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THE UNRESTRICTED SECTION PROPERTIES OF SEQUENCES

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1. Introduction. An unrestricted section of a sequence x is any sequence of the form $\sum_{k\in F} x_k\delta^k$, where F is some finite subset of the natural numbers. The notion of boundedness of the set of unrestricted sections of a sequence in a K-space was studied in [**10**], and called *unconditional section boundedness* (UAB). It was shown in [**10**] (Theorem 7) that the class of FK-spaces in which every element has UAB consists of those FK-spaces that are invariant under coordinatewise multiplication by the convergent sequences.

In this paper we investigate further the topological properties of the set of unrestricted sections. Theorem 1 shows that the property UAB is related to the classical Köthe α-dual of a sequence space in the same way in which the more familiar section properties FAK and AB are related to the β-dual and γ-dual, respectively. Theorem 2 states that, for barrelled K-spaces with ordinary section boundedness, the property UAB holds if and only if the α-, β-, and γ-duals coincide. Theorem 3 identifies, for any FK-space, the set of sequences x for which the series $\sum x_k\delta^k$ is subseries, or unconditionally, convergent. Theorem 4 contains the fact that, in FK-spaces with UAB, the closure of the set of finitely nonzero sequences is c_0E. Finally we observe, in Theorem 5, that the FK-spaces in which the series $\sum \delta^k$ converges unconditionally in the weak topology are the FK-spaces that are both conservative and conull, and thus are the same FK-spaces as those studied by Snyder in [**11**].

2. Preliminaries. We employ, for the most part, the standard notation, letting ω denote the linear space of all sequences, φ the subspace of finitely nonzero sequences, δ^k the sequence with 1 in the k^{th} coordinate and 0 elsewhere. The coordinatewise product of two sequences t and x will be denoted by $t \cdot x$ or $t(x)$ and, if D and E are sets of sequences, we write $D \cdot E = D(E) = \{t(x) : t \in D, x \in E\}$. The usual sections of a sequence x can then be written as $P(x) = \{P_n(x)\}$, where $P_n = \sum_{k=1}^{n} \delta^k$, and the set of unrestricted sections as $H(x) = \{h_F(x) : F \in \Phi\}$, where $h_F = \sum_{k\in F} \delta^k$ and Φ denotes the collection of all finite subsets of the positive integers. For the various section subspaces and section properties associated with P and H, we use the notation of [**5**] and [**10**]. Thus, for example, in a K-space E,

$$E_{\mathrm{AB}} = \{x \in \omega : P(x) \text{ is bounded in } E\} \quad \text{and}$$
$$E_{\mathrm{USAK}} = \{x \in E : H(x) \rightarrow x \text{ weakly}\},$$

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the convergence in this latter case being unconditional convergence in the weak topology (see [**8**], p. 146 or [**10**], p. 700). Further, we let $E_f = \{f(\delta^k) : f \in E'\}$ and $E_{AD} = \bar{\varphi}$ (the closure of φ in E). The theory of K-spaces and FK-spaces can be found in [**4**], [**6**], and [**12**].

3. UAB **and the α-dual.** The statements in this section concerning the properties AB and FAK are known. They are included here for completeness of presentation.

LEMMA 1. *Let E be any* K-*space. Then*
(*a*) $E_{AB} = (E_f)^\gamma$
(*b*) $E_{FAK} = (E_f)^\beta$
(*c*) $E_{UAB} = (E_f)^\alpha$

Proof. Parts (*a*) and (*b*) are special cases of Theorems 1 and 2, respectively, of [**5**]. Statement (*c*) follows directly from Theorem 1 of [**10**].

THEOREM 1. *If E is any barrelled* K-*space, then*
(*a*) $E \subseteq E_{AB}$ *if and only if* $E_{AB} = E^{\gamma\gamma}$
(*b*) $E \subseteq E_{FAK}$ *if and only if* $E_{FAK} = E^{\beta\beta}$
(*c*) $E \subseteq E_{UAB}$ *if and only if* $E_{UAB} = E^{\alpha\alpha}$

Proof. Part (*a*) is included in Theorem 4 of [**2**] and (*b*) follows from Theorems 2 and 5 of [**2**]. To prove (*c*) note that if $E \subseteq E_{UAB}$, then $E \subseteq E_{AB}$ and therefore (again by Theorem 5 of [**2**]) $E_f = E^\gamma$. Consequently, we have, using Lemma 1,

$$E^\alpha \supseteq (E_{UAB})^\alpha = (E_f)^{\alpha\alpha} = (E^\gamma)^{\alpha\alpha} \supseteq E^\gamma$$

Thus $E^\gamma = E^\alpha$ and $E_{UAB} = E^{\alpha\alpha}$. The converse implication in (*c*) is clear.

From the proof above we can give the following topological characterization of those spaces for which the α-, β-, and γ-duals coincide.

THEOREM 2. *Let E be a barrelled* K-*space with $E \subseteq E_{AB}$. A necessary and sufficient condition for $E^\alpha = E^\beta = E^\gamma$ is that $E \subseteq E_{UAB}$.*

Proof. If $E \subseteq E_{UAB}$ then, as in the proof of Theorem 1, $E^\gamma = E^\alpha$ $(= E^\beta)$. Conversely, if $E^\alpha = E^\beta = E^\gamma$ and $E \subseteq E_{AB}$, then $E_f = E^\gamma$ and $E_{UAB} = (E_f)^\alpha = E^{\gamma\alpha} = E^{\alpha\alpha} \supseteq E$.

COROLLARY 2.1. *Let E be any* FK-*space with ordinary section boundedness. A necessary and sufficient condition for $E^\alpha = E^\beta = E^\gamma$ is that $E = c \cdot E$, where c denotes the set of convergent sequences.*

Proof. For FK-spaces the condition $E \subseteq E_{UAB}$ is equivalent to $E = c \cdot E$ ([**10**], Theorem 7).

4. Unconditional Section Convergence. For any FK-space E, the space bv_0 of null sequences of bounded variation acts as a set of multipliers between

E_{AB} and E_{AK}, that is, $E_{AK} = bv_0 \cdot E_{AB}$ (see [**4**] and [**5**]). If Cesàro sections are considered the corresponding space of multipliers is the space q_0 of quasi-convex null sequences [**1**]. We are able to improve upon a result in [**10**] and show that, for unrestricted sections, the appropriate multiplier space is c_0.

We first use a standard technique to topologize E_{UAB}.

Definition 1. If p is a seminorm on a K-space E, then p_H is the seminorm on E_{UAB} defined by

$$p_H(x) = \sup_{h \in H} p(h(x)).$$

LEMMA 2. *Let $(E, \mathscr{T})$ be a K-space with topology $\mathscr{T}$ generated by the collection of seminorms $\mathscr{P}$. Then $(E_{UAB}, \mathscr{T}_H)$ is a sequentially complete K-space, where $\mathscr{T}_H$ is the topology generated by the collection of seminorms $\mathscr{P}_H = \{p_H : p \in \mathscr{P}\}$.*

Proof. The proof of Theorem 6 of [**2**] can be used with $\mathscr{P}_T$ replaced by $\mathscr{P}_H$. (Note that if p is the coordinate seminorm $p(x) = |x_k|$, then also $p_H(x) = |x_k|$.)

COROLLARY. *If $(E, \mathscr{T})$ is a metrizable K-space, then $(E_{UAB}, \mathscr{T}_H)$ is an FK-space.*

Proof. An FK-space is a sequentially complete metrizable K-space, and a K-space is metrizable if (and only if) its topology is generated by a countable collection of seminorms.

LEMMA 3. *If $(E, \mathscr{T})$ is any K-space, then $(E_{UAB}, \mathscr{T}_H)$ is a sequentially complete K-space in which every element has unconditional section boundedness.*

Proof. If $x \in E_{UAB}$ and p is one of the seminorms for $\mathscr{T}$, we have

$$\sup_{h' \in H} p_H(h'(x)) = \sup_{h' \in H} \sup_{h \in H} p(h(h'(x))) = \sup_{h \in H} p(h(x)) < +\infty$$

We also need the following result, which relates unconditional section convergence in the two spaces $(E, \mathscr{T})$ and $(E_{UAB}, \mathscr{T}_H)$.

LEMMA 4. *If $(E, \mathscr{T})$ is any FK-space, then $E_{UAK} \subseteq (E_{UAB}, \mathscr{T}_H)_{UAK}$.*

Proof. Let $x \in E_{UAK}$, let p be one of the generating seminorms of $\mathscr{T}$, and let $\epsilon > 0$. There exists $F_0 \in \Phi$ such that if $F \in \Phi$ and $F \supseteq F_0$, then

$$p(h_F(x) - x) < \epsilon/4.$$

If $G \in \Phi$ and $G \cap F_0 = \emptyset$, we have

$$p(h_G(x)) = p(h_{F'}(x) - h_{F_0}(x)) \quad (\text{where } F' \supseteq F_0) \leqq p(h_{F'}(x) - x) + p(x - h_{F_0}(x)) < \epsilon/2.$$

For (any) $h \in H$,

$$\begin{aligned} p_H(h(x) - x) &= \sup_{h' \in H} p(h'(h(x)) - h'(x)) \\ &= \sup_{h' \in H} p\Big(\textstyle\sum_{k \in F_0} (h(h'(x)))_k \delta^k + \sum_{k \notin F_0} (h(h'(x)))_k \delta^k \\ &\qquad - \sum_{k \in F_0} (h'(x))_k \delta^k - \sum_{k \notin F_0} (h'(x))_k \delta^k\Big). \end{aligned}$$

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Thus, whenever $F \supseteq F_0$,

$$p_H(h_F(x) - x) = \sup_{h' \in H} p(\textstyle\sum_{k \notin F_0} (h_F(h'(x)))_k \delta^k - \sum_{k \notin F_0} (h'(x))_k \delta^k)$$

$$\leqq 2 \sup_{\substack{h \in H \\ h_k = 0, k \in F_0}} p(h(x)) < 2(\epsilon/2) = \epsilon$$

Therefore $x \in (E_{\mathrm{UAB}}, \mathscr{T}_H)_{\mathrm{UAK}}$.

We are now able to prove

THEOREM 3. *For any* FK-*space* E,

$$\{x \in E : \textstyle\sum x_k \delta^k \text{ is subseries convergent}\} = c_0 \cdot E_{\mathrm{UAB}} = E_{\mathrm{UAK}}.$$

Proof. Since subseries convergence and unconditional convergence are the same in sequentially complete spaces, we need only show that $E_{\mathrm{UAK}} = c_0 \cdot E_{\mathrm{UAB}}$. Combining Theorems 4 and 5 of [**10**] with Lemmas 3 and 4 above, we can write

$$c_0 \cdot E_{\mathrm{UAB}} \subseteq E_{\mathrm{UAK}} \subseteq (E_{\mathrm{UAB}}, \mathscr{T}_H)_{\mathrm{UAK}} = c_0 \cdot E_{\mathrm{UAB}}$$

and the result follows.

COROLLARY 3.1. *If* E *is an* FK-*space with ordinary section boundedness* (*i.e., if* $E \subseteq E_{\mathrm{AB}}$), *then*

$$\{x \in E : \textstyle\sum x_k \delta^k \text{ is subseries convergent}\} = c_0 \cdot E^{\gamma\alpha}$$

Proof. If $E \subseteq E_{\mathrm{AB}}$ then $E_f = E^\gamma$ ([**2**], Theorem 5) and it is always true that $E_{\mathrm{UAB}} = (E_f)^\alpha$ (Lemma 1(*c*)).

COROLLARY 3.2. *If* E *is an* FK-*space, then* E_{UAK} *is solid.*

E_{AK} need not be solid. Corollary 3.2 also follows from [3] ((1)(*d*), p. 59).

5. Some remarks on weak unconditional section convergence. If every sequence in an FK-space E has ordinary section boundedness, then the closure E_{AD} of the set of finitely nonzero sequences is E_{AK} ([**4**], Proposition 1). Parallel to this result we show that if every sequence has unconditional section boundedness, the closure is E_{UAK}. A consequence is that, for such spaces, weak section convergence implies strong unconditional section convergence.

THEOREM 4. *Let* E *be an* FK-*space with* $E \subseteq E_{\mathrm{UAB}}$. *Then* $E_{\mathrm{AD}} = E_{\mathrm{SAK}} = E_{\mathrm{USAK}} = E_{\mathrm{UAK}} = c_0 \cdot E$.

Proof. The inclusions $E_{\mathrm{UAK}} \subseteq E_{\mathrm{USAK}} \subseteq E_{\mathrm{SAK}}$ are clear and $E_{\mathrm{UAK}} = c_0 \cdot E$ is the content of Theorem 5 of [**10**]. If $x \in E_{\mathrm{SAK}}$ then x belongs to the weak closure of $P(x)$, hence to the weak closure of the convex hull of $H(x)$, which is $C_\phi{}^+$ (see [**10**], Lemma 1). Since the weak and strong closures of convex sets are the same, and since $C_\phi{}^+ \subseteq \varphi$, it follows that $x \in E_{\mathrm{AD}}$. Thus $E_{\mathrm{SAK}} \subseteq E_{\mathrm{AD}}$.

We complete the proof by showing that $E_{\mathrm{AD}} \subseteq E_{\mathrm{UAK}}$. If $x \in E_{\mathrm{AD}}$ and if U is a basic absolutely convex neighborhood of zero for which $H(U) \subseteq U$ ([**9**], Theorem 3, p. 69), then there exists $y \in \varphi$ such that $x - y \in \frac{1}{2}U$. There exists N so that $h_F(y) = y$ whenever $F \in \Phi$ with $F \supseteq \{1, 2, \ldots, N\}$. Then, for such F, we have

$$x - h_F(x) = (x - y) + h_F(y - x) \in \tfrac{1}{2}U + \tfrac{1}{2}U = U,$$

and it follows that $x \in E_{\mathrm{UAK}}$.

Theorem 4 includes the fact that if $E \subseteq E_{\mathrm{UAB}}$, then $E_{\mathrm{UAK}} = E_{\mathrm{AK}}$. The converse does not hold. For example, let $E = c_0 \oplus \mathrm{span}\,\{2^i\}$ with topology generated by p and q, where $p(x) = p(y + \lambda 2^i) = ||y||_\infty$ and

$$q(x) = q(y + \lambda 2^i) = |\lambda|.$$

Then E is an FK-space that does not have ordinary section boundedness ($\{2^{-i}\}\{2^i\} \notin E$ and $\{2^{-i}\} \in bv$), whereas $E_{\mathrm{AK}} = E_{\mathrm{UAK}} = c_0$.

Recall that an FK-space E is *conservative* in case $E \supseteq c$ and *conull* in case $1 \in E_{\mathrm{SAK}}$ [**11**].

THEOREM 5. *For an* FK-*space* E, *the following are equivalent:*
(*i*) $1 \in E_{\mathrm{USAK}}$
(*ii*) *E is both conservative and conull.*

Proof. (*i*) $\Rightarrow$ (*ii*). If $1 \in E_{\mathrm{USAK}}$ then clearly $1 \in E_{\mathrm{SAK}}$ and so E is conull. Since $E_{\mathrm{USAK}} \subseteq E_{\mathrm{UAB}}$ (This follows, for example, from [**8**], Theorem 1), we then have $1 \in E_{\mathrm{UAB}}$, and therefore $E \supseteq c_0$ ([**10**], Theorem 1 and Corollary 4.2). It follows that E is conservative. (*ii*) $\Rightarrow$ (*i*). If $1 \in E_{\mathrm{SAK}}$ and E is conservative then, again by Corollary 4.2 of [**10**], $1 \in E_{\mathrm{UAB}}$. Thus

$$1 \in E_{\mathrm{SAK}} \cap E_{\mathrm{UAB}} = E_{\mathrm{USAK}}$$

([**10**], Theorem 2).

REFERENCES

1. M. Buntinas, *Convergent and bounded Cesàro sections in* FK-*spaces*, Math. Z. *121* (1971), 191–200.
2. ——— *On Toeplitz sections in sequence spaces*, Proc. Cambridge Philos. Soc. *78* (1975), 451–460.
3. M. M. Day, *Normed linear spaces* (Berlin-Göttingen-Heidelberg, 1962).
4. D. J. H. Garling, *On topological sequence spaces*, Proc. Cambridge Philos. Soc. *63* (1967), 997–1019.
5. G. Goes, *Summen von* FK-*räumen, funktionale abschnittskonvergenz und Umkehrsätze*, Tôhoku Math. J. *26* (1974), 487–504.
6. C. Goffman and G. Pedrick, *First course in functional analysis* (Englewood Cliffs, N.J., 1965).
7. C. W. McArthur, *On a theorem of Orlicz and Pettis*, Pacific J. Math. *22* (1967), 297–302.

8. A. P. Robertson, *On unconditional convergence in topological vector spaces*, Proc. Roy. Soc. Edin., Sect. A *68* (II) (1969), 145–157.
9. A. P. Robertson and W. J. Robertson, *Topological vector spaces* (Cambridge, 1964).
10. J. J. Sember, *On unconditional section boundedness in sequence spaces*, Rocky Mountain J. of Math. Vol. *7*, No. 4, (1977).
11. A. K. Snyder, *Conull and coregular* FK-*spaces*, Math Z. *90*, (1965), 376–381.
12. A. Wilansky, *Functional analysis* (New York, 1964).

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