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LANDAU'S THEOREM, FIELDS OF VALUES FOR CHARACTERS, AND SOLVABLE GROUPS

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Abstract

When G is a finite solvable group, we prove that $|G|$ can be bounded by a function in the number of irreducible characters with values in fields where $\mathbb{Q}$ is extended by prime power roots of unity. This gives a character theory analog for solvable groups of a theorem of Héthelyi and Külshammer that bounds the order of a finite group in terms of the number of conjugacy classes of elements of prime power order. In particular, we obtain for solvable groups a generalization of Landau's theorem.

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1. Introduction

In this note, all groups are finite. In [9], Landau proved that for every positive integer k, there are finitely many finite groups that have at most k conjugacy classes. This is equivalent to saying that the order of a finite group G can be bounded in terms of a function of the number of its conjugacy classes. In [2, Theorem 1.1], Héthelyi and Külshammer showed that in fact the order of the group is bounded in terms of the number of its conjugacy classes of elements of prime power order.

It is well known that there is a duality between conjugacy classes and ordinary characters. In particular, the number of conjugacy classes equals the number of irreducible characters, so Landau's theorem could equivalently be stated as saying that if G is a group, then $|G|$ is bounded by a function in terms of $|\mathrm{Irr}(G)|$. Thus, it makes sense to ask if there is a character-theoretic version of the Héthelyi–Külshammer theorem. We will show that there is for solvable groups.

If p is a prime, we define $\mathbb{Q}_p$ to be the field obtained by adjoining all p^ath roots of unity for all positive integers a to $\mathbb{Q}$. We say that a character is *PP*-valued if there is a prime p so that the values of the character all lie in $\mathbb{Q}_p$. Hence, the set of *PP*-valued irreducible characters of G is the union over all the primes p dividing $|G|$ of the $\mathbb{Q}_p$-valued irreducible characters of G.

THEOREM 1.1. *For every positive integer k, there are at most finitely many solvable finite groups having exactly k PP-valued irreducible characters.*

We propose that the PP-valued irreducible characters should correspond to the conjugacy classes of prime power elements. To give some evidence that this might make sense, we prove the following theorem.

THEOREM 1.2. *If G is a solvable group, then the number of prime power conjugacy classes in G is less than or equal to the number of PP-valued irreducible characters of G.*

It is tempting to believe that the number of prime power conjugacy classes should equal the number of PP-valued irreducible characters of G, but we will present a number of examples where these two sets do not have equal sizes. When we restrict ourselves to groups of odd order, we obtain equality.

THEOREM 1.3. *If $|G|$ is odd, then the number of prime power conjugacy classes in G equals the number of PP-valued irreducible characters of G.*

We note that the theorem of Héthelyi and Külshammer is proved for all finite groups and, unfortunately, our results are only for solvable groups. Our proof of Theorem 1.1 relies on facts that are true only for solvable groups. However, we believe that one should be able to remove the solvability hypothesis from Theorem 1.1, but different tools will need to be developed to do this. At this time, it is also an open question whether the solvability hypothesis is needed for Theorem 1.2.

We will prove Theorem 1.2 one prime at a time. On the other hand, we will present examples that show that it is not possible to prove Theorem 1.1 one prime at a time. We will show that the number of chief factors that are p-groups for a p-solvable group is bounded by the number of $\mathbb{Q}_p$-valued irreducible characters.

THEOREM 1.4. *If G is a p-solvable group for a prime p, then the number of p-chief factors in a chief series for G is at most the number of nonprincipal $\mathbb{Q}_p$-valued irreducible characters of G. Furthermore, if G is a solvable group, then the number of factors in a chief series for G is less than or equal to the number of nonprincipal irreducible PP-valued characters.*

2. *PP*-valued characters

We work one prime at a time. Let p be a prime. The following theorem is the key to this section.

THEOREM 2.1. *Let p be a prime, let G be a p-solvable group, and let N be a minimal normal subgroup of G that is a p-group. If $\lambda \in \mathrm{Irr}(N)$, then λ^G has an irreducible constituent χ such that χ is $\mathbb{Q}_p$-valued.*

To prove this, we will use the p-special characters defined by Gajendragadkar in [1]. If G is a p-solvable group and $\chi \in \mathrm{Irr}(G)$, then following [5] we say that χ is *p-special* if $\chi(1)$ is a power of p and, for every subnormal subgroup S of G, the irreducible constituents of χ_S have p-power determinantal order. The results we need regarding p-special characters can be found in [11, Section 21] among other places. We note that these characters are only defined when the group is p-solvable, and this is the main barrier we have in removing the solvable hypothesis from Theorems 1.1 and 1.2. The proof of the following lemma is due to Isaacs, who apparently has not published this result. However, basically the same proof is used in [4, Lemma 2.4] to obtain a weaker conclusion than the conclusion we need here.

Lemma 2.2. *Let $\theta \in \mathrm{Irr}(N)$, where N is a normal subgroup of G and G is a p-solvable group. Assume that θ is p-special and that it lies in a G-orbit whose size is a power of p. Then there exists a p-special character in $\mathrm{Irr}(G \mid \theta)$.*

Proof. We proceed by induction on $|G : N|$. Since there is nothing to prove if $N = G$, we can assume that $N < G$, and we choose a minimal normal subgroup M/N of G/N. The index in G of the stabilizer of θ is a power of p, so the stabilizer of θ contains some Hall p-complement K of G. We will show that $\mathrm{Irr}(M \mid \theta)$ contains a K-invariant p-special character γ. We assume this fact for the moment. We see that since K stabilizes γ, the size of the G-orbit of γ is a power of p. Applying the inductive hypothesis, we see that $\mathrm{Irr}(G \mid \gamma)$ contains a p-special character χ and, since χ_N has θ as an irreducible constituent, we are done.

Thus, we need to show that $\mathrm{Irr}(M \mid \theta)$ contains a K-invariant p-special character. If M/N is a p'-group, then $M \leq NK$, and so θ is M-invariant. We deduce that $\mathrm{Irr}(M \mid \theta)$ contains a unique p-special character by [11, Proposition 21.5(iv)] and, by its uniqueness, this character will be K-invariant, as desired. In the case where M/N is a p-group, every member of $\mathrm{Irr}(M \mid \theta)$ is p-special by [11, Lemma 21.4]. At least one of these characters is K-invariant by an application of a lemma of Glauberman [3, Theorem 13.28], and this completes the proof. □

We can now prove Theorem 2.1.

Proof of Theorem 2.1. Note that λ has degree one and the restriction of λ to any subgroup of N has order dividing p, so λ is p-special. Let T be the stabilizer of λ in G. Obviously, the T-orbit of λ has size 1, which is a power of p. We now apply Lemma 2.2 to see that there exists a p-special character ψ such that ψ is an irreducible constituent of λ^T. We then use Clifford's theorem [3, Theorem 6.11] to see that $\chi = \psi^G$ is an irreducible constituent of λ^G. Finally, we apply [11, Corollary 21.11] to conclude that ψ is $\mathbb{Q}_p$-valued, and it follows immediately that χ is $\mathbb{Q}_p$-valued. □

We now obtain Theorems 1.1 and 1.4 as corollaries.

Proof of Theorem 1.1. Suppose that the number of PP-valued irreducible characters of G is n. We will show by induction on n that $|G| \leq F(n)$ for some (finite) function F.

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We first claim that $F(1) = 1$. To see this, observe that if $G > 1$, then, since G is solvable, $G/\mathbf{O}^p(G) > 1$ for some prime p, and note that all the irreducible characters in $\mathrm{Irr}(G/\mathbf{O}^p(G))$ are $\mathbb{Q}_p$-valued [3, Lemma 2.15]. Hence, if G is solvable and $|G| > 1$, then G has at least two PP-valued irreducible characters. This proves that $F(1) = 1$ and works to start the induction.

Let N be a minimal normal subgroup of G. Since G is solvable, we know that N has order q for some prime power q. Theorem 2.1 implies that the number of irreducible PP-valued characters of G/N is at most $n - 1$, so by the inductive hypothesis, we have $|G/N| \le F(n - 1)$. Then $|G| \le qF(n - 1)$, so we need a bound on q. Since G/N has order at most $F(n - 1)$, no G-orbit on $\mathrm{Irr}(N)$ can have size bigger than that, so the number m of orbits satisfies $m \ge q/F(n - 1)$. On the other hand, we know from Theorem 2.1 that each G-orbit contributes at least one PP-valued irreducible character to the total, and thus $m \le n$. Then $n \ge m \ge q/F(n - 1)$, and thus $q \le nF(n - 1)$. This yields $|G| \le nF(n - 1)^2$, so we can recursively define $F(n) = nF(n - 1)^2$. □

We note that the estimates used in the proof of Theorem 1.1 were crude; almost certainly one can improve the actual function bounding $|G|$ in terms of the number of PP-valued characters.

Proof of Theorem 1.4. We work by induction on $|G|$. If G is trivial, then the conclusion is trivial. Thus, we may assume that $G > 1$. Let N be a minimal normal subgroup of G. By the inductive hypothesis, the number of p-chief factors of G/N is at most the number of nonprincipal $\mathbb{Q}_p$-valued irreducible characters of G/N and, when G is solvable, the total number of chief factors of G/N is at most the number of nonprincipal PP-valued irreducible characters of G/N. If N is a p'-group, this will yield the first conclusion. Thus, for the first conclusion, we may assume that N is a p-group. In light of Theorem 2.1, we see that G/N has at least one fewer nonprincipal $\mathbb{Q}_p$-valued irreducible character when N is a p-group. When G is solvable, we know that N is a p-group for some prime p; so G/N will have at least one fewer nonprincipal PP-valued irreducible character than G and, using the inductive hypothesis, this proves the result. □

3. Conjugacy classes of elements of prime power order

We now explore the relationship between the PP-valued irreducible characters and conjugacy classes of elements of prime power order. We say that g is a p-element of G if $g \in G$ has p-power order.

To prove Theorems 1.2 and 1.3, we will use the B_p-characters defined by Isaacs in [5, Definition 5.1]. The B_p-characters can be thought of as a generalization of the p-special characters. Since the definition of B_p-characters is somewhat complicated, we do not repeat it here, but refer the interested reader to [5] or to the expository accounts in [6] and [8].

Given a p-solvable group G, the set $\mathrm{B}_p(G)$ is a subset of $\mathrm{Irr}(G)$. The first conclusion of the next theorem is based on two results in [5]. The second conclusion comes from a result in [7]. One should note the relationship of this result with [5, Theorem 12.3] and [7, Lemma 3.3].

Theorem 3.1. *Let p be a prime and let G be a p-solvable group. Then the number of conjugacy classes of p-elements equals $|\mathrm{B}_p(G)|$ and $\mathrm{B}_p(G)$ is a subset of the $\mathbb{Q}_p$-valued irreducible characters of G. Furthermore, if $|G|$ is odd, then $\mathrm{B}_p(G)$ is the set of $\mathbb{Q}_p$-valued irreducible characters of G.*

Proof. Corollary 12.1 of [5] shows that every character in $\mathrm{B}_p(G)$ is $\mathbb{Q}_p$-valued. In [5, Theorem 9.3], it is proved that the number of p-conjugacy classes equals the number of characters in $\mathrm{B}_p(G)$. Since $\mathrm{B}_p(G)$ is a subset of the $\mathbb{Q}_p$-valued irreducible characters of G, this implies that the number of p-conjugacy classes is less than or equal to the number of $\mathbb{Q}_p$-valued irreducible characters of G. This proves the first conclusion.

To prove the second conclusion, we use the work in [7]. Suppose now that $|G|$ is odd; we show that $\mathrm{B}_p(G)$ is the set of $\mathbb{Q}_p$-valued irreducible characters of G. Since we have shown that every character in $\mathrm{B}_p(G)$ has values in $\mathbb{Q}_p$, it suffices to show that every $\mathbb{Q}_p$-valued character in $\mathrm{Irr}(G)$ lies in $\mathrm{B}_p(G)$. Let E be the field obtained by adjoining a primitive $|G|$th root of unity to $\mathbb{Q}$. Following [7, page 551], E has an automorphism τ that fixes the pth power roots of unity and acts like complex conjugation on the p'th roots of unity. In particular, the fixed field for τ will contain $E \cap \mathbb{Q}_p$.

Suppose that $\chi \in \mathrm{Irr}(G)$ is $\mathbb{Q}_p$-valued. Since the values of χ also lie in E, we conclude that the values of χ all lie in $E \cap \mathbb{Q}_p$ and as $E \cap \mathbb{Q}_p$ is a subfield of the fixed field of τ, we see that τ fixes all the values of χ. In particular, we have $\chi^\tau = \chi$. Applying [7, Lemma 3.1], we see that $\chi \in \mathrm{B}_p(G)$. Thus, this proves that χ lies in $\mathrm{B}_p(G)$ if and only if χ is $\mathbb{Q}_p$-valued. Therefore, the number of p-conjugacy classes will equal the number of $\mathbb{Q}_p$-valued irreducible characters of G. □

We note that one cannot bound the order of a Sylow p-subgroup of a solvable group G in terms of a function of the number of $\mathbb{Q}_p$-valued irreducible characters of G. Fix a prime p and let n be any positive integer. Let F be the field of order p^n, let E be the additive group of F, and let C be the multiplicative group of F so that $|E| = p^n$ and $|C| = p^n - 1$. It is easy to see that multiplication in F defines a group action of C on E, and we let G be the semi-direct product resulting from C acting on E. It is not difficult to see that G has a unique faithful irreducible character and that it will be $\mathbb{Q}_p$-valued. If $p = 2$, then it is not difficult to see that 1_G is the only $\mathbb{Q}_2$-valued character in $\mathrm{Irr}(G/E)$ and, so, G has exactly two $\mathbb{Q}_2$-valued irreducible characters. On the other hand, if p is odd, then one can see that $\mathrm{Irr}(G/E)$ has exactly two $\mathbb{Q}_p$-valued characters and, so, $\mathrm{Irr}(G)$ will have exactly three irreducible characters that are $\mathbb{Q}_p$-valued. Since n is arbitrary, this gives the desired conclusion.

We have shown that the sets $\mathrm{B}_p(G)$ for the various primes p contain PP-valued characters. To get a lower bound on the number of PP-valued characters, we count the sizes of the sets $\mathrm{B}_p(G)$ for the different primes p. We next show that the different sets $\mathrm{B}_p(G)$ intersect only in the principal character.

LEMMA 3.2. *Let p and q be distinct primes, and let G be a group that is both p-solvable and q-solvable. Then* $\mathrm{B}_p(G) \cap \mathrm{B}_q(G) = \{1_G\}$.

PROOF. We work by induction on $|G|$. If $G = 1$, then the result is trivial. Thus, we may assume that $G > 1$. Hence, we can find a maximal normal subgroup N in G. Consider a character $\chi \in \mathrm{B}_p(G) \cap \mathrm{B}_q(G)$. Let θ be an irreducible constituent of χ_N. Applying [5, Corollary 7.5], we know that θ lies in both $\mathrm{B}_p(N)$ and $\mathrm{B}_q(N)$. By the inductive hypothesis, it follows that $\theta = 1_N$. Hence, $\chi \in \mathrm{Irr}(G/N)$. In [10], we prove that $\mathrm{B}_p(G/N) = \mathrm{Irr}(G/N) \cap \mathrm{B}_p(G)$ and $\mathrm{B}_q(G/N) = \mathrm{Irr}(G/N) \cap \mathrm{B}_q(G)$. It follows that $\chi \in \mathrm{B}_p(G/N) \cap \mathrm{B}_q(G/N)$. Since G is a p-solvable group, we know that G/N is either a p-group or a p'-group. Suppose that G/N is a p-group; then it follows that $\mathrm{B}_q(G/N) = \{1_G\}$ (see [5, Corollary 5.3]). Otherwise, G/N is a p'-group; so $\mathrm{B}_p(G/N) = \{1_G\}$ (again this is [5, Corollary 5.3]). In both cases, we obtain $\chi = 1_G$. □

This next theorem includes both Theorems 1.2 and 1.3.

THEOREM 3.3. *If G is a solvable group, then the number of prime power conjugacy classes in G is less than or equal to the number of PP-valued irreducible characters of G. Furthermore, if $|G|$ is odd, then this is an equality.*

PROOF. We know from Theorem 3.1 that for each prime p the number of p-conjugacy classes equals the number of characters in $\mathrm{B}_p(G)$. Thus, the number of nonidentity p-conjugacy classes equals the number of nonprincipal characters in $\mathrm{B}_p(G)$. Observe that the set of nonidentity prime power classes is a disjoint union of the sets of nonidentity p-classes for the various primes p that divide $|G|$. In light of Lemma 3.2, the sets $\mathrm{B}_p(G) \setminus \{1_G\}$ are disjoint for the various primes p. We conclude that the number of prime power classes equals the size of $\bigcup_p \mathrm{B}_p(G)$. Since all the characters in $\mathrm{B}_p(G)$ are PP-valued for all the primes p, we obtain the first conclusion. Finally, when $|G|$ is odd, we have seen that every PP-valued irreducible character of G lies in $\mathrm{B}_p(G)$ for some prime p, and this yields the second conclusion. □

We now present some examples to see that the number of PP-valued irreducible characters need not equal the number of prime power conjugacy classes. Recall that a group is called *rational* if all the irreducible characters are rational valued. Thus, if G is a rational group, then all of the irreducible characters will be PP-valued, so it suffices to find rational groups that have elements whose orders are not prime powers. It is not difficult to see that any symmetric group of degree at least 5 will fit the bill. Since we are discussing solvable groups in this note, we feel obligated to present a solvable example. As such, take $G = S_3 \times Z_2$. It is obvious that G is a rational group so all the irreducible characters are PP-valued, but it has an element of order 6 so not all the conjugacy classes contain prime power elements.

In light of Theorem 3.1 and Lemma 3.2, when G is a solvable group, the number of conjugacy classes of prime power elements equals $|\bigcup_p \mathrm{B}_p(G)|$. Thus, the existence of a function in terms of $|\bigcup_p \mathrm{B}_p(G)|$ that bounds $|G|$ is exactly the theorem of Héthelyi–Külshammer on solvable groups. With this in mind, one could argue for solvable groups that $\bigcup_p \mathrm{B}_p(G)$ is the analog of conjugacy classes of prime power elements.

However, as we have stated, we would like to remove the hypothesis that G is solvable from our results and, at this time, we do not know of a natural generalization of $\bigcup_p \mathrm{B}_p(G)$ to nonsolvable groups.

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