&\ge(1-C\varepsilon)\bar{F}(x)\sum_{i=1}^{\infty}\mathbb{E}\left(\nu\left(e^{\mathbf{L}(\tau_{i-1})}\boldsymbol{\rho},\infty\right]\right)-C\varepsilon\bar{F}(x).\end{aligned}$$

From Markov's inequality, Lemma 6, and the relation (3), we obtain

$$\sum_{k=1}^{d}\mathbb{P}\left(c_k Z_k>\rho_k v_4 x\right)\le Cx^{-\beta}\sum_{k=1}^{d}\mathbb{E}Z_k^{\beta}\le C\varepsilon\bar{F}(x). \tag{40}$$

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Then we find

$$
\begin{aligned}
I_1(x) &\geq \sum_{i=1}^{n_3} \mathbb{P}\left(\boldsymbol{A}_i e^{-\mathbf{L}(\tau_i)} > \boldsymbol{\rho} x + \boldsymbol{cZ}, \boldsymbol{cZ} \leq \boldsymbol{\rho} v_4 x\right) \\
&\geq \sum_{i=1}^{n_3} \mathbb{P}\left(\boldsymbol{A}_i e^{-\mathbf{L}(\tau_i)} > \boldsymbol{\rho}(1+v_4)x, \boldsymbol{cZ} \leq \boldsymbol{\rho} v_4 x\right) \geq \sum_{i=1}^{n_3} \mathbb{P}\left(\boldsymbol{A}_i e^{-\mathbf{L}(\tau_i)} > \boldsymbol{\rho}(1+v_4)x\right) \\
&- \sum_{i=1}^{n_3} \mathbb{P}\left\{\boldsymbol{A}_i e^{-\mathbf{L}(\tau_i)} > \boldsymbol{\rho} x, \bigcup_{k=1}^{d} (c_k Z_k > \rho_k v_4 x)\right\} \\
&\geq (1 - C\varepsilon)\bar{F}(x) \sum_{i=1}^{\infty} \mathbb{E}\left(\nu\left\{\left(e^{\mathbf{L}(\tau_{i-1})}\boldsymbol{\rho}, \infty\right]\right\}\right).
\end{aligned} \tag{41}
$$

Combining the relations (38), (39), and (41) yields the relation (31) for $\Psi_{\max}(x, \infty)$.

Now we establish the relation (32) for $\Psi_{\min}(x, \infty)$. We obtain by the inclusion–exclusion principle and Lemma 5 that

$$
\begin{aligned}
&\Psi_{\min}(x, \infty) \\
&\leq \mathbb{P}\left(\bigcup_{k=1}^{d}\left\{\sum_{i=1}^{\infty} A_{ki} e^{-L_k(\tau_i)} > \rho_k x\right\}\right) \\
&= \sum_{k=1}^{d} (-1)^{k-1} \sum_{1 \leq m_1 < \cdots < m_k \leq d} \mathbb{P}\left(\sum_{i=1}^{\infty} \boldsymbol{A}_i e^{-\mathbf{L}(\tau_i)} > \left(\sum_{l=1}^{k} \rho_{m_l} e_{m_l}\right) x\right) \\
&\sim \bar{F}(x) \sum_{k=1}^{d} (-1)^{k-1} \sum_{1 \leq m_1 < \cdots < m_k \leq d} \sum_{i=1}^{\infty} \mathbb{E}\left(\nu\left\{\left(e^{\mathbf{L}(\tau_{i-1})} \sum_{l=1}^{k} \rho_{m_l} e_{m_l}, \infty\right]\right\}\right).
\end{aligned} \tag{42}
$$

By the inclusion–exclusion principle and (40), we get

$$
\begin{aligned}
&\Psi_{\min}(x, \infty) \\
&\geq \mathbb{P}\left(\bigcup_{k=1}^{d}\left\{\sum_{i=1}^{n_3} A_{ki} e^{-L_k(\tau_i)} > \rho_k x + c_k Z_k\right\}, \bigcap_{k=1}^{d} \{c_k Z_k \leq \rho_k v_4 x\}\right) \\
&\geq \mathbb{P}\left(\bigcup_{k=1}^{d}\left\{\sum_{i=1}^{n_3} A_{ki} e^{-L_k(\tau_i)} > \rho_k (1+v_4) x\right\}\right) \\
&\quad - \mathbb{P}\left(\bigcup_{k=1}^{d}\left\{\sum_{i=1}^{n_3} A_{ki} e^{-L_k(\tau_i)} > \rho_k x + c_k Z_k\right\}, \left(\bigcap_{k=1}^{d} \{c_k Z_k \leq \rho_k v_4 x\}\right)^c\right)
\end{aligned}
$$

$$\geq (1-C\varepsilon)\bar{F}(x)\sum_{k=1}^{d}(-1)^{k-1}\sum_{1\leq m_1<\cdots<m_k\leq d}\sum_{i=1}^{\infty}\mathbb{E}\left(\nu\left\{\left(e^{\mathbf{L}(\tau_{i-1})}\sum_{l=1}^{k}\rho_{m_l}e_{m_l},\,\infty\right]\right\}\right)$$

$$-\sum_{k=1}^{d}\mathbb{P}\left(c_kZ_k>\rho_k v_4 x\right)$$

$$\geq (1-C\varepsilon)\bar{F}(x)\sum_{k=1}^{d}(-1)^{k-1}\sum_{1\leq m_1<\cdots<m_k\leq d}\sum_{i=1}^{\infty}\mathbb{E}\left(\nu\left\{\left(e^{\mathbf{L}(\tau_{i-1})}\sum_{l=1}^{k}\rho_{m_l}e_{m_l},\,\infty\right]\right\}\right)$$

$$+o(1)\bar{F}(x). \tag{43}$$

Combining (42) and (43) gives the relation (32).

To establish the relation (31) for $\Psi_{\mathrm{ult}}(x,\infty)$, we note that by Lemma 5,

$$\Psi_{\mathrm{ult}}(x,\infty)=\mathbb{P}\left\{\inf_{0<s<\infty}\left(\rho_1x+c_1\int_0^s e^{-L_1(u)}\mathrm{d}u-\sum_{i=1}^{N(s)}A_{1i}e^{-L_1(\tau_i)}\right)<0,\ldots,\right.$$
$$\left.\inf_{0<s<\infty}\left(\rho_dx+c_d\int_0^s e^{-L_d(u)}\mathrm{d}u-\sum_{i=1}^{N(s)}A_{di}e^{-L_d(\tau_i)}\right)<0\right\}$$
$$\leq\mathbb{P}\left\{\sum_{i=1}^{\infty}A_{1i}e^{-L_1(\tau_i)}>\rho_1x,\ldots,\sum_{i=1}^{\infty}A_{di}e^{-L_d(\tau_i)}>\rho_dx\right\}$$
$$=\mathbb{P}\left\{\sum_{i=1}^{\infty}\mathbf{A}_ie^{-\mathbf{L}(\tau_i)}>\boldsymbol{\rho}x\right\}\sim\sum_{i=1}^{\infty}\bar{F}(x)\mathbb{E}\left(\nu\left\{\left(e^{\mathbf{L}(\tau_{i-1})}\boldsymbol{\rho},\,\boldsymbol{\infty}\right]\right\}\right). \tag{44}$$

Analogously to the reasoning for the lower bound of $\Psi_{\max}(x,\ \infty)$, it follows that

$$\Psi_{\mathrm{ult}}(x,\infty)=\mathbb{P}\left\{\sup_{0<s<\infty}R(t)>\boldsymbol{\rho}x\right\}$$
$$\geq\mathbb{P}\left\{\sum_{i=1}^{\infty}A_{1i}e^{-L_1(\tau_i)}-c_1Z_1>\rho_1x,\ \ldots,\ \sum_{i=1}^{\infty}A_{di}e^{-L_d(\tau_i)}-c_dZ_d>\rho_dx\right\}$$
$$\geq\mathbb{P}\left\{\sum_{i=1}^{n_3}\mathbf{A}_ie^{-\mathbf{L}(\tau_i)}>\boldsymbol{\rho}x+\mathbf{c}\mathbf{Z}\right\}$$
$$\geq(1-C\varepsilon)\bar{F}(x)\sum_{i=1}^{\infty}\mathbb{E}\left(\nu\left\{\left(e^{\mathbf{L}(\tau_{i-1})}\boldsymbol{\rho},\,\boldsymbol{\infty}\right]\right\}\right). \tag{45}$$

Combining the relations (44) and (45) gives the relation (31) for $\Psi_{\mathrm{ult}}(x,\ \infty)$. This completes the proof. □

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References

[1] ASIMIT, A. V. AND BADESCU, A. L. (2010). Extremes on the discounted aggregate claims in a time dependent risk model. *Scand. Actuarial J.* **2010**, 93–104.

[2] ASIMIT, A. V. AND JONES, B. L. (2008). Dependence and the asymptotic behavior of large claims reinsurance. *Insurance Math. Econom.* **43**, 407–411.

[3] ASMUSSEN, S. AND ALBRECHER, H. (2010). *Ruin Probabilities*. World Scientific, Singapore.

[4] BASRAK, B., DAVIS, R. A. AND MIKOSCH, T. (2002). Regular variation of GARCH processes. *Stoch. Process. Appl.* **99**, 95–115.

[5] BINGHAM, N. H., GOLDIE, C. M. AND TEUGELS, J. L. (1987). *Regular Variation*. Cambridge University Press.

[6] CHEN, Y. (2017). Interplay of subexponential and dependent insurance and financial risks. *Insurance Math. Econom.* **77**, 78–83.

[7] CHEN, Y., WANG, J. AND ZHANG, W. (2021). Tail distortion risk measure for portfolio with multivariate regularly variation. *Commun. Math. Statist.* **10**, 263–285.

[8] CHEN, Y. AND YANG, Y. (2019). Bivariate regular variation among randomly weighted sums in general insurance. *Europ. Actuarial J.* **9**, 301–322.

[9] CHEN, Y. AND YUAN, Z. (2017). A revisit to ruin probabilities in the presence of heavy-tailed insurance and financial risks. *Insurance Math. Econom.* **73**, 75–81.

[10] CHENG, M., KONSTANTINIDES, D. G. AND WANG, D. (2022). Uniform asymptotic estimates in a time-dependent risk model with general investment returns and multivariate regularly varying claims. *Appl. Math. Comput.* **434**, article no. 127436.

[11] FOUGERES, A. AND MERCADIER, C. (2012). Risk measures and multivariate extensions of Breiman's theorem. *J. Appl. Prob.* **49**, 364–384.

[12] HEYDE, C. C. AND WANG, D. (2009). Finite-time ruin probability with an exponential Lévy process investment return and heavy-tailed claims. *Adv. Appl. Prob.* **41**, 206–224.

[13] JI, L. (2020). On the cumulative Parisian ruin of multi-dimensional Brownian motion risk models. *Scand. Actuarial J.* **9**, 819–842.

[14] JOE, H. AND LI, H. (2011). Tail risk of multivariate regular variation. *Methodology Comput. Appl. Prob.* **13**, 671–693.

[15] Klüppelberg, C. and Kostadinova, R. (2008). Integrated insurance risk models with exponential Lévy investment. *Insurance Math. Econom.* **42**, 560–577.

[16] KONSTANTINIDES, D. G. AND LI, J. (2016). Asymptotic ruin probabilities for a multidimensional renewal risk model with multivariate regularly varying claims. *Insurance Math. Econom.* **69**, 38–44.

[17] KONSTANTINIDES, D. G. (2018). *Risk Theory: A Heavy Tail Approach*. World Scientific, Singapore.

[18] LI, H. AND SUN Y. (2009). Tail dependence for heavy-tailed scale mixtures of multivariate distributions. *J. Appl. Prob.* **46**, 925–937.

[19] LI, J. (2016). Uniform asymptotics for a multi-dimensional time-dependent risk model with multivariate regularly varying claims and stochastic return. *Insurance Math. Econom.* **71**, 195–204.

[20] LI, J. (2018). On the joint tail behavior of randomly weighted sums of heavy-tailed random variables. *J. Multivariate Anal.* **164**, 40–53.

[21] LI, J. AND TANG, Q. (2015). Interplay of insurance and financial risks in a discrete-time model with strongly regular variation. *Bernoulli* **21**, 1800–1823.

[22] LI, J., TANG, Q. AND WU, R. (2010). Subexponential tails of discounted aggregate claims in a time-dependent renewal risk model. *Adv. Appl. Prob.* **42**, 1126–1146.

[23] MAULIK, K. AND ZWART, B. (2006). Tail asymptotics for exponential functionals of Lévy process. *Stoch. Process. Appl.* **116**, 156–177.

[24] NORBERG, R. (1999). Ruin problems with assets and liabilities of diffusion type. *Stoch. Process. Appl.* **81**, 255–269.

[25] NELSEN, R. B. (2007). *An Introduction to Copulas*. Springer, New York.

[26] PAULSEN, J. (2008). Ruin models with investment income. *Prob. Surveys* **5**, 416–434.

[27] RESNICK, S. I. (2007). *Heavy-Tail Phenomena: Probabilistic and Statistical Modelling*. Springer, New York.

[28] TANG, Q., WANG, G. AND YUEN, K. C. (2010). Uniform tail asymptotics for the stochastic present value of aggregate claims in the renewal risk model. *Insurance Math. Econom.* **46**, 362–370.

[29] TANG, Q. AND YANG, Y. (2019). Interplay of insurance and financial risks in a stochastic environment. *Scand. Actuarial J.* **5**, 432–451.

[30] TANG, Q. AND YUAN, Z. (2012). A hybrid estimate for the finite-time ruin probability in a bivariate autoregressive risk model with application to portfolio optimization. *N. Amer. Actuarial J.* **16**, 378–397.

[31] TANG, Q. AND YUAN, Z. (2016). Interplay of insurance and financial risks with bivariate regular variation. In *Extreme Value Modeling and Risk Analysis: Methods and Applications*, eds D. K. Dey and J. Yan, CRC Press, Boca Raton, FL, pp. 419–438.

[32] TANG, Q. AND TSITSIASHVILI, G. (2003). Precise estimates for the ruin probability in finite horizon in a discrete-time model with heavy-tailed insurance and financial risks. *Stoch. Process. Appl.* **108**, 299–325.

[33] YANG, Y. AND KONSTANTINIDES, D. G. (2015). Asymptotics for ruin probabilities in a discrete-time risk model with dependent financial and insurance risks. *Scand. Actuarial J.* **8**, 641–659.

[34] ZHU, L. AND LI, H. (2012). Asymptotic analysis of multivariate tail conditional expectations. *N. Amer. Actuarial J.* **16**, 350–363.