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A GENERALISATION OF KRAMER'S THEOREM AND ITS APPLICATIONS

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The main purpose of this paper is to generalise a supersolvability theorem of O. U. Kramer to a saturated formation containing the class of supersolvable groups. As applications, we generalise some results recently obtained by some scholars.

1. Introduction

As two dual concepts of a finite group, the maximal subgroups and the minimal subgroups have been studied by many scholars in determining the structure of a finite group. For instance, B. Huppert's well known theorem shows that a finite group G is supersolvable if and only if every maximal subgroup of G has prime index in G ([**3**]). A theorem of O. U. Kramer shows that a finite solvable group G is supersolvable if and only if, for every maximal subgroup M of G, either $M \geq F(G)$, the Fitting subgroup of G, or $M \cap F(G)$ is a maximal subgroup of $F(G)$ (see [**4**, Theorem 1.3.3]). Buckley in [**2**] proved that a finite group G of odd order is supersolvable if all minimal subgroups of G are normal in G. The main purpose of this paper is to generalise this theorem of Kramer to a saturated formation containing the class of supersolvable groups. Ballester Bolinches, Wang and Guo introduced the concept of c-supplementation of a finite group in [**1**], which is weaker than c-normality or suplementation. They generalised Buckley's theorem by replacing normality with c-supplementation. More recently, Li and Guo in [**6**] obtained two supersolvability theorems on complemented subgroups of finite groups. By using the theory of formations, Wei in [**9**] obtained two results with respect to c-normal subgroups of finite groups. As applications of our main result, we generalise the above theorems to a saturated formation containing the class of supersolvable groups by minimising the number of c-supplemented minimal subgroups or replacing complementation and c-normality with c-supplementation.

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Let $\mathcal{F}$ be a class of finite groups. We call $\mathcal{F}$ a formation provided:

(1) If $G \in \mathcal{F}$ and $N \triangleleft G$, then $G/N \triangleleft \mathcal{F}$;

(2) If $N_1, N_2 \triangleleft G$ such that $G/N_1, G/N_2 \in \mathcal{F}$, then $G/(N_1 \cap N_2) \in \mathcal{F}$.

A formation $\mathcal{F}$ is said to be saturated if $G/\Phi(G) \in \mathcal{F}$ implies that $G \in \mathcal{F}$ (refer to [7]).

All groups considered in this paper are finite; $\mathcal{U}$ and $\pi(G)$ denote, respectively, the class of all supersolvable groups and the set of prime divisors of $|G|$.

2. Preliminaries

Definition 2.1: ([1]) A subgroup H of a group G is said to be c-supplemented in G if there exists a subgroup N of G such that $G = HN$ and $H \cap N \leqslant H_G = Core_G(H)$. We say that N is a c-supplement of H in G.

Recall that a subgroup H of G is c-normal in G if there exists a normal subgroup N of G such that $G = HN$ and $H \cap N \leqslant H_G$ ([8]). Also a subgroup H of G is complemented in G if there exists a subgroup N of G such that $G = HN$ and $H \cap N = 1$.

A c-normal or complemented subgroup must be a c-supplemented subgroup. But examples in [1] showed that the converses are not true.

Lemma 2.2. ([1, Lemma 2.1]) *Let G be a group. Then*

(1) *If H is c-supplemented in $G, H \leqslant M \leqslant G$, then H is c- supplemented in M.*

(2) *Let $K \triangleleft G$ and $K \leqslant H$. Then H is c-supplemented in G if and only if H/K is c-supplemented in G/K.*

(3) *Let π be a set of primes, H a π subgroup of G and K a normal π' subgroup of G. If H is c-supplemented in G, then HK/K is c-supplemented in G/K. If furthermore K normalises H, then the converse also holds.*

(4) *Let $H \leqslant G$ and $L \leqslant \Phi(H)$. If L is c-supplemented in G, then $L \triangleleft G$ and $L \leqslant \Phi(G)$.*

Lemma 2.3. (Gaschutz, refer to [3].) *Let G be a group. Suppose that H and D are normal subgroups of G, and also $D \leqslant H, D \leqslant \Phi(G)$. Then H/D is nilpotent if and only if H is nilpotent.*

Lemma 2.4. ([5, Lemma 2.3].) *Let H be a non-identity solvable normal subgroup of G. If every minimal normal subgroup of G which is contained in H is not contained in $\Phi(G)$, then the Fitting subgroup $F(H)$ of H is the direct product of minimal normal subgroups of G which are contained in H.*

Lemma 2.5. *Let p be a prime, x a p-element of G and m an integer. If*

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$\langle x^{pm}\rangle$ is c-supplemented in G, then $\langle x^{pm}\rangle$ is normal in G. In particular, if $\langle x^{pm}\rangle$ is complemented in G, then $x^p = 1$.

PROOF: By Definition 2.1, there is a subgroup N of G with $G = \langle x^{pm}\rangle N$ and $\langle x^{pm}\rangle \cap N \leqslant \langle x^{pm}\rangle_G$. Then $x = (x^{pm})^n y$, that is, $x^{1-pmn} = y$, for some integer n and some element y in N. Furthermore, $\langle x^{pm}\rangle \leqslant \langle x\rangle = \langle x^{1-pmn}\rangle \leqslant N$. Hence $\langle x^{pm}\rangle = \langle x^{pm}\rangle_G \lhd G$. ∎

LEMMA 2.6. *A group G is 2-nilpotent if every cyclic subgroup of order 2 or 4 of G is c-supplemented in G.*

PROOF: Suppose that G is not 2-nilpotent, so that G contains a minimal non-2-nilpotent subgroup H. Then by a theorem of Ito ([**3**, IV, 5.4 Satz]), every proper subgroup of H is nilpotent and $H = [H_2]H_p$ with $H_2 \in Syl_2(H)$ and $H_p \in Syl_p(H)$ $(p \neq 2)$, and the exponent of H_2 is at most 4. Let x be an element of H_2; then $o(x) = 2$ or 4. Since $\langle x\rangle$ is c-supplemented in H by Lemma 2.2(1), there is a subgroup N of H with $H = \langle x\rangle N$ and $\langle x\rangle \cap N \leqslant \langle x\rangle_H$ by Definition 2.1. Again, by Lemma 2.5, $\langle x^2\rangle \lhd H$, so $\langle x^2\rangle \leqslant \langle x\rangle_H$ and $\langle x^2\rangle N$ is a group. If $\langle x^2\rangle N = H$, then $\langle x\rangle = \langle x^2\rangle\big(\langle x\rangle \cap N\big) \leqslant \langle x\rangle_H$, that is, $\langle x\rangle = \langle x\rangle_H \lhd H$. In this case, if $\langle x\rangle H_p = H$, then $\langle x\rangle = H_2$ is cyclic, H is certainly 2-nilpotent, which is contrary to the above hypothesis of H. If $\langle x\rangle H_p < H$, then $\langle x\rangle H_p$ is nilpotent, which implies that H_2 normalises H_p by the arbitrariness of x in H_2. Furthermore, $H_p \lhd H$ and so H is nilpotent, a contradiction. Hence $\langle x^2\rangle N < H$ and $\langle x^2\rangle N$ is nilpotent. Note that $\big|H : \langle x^2\rangle N\big| = 2$, so $\langle x^2\rangle N \lhd H$. Then H_p char $\langle x^2\rangle N$ as is easy to see, so $H_p \lhd H$ and H is nilpotent, a final contradiction. ∎

LEMMA 2.7. ([**2**, Theorem 1].) *Let G be a PN-group (that is, a finite group in which every minimal subgroup is normal) of exponent p^n with p an odd prime. Let $1 \leqslant k \leqslant n$. Then*

(1) *$G/\Omega_k(G)$ is a PN-group of exponent p^{n-k};*
(2) *$\Omega_k(G) = x \in G \mid x^{p^k} = 1$;*
(3) *$1 \leqslant \Omega_1(G) \leqslant \Omega_2(G) \leqslant \cdots \leqslant \Omega_n(G) = G$ is a central series and hence class of $G \leqslant n$;*
(4) *$(xy)^{p^{n-1}} = x^{p^{n-1}} y^{p^{n-1}}$ for all x, y in G.*

LEMMA 2.8. *Let M be a maximal subgroup of G, P a normal p- subgroup of G such that $G = PM$, where p a prime. Then*

(1) *$P \cap M$ is a normal subgroup of G.*
(2) *If $p > 2$ and all minimal subgroups of P are normal in G, then M has index p in G.*

PROOF: (1) Clearly, $P \cap M < P$. Let P_1 be a subgroup of P such that $P \cap M$ is a maximal subgroup of P_1. Then $P_1 \not\leqslant M$, otherwise $P \cap M < P_1 \leqslant P \cap M$, a

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contradiction. Now that $P \cap M$ is normal in both P_1 and M, we have $M < \langle P_1, M\rangle \leqslant N_G(P \cap M)$. By the maximality of M in $G, N_G(P \cap M) = G$, that is, $P \cap M \lhd G$ as desired.

(2) By Lemma 2.7(2), $\Omega_1(P) = x \in P \mid x^p = 1$. So $\Omega_1(P)$ is normal in G. We consider the following two cases:

CASE 1: $\Omega_1(P) \not\leqslant M$. In this case, there exists an element x in $\Omega_1(P)$ such that x is not in M. By hypothesis, $\langle x\rangle$ is normal in G and so $G = \langle x\rangle M$ with $\langle x\rangle \cap M = 1$, which implies that $|G : M| = \big|\langle x\rangle\big| = p$.

CASE 2: $\Omega_1(P) \leqslant M$. We shall show that $G/\Omega_1(P)$ satisfies the hypothesis of the Lemma. Obviously, $G/\Omega_1(P) = \big(P/\Omega_1(P)\big)\big(M/\Omega_1(P)\big)$, where $P/\Omega_1(P)$ normal and $M/\Omega_1(P)$ maximal in $G/\Omega_1(P)$. Now, let $\langle x\rangle\Omega_1(P)/\Omega_1(P)$ be a minimal subgroup of $P/\Omega_1(P)$, where x is an element of P; then $x^p \in \Omega_1(P)$. Furthermore, $x^{p^2} = 1$ and so $\langle x^p\rangle$ is normal in G by hypothesis. Let g be an element of G. Then $(x^g)^p = (x^p)^g = (x^p)^t = (x^t)^p$ for some integer t. Since both x^g and x^t lie in $\Omega_2(P)$, it follows that $(x^g x^{-t})^p = (x^g)^p(x^{-t})^p = 1$ by Lemma 2.7(4), which implies that $x^g x^{-t}$ lies in $\Omega_1(P)$. Set $x^g x^{-t} = u \in \Omega_1(P)$. Then $x^g = ux^t \in \langle x\rangle\Omega_1(P)$ and so $\langle x\rangle > \Omega_1(P)/\Omega_1(P) \lhd G/\Omega_1(P)$. By induction, $\big|G/\Omega_1(P) : M/\Omega_1(P)\big| = p$, that is, $|G : M| = p$. The proof of Lemma 2.8 is complete.

3. Main result

THEOREM 3.1. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}, G$ a group with a solvable normal subgroup H such that $G/H \in \mathcal{F}$. If for any maximal subgroup M of G, either $F(H) \leqslant M$ or $F(H) \cap M$ is a maximal subgroup of $F(H)$, then $G \in \mathcal{F}$. The converse also holds, in the case where $\mathcal{F} = \mathcal{U}$.*

PROOF: Suppose that the "if" part is false and let G be a counterexample of minimal order. Then we have

(1) $H \cap \Phi(G) = 1$.

If not, then $1 \neq H \cap \Phi(G) \lhd G$. Let R be a minimal normal subgroup of G that is contained in $H \cap \Phi(G)$. Then R is an elementary Abelian p-group for some prime p and hence $R \leqslant F(H)$. We shall show G/R satisfies the hypothesis of the theorem:

(1.1) $(G/R)/(H/R) \cong G/H \in \mathcal{F}$.

(1.2) For any maximal subgroup M/R of G/R, either $F(H/R) \leqslant M/R$ or $F(H/R) \cap (M/R)$ is maximal in $F(H/R)$.

By Lemma 2.3, $F(H/R) = F(H)/R$. If $F(H/R) \not\leqslant M/R$, then $F(H) \not\leqslant M$. Since M is maximal in $G, F(H) \cap M$ is maximal in $F(H)$ by hypothesis. Therefore $F(H/R) \cap (M/R) = (F(H) \cap M)/R$ is maximal in $F(H/R)$. By the minimality of

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$G, G/R \in \mathcal{F}$. Since $G/\Phi(G) \cong (G/R)/(\Phi(G)/R) \in \mathcal{F}$ and $\mathcal{F}$ is a saturated formation, it follows that $G \in \mathcal{F}$, a contradiction.

(2) $F(H) = R_1 \times \cdots \times R_m$, where all R_i normal in G of prime order.

From (1) and Lemma 2.4, $F(H) = R_1 \times \cdots \times R_m$, where $R_i (i = 1, \ldots, m)$ are minimal normal subgroups of G. Now that $H \cap \Phi(G) = 1$, for each $i, i = 1, 2, \ldots, m$, there is a maximal subgroup M_i of G with $G = R_i M_i$ and $R_i \cap M_i = 1$. Moreover, $F(H) = R_i(F(H) \cap M_i)$ as is easy to see. By hypothesis, $F(H) \cap M_i$ is maximal in $F(H)$ and, since $F(H)$ is nilpotent, $F(H) \cap M_i$ has prime index in $F(H)$. Note that $R_i \cap M_i = 1$, so R_i has prime order for $i = 1, 2, \ldots, m$.

(3) $G/F(H) \in \mathcal{F}$.

Because $G/C_G(R_i)$ is isomorphic to a subgroup of $\mathrm{Aut}(R_i)$, $G/C_G(R_i)$ is cyclic and so it lies in $\mathcal{U}$ for each i. This implies that $G/(\bigcap_{i=1}^{n} C_G(R_i)) \in \mathcal{U}$. Again, $C_G(F(H)) = \bigcap_{i=1}^{n} C_G(R_i)$, so we have $G/C_G(F(H)) \in \mathcal{U} \subseteq F$. Since both $G/C_G(F(H))$ and G/H lie in $\mathcal{F}$, so does $G/\left(H \cap C_G(F(H))\right) = G/C_H(F(H))$. Since $F(H)$ is Abelian, $F(H) \leqslant C_H(F(H))$. On the other hand, $C_H(F(H)) \leqslant F(H)$ as H is solvable. Thus $F(H) = C_H(F(H))$ and so $G/F(H) \in \mathcal{F}$.

(4) $m = 1$, that is, $F(H) = R_1$.

For each $i, G/R_i$ satisfies the hypothesis of the theorem:

(4.1) From (3), $(G/R_i)/(F(H)/R_i) \cong G/F(H) \in \mathcal{F}$.

(4.2) For any maximal subgroup M/R_i of G/R_i, $(F(H)/R_i) \cap (M/R_i)$ is maximal in $F(H)/R_i$ if $F(H)/R_i \not\subseteq M/R_i$.

In fact, M is maximal in G and $F(H) \not\subseteq M$, so $F(H) \cap M$ is maximal in $F(H)$ by hypothesis. Hence $(F(H)/R_i) \cap (M/R_i) = (F(H) \cap M)/R_i$ is maximal in $F(H)/R_i$.

By the minimality of $G, G/R_i \in \mathcal{F}$. Hence $G/(\bigcap_{i=1}^{m} R_i) \in \mathcal{F}$. This implies that $G \in \mathcal{F}$ if $m \neq 1$, a contradiction. (4) is true.

(5) Final contradiction.

First, we shall show that R_1 is the only minimal normal subgroup of G. Suppose that $N \neq R_1$ is another minimal normal subgroup of G and we consider G/N. Then R_1N/N is a normal subgroup of G/N and $(G/N)/(R_1N/N)$ is isomorphic to G/R_1N, which is in $\mathcal{F}$ because G/R_1 is in $\mathcal{F}$ by (3) and (4). For any maximal subgroup M/N of G/N not containing R_1N/N, since $R_1N/N \cong R_1$ has prime order, $(R_1N/N) \cap (M/N)$ is an identity group, which is certainly maximal in R_1N/N. By the minimal choice of G, $G/N \in \mathcal{F}$, so $G \in \mathcal{F}$, a contradiction. Hence R_1 is the unique minimal normal subgroup of G. By (1), $\Phi(G) = 1$. Let M be a maximal subgroup of G such that $R_1 \not\subseteq M$. Then $G = R_1M$ and $R_1 \cap M = 1$. If $R_1 < C_G(R_1)$, then $1 < C_G(R_1) \cap M \triangleleft R_1M = G$. By

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the unique minimal normality of R_1, $R_1 \leqslant C_G(R_1) \cap M \leqslant M$, a contradiction. Hence $R_1 = C_G(R_1)$. Thus $G/R_1 = G/C_G(R_1)$ is cyclic of order dividing $|R_1| - 1$ and so $G \in \mathcal{U} \subseteq F$, a final contradiction.

In the case where $\mathcal{F} = \mathcal{U}$, if $G \in \mathcal{F}$, that is, G is supersolvable, then by Huppert's Theorem, any maximal subgroup M of G has prime index in G. And for any normal subgroup H of G, since $F(H)$ is normal in $G, G = F(H)M$ if $F(H)$ is not contained in M. This shows that $|F(H) : F(H) \cap M| = |G : M|$ is a prime and hence $F(H) \cap M$ is a maximal subgroup of $F(H)$.

The proof of Theorem 3.1 is complete. ∎

4. Applications

Theorem 4.1. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a group with a solvable normal subgroup H such that $G/H \in \mathcal{F}$. If all minimal subgroups and all cyclic subgroups with order 4 of $F(H)$ are c-supplemented in G, then $G \in \mathcal{F}$.*

Proof: For any maximal subgroup M of G not containing $F(H)$, we only need to prove that $F(H) \cap M$ is a maximal subgroup of $F(H)$. First, since $F(H) \not\subseteq M$, there exists a prime p such that $O_p(H) \not\subseteq M$. Then $G = O_p(H)M$ as $O_p(H)$ is normal in G. We consider the following two cases:

Case 1: $p > 2$. If $O_p(H)$ has at least one minimal subgroup $\langle x \rangle$ non-normal in G, then by hypothesis, $\langle x \rangle$ is c-supplemented in G, that is, there is a subgroup K with $G = \langle x \rangle K$ and $\langle x \rangle \cap K = 1$. Furthermore, K is a maximal subgroup of G and $O_p(H) \cap K$ is a normal subgroup of G by Lemma 2.8(1). Again, $O_p(H) = O_p(H) \cap \langle x \rangle K = \langle x \rangle (O_p(H) \cap K)$. If $O_p(H) \cap K \leqslant M$, then $G = O_p(H)M = \langle x \rangle M$ with $\langle x \rangle \cap M = 1$. This deduces $|F(H) : F(H) \cap M| = |F(H)M : M| = |G : M| = |\langle x \rangle| = p$. Hence $F(H) \cap M$ is a maximal subgroup of $F(H)$. If $O_p(H) \cap K \not\subseteq M$, then $G = (O_p(H) \cap K)M$, where x not in $O_p(H) \cap K$. With the same argument we may assume that all minimal subgroups of $O_p(H) \cap K$ are normal in G. By Lemma 2.8(2), $|F(H) : F(H) \cap M| = |G : M| = p$, so $F(H) \cap M$ is a maximal subgroup of $F(H)$.

Case 2: $p = 2$. Let $\pi(G) = p_1, p_2, \dots, p_n, M_{p_i}$ be a Sylow p_i-subgroup of M, where $i = 1, 2, \dots, n$ and $p_1 = 2$. Then we know easily that $O_2(H)M_2 = G_2$ is a Sylow 2-subgroup of G. Now, let P_1 be a maximal subgroup of G_2 containing M_2 and, set $P_2 = P_1 \cap O_2(H)$. Then $P_1 = P_2M_2$. Moreover, $P_2 \cap M_2 = O_2(H) \cap M_2$, so $|O_2(H) : P_2| = |O_2(H)M_2 : P_2M_2| = |G_2 : P_1| = 2$. Again, for each $i \neq 1, O_2(H)M_{p_i}$ is 2-nilpotent by Lemma 2.2(1) and Lemma 2.6, so $O_2(H)M_{p_i} = O_2(H) \times M_{p_i}$. Furthermore, $P_2M_{p_i}$ forms a group, where $i = 1, 2, \dots, n$. Hence $P_2\langle M_{p_1}, M_{p_2}, \dots, M_{p_n} \rangle = P_2M$ also forms a group. Since $|O_2(H) : P_2| = 2$ and $P_2 \cap M = O_2(H) \cap M$, it

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follows that $P_2M < O_2(H)M = G$. By the maximality of M in $G, P_2M = M$ and hence $P_2 \leqslant M$. Thus $O_2(H) \cap M = P_2 \cap M = P_2$ and $|G : M| = |O_2(H) : O_2(H) \cap M| = |O_2(H) : P_2| = 2$. This implies that $F(H) \cap M$ is a maximal subgroup of $F(H)$.

By Theorem 3.1, $G \in \mathcal{F}$. The proof of Theorem 4.1 is complete. ∎

COROLLARY 4.2. ([1, Theorem 4.1].) *Let G be a group and let H be the supersolvable residual of G. If all minimal subgroups and all cyclic subgroups with order 4 of H are c-supplemented in G, then G is supersolvable.*

PROOF: H is 2-nilpotent by Lemma 2.6, so it is solvable, and G is supersolvable by Theorem 4.1. ∎

COROLLARY 4.3. ([6, Theorem 1.1].) *Suppose that G is a solvable group with a normal subgroup H such that G/H is supersolvable. If all minimal subgroups of $F(H)$ are complemented in G, then G is supersolvable.*

PROOF: By hypothesis and Lemma 2.5, every Sylow subgroup of $F(H)$ is elementary Abelian. That is $F(H)$ has not any element of order p^2 for any $p \in \pi(F(H))$. Corollary 4.3 is certainly true by Theorem 4.1. ∎

COROLLARY 4.4. ([9, Theorem 2].) *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a group with a solvable normal subgroup H such that $G/H \in \mathcal{F}$. If all minimal subgroups and all cyclic subgroups with order 4 of $F(H)$ are c-normal in G, then $G \in \mathcal{F}$.*

THEOREM 4.5. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a group with a solvable normal subgroup H such that $G/H \in \mathcal{F}$. If all maximal subgroups of all Sylow subgroups of $F(H)$ are c-supplemented in G, then $G \in \mathcal{F}$.*

PROOF: For any maximal subgroup M of G not containing $F(H)$, we shall show $F(H) \cap M$ is a maximal subgroup of $F(H)$. First, since $F(H) \not\subseteq M$, there is a prime p with $O_p(H) \not\subseteq M$. Then $G = O_p(H)M$ as $O_p(H)$ is normal in G. Let M_p be a Sylow p-subgroup of M. Then we see easily that $O_p(H)M_p = G_p$ is a Sylow p-subgroup of G. Now, let P_1 be a maximal subgroup of G_p containing M_p and, set $P_2 = P_1 \cap O_p(H)$. Then $P_1 = P_2M_p$. Moreover, $P_2 \cap M_p = O_p(H) \cap M_p$, so $|O_p(H) : P_2| = |O_p(H)M_p : P_2M_p| = |G_p : P_1| = p$, that is, P_2 is a maximal subgroup of $O_p(H)$. Hence $P_2(O_p(H) \cap M)$ is a subgroup of $O_p(H)$. By the maximality of P_2 in $O_p(H), P_2(O_p(H) \cap M) = P_2$ or $O_p(H)$.

(1) If $P_2(O_p(H) \cap M) = O_p(H)$, then $G = O_p(H)M = P_2M$. Note that $O_p(H) \cap M = P_2 \cap M$, so $O_p(H) = P_2$, a contradiction. Hence

(2) $P_2 = P_2(O_p(H) \cap M)$, that is, $O_p(H) \cap M \leqslant P_2$. By Lemma 2.8(1), $O_p(H) \cap M \lhd G$, so $O_p(H) \cap M \leqslant (P_2)_G$. On the other hand, since P_2 is c-supplemented in G, there exists a subgroup N of G such that $G = P_2N$ and $P_2 \cap N \leqslant (P_2)_G$ by Definition

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2.1. Set $K = (P_2)_G N$; then $P_2 \cap K = P_2 \cap (P_2)_G N = (P_2)_G (P_2 \cap N) = (P_2)_G$. Now, we consider the following two cases:

CASE 1: $K < G$. Suppose that K_1 is a maximal subgroup of G containing K. Then $O_p(H) \cap K_1 \lhd G$, which implies that $(O_p(H) \cap K_1)M$ is a group. If $(O_p(H) \cap K_1)M = G = O_p(H)M$, then $O_p(H) \cap K_1 = O_p(H)$ because $(O_p(H) \cap K_1) \cap M = O_p(H) \cap M$. This implies that $O_p(H) \leqslant K_1$ and therefore $G = O_p(H)K_1 = K_1$, which is contray to the above hypothesis on K_1. Thus $(O_p(H) \cap K_1)M = M, O_p(H) \cap K_1 \leqslant M$. Furthermore, $P_2 \cap K \leqslant O_p(H) \cap K \leqslant O_p(H) \cap M \leqslant (P_2)_G = P_2 \cap K$, that is, $O_p(H) \cap K = O_p(H) \cap M = P_2 \cap K$. This is contrary to $G = P_2 K = O_p(H)K$.

CASE 2: $K = G$. In this case, $P_2 \lhd G$. By the maximality of M in $G, M = P_2 M$ or $P_2 M = G$. With the same argument in (1), we see $P_2 M \neq G$, so $M = P_2 M$, that is, $P_2 \leqslant M$. Thus $O_p(H) \cap M = P_2 \cap M = P_2$ and hence $|F(H) : F(H) \cap M| = |G : M| = |O_p(H) : O_p(H) \cap M| = p$. This means that $F(H) \cap M$ is a maximal subgruop of $F(H)$. By Theorem 3.1, $G \in \mathcal{F}$. The proof of Theorem 4.5 is complete. □

COROLLARY 4.6. ([**6**, Theorem 1.2].) *Suppose that G is a solvable group with a normal subgroup H such that G/H is supersolvable. If all maximal subgroups of every Sylow subgroup of $F(H)$ are complemented in G, then G is supersolvable.*

COROLLARY 4.7. ([**9**, Theorem 1].) *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a group with a solvable normal subgroup H such that $G/H \in \mathcal{F}$. If all maximal subgroups of all Sylow subgroups of $F(H)$ are c-normal in G, then $G \in \mathcal{F}$.*

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