Can. J. Math., Vol. XXIV, No. 5, 1972, pp. 873–890

THE STRICT TOPOLOGY IN A COMPLETELY REGULAR SETTING: RELATIONS TO TOPOLOGICAL MEASURE THEORY

STEVEN E. MOSIMAN AND ROBERT F. WHEELER

1. Introduction. Let X be a locally compact Hausdorff space, and let $C^*(X)$ denote the space of real-valued bounded continuous functions on X. An interesting and important property of the strict topology β on $C^*(X)$ was proved by Buck [**2**]: the dual space of $(C^*(X), \beta)$ has a natural representation as the space of bounded regular Borel measures on X.

Now suppose that X is completely regular (all topological spaces are assumed to be Hausdorff in this paper). Again it seems natural to seek locally convex topologies on the space $C^*(X)$ whose dual spaces are (*via* the integration pairing) significant classes of measures. Motivated by this idea, Sentilles [**24**] has considered locally convex topologies β_0, β, and β_1 on $C^*(X)$ which yield as dual spaces the tight, τ-additive, and σ-additive Baire measures of Varadarajan [**30**]. If X is locally compact, the topologies β_0 and β coincide and are precisely the original strict topology of Buck.

The topology β_0 on $C^*(X)$ has an intuitively appealing description: it is the finest locally convex topology which, when restricted to sets bounded in the supremum norm, coincides with the compact-open topology. However, it is defective with respect to a desirable property of the dual space: a weak*-compact set of tight measures need not be β_0-equicontinuous. This bears on a question posed by Buck in the locally compact setting: when is $(C^*(X), \beta)$ a Mackey space?

A partial answer can be found in the work of LeCam [**15**] and, independently, Conway [**4**]: if X is σ-compact locally compact, then $(C^*(X), \beta)$ is a strong Mackey space (i.e., weak*-compact subsets of the dual space are β-equicontinuous). We note that the results of Conway are actually somewhat stronger than this: "weak*-compact" can be replaced in the previous sentence by "weak*-countably compact", and "σ-compact" can be replaced by "paracompact".

The first modification will not be employed here for the sake of simplicity; the second will not be emphasized because it relies on the special decomposition property of paracompact locally compact spaces [**6**, p. 241].

Using the Conway-LeCam result, Sentilles showed that $(C^*(X), \beta_1)$ is always a strong Mackey space. The difficulty with β_1 is its definition, which is

---

Received September 8, 1971 and in revised form, March 15, 1972. The research of the first named author was supported by NDEA Fellowship 67-06345.2.

phrased in terms of the Stone-Čech compactification βX of a given space X and has not yet been described in terms of convergence of nets or sequences of continuous functions on X.

Spaces X for which β_0, β, and β_1 coincide on $C^*(X)$ thus possess the virtues of both β_0 and β_1 and avoid their disadvantages. We refer to such spaces as *β-simple*, and investigate topological conditions on X which are related to β-simplicity. A space is β-simple if and only if (1) every σ-additive measure is tight and (2) every weak*-compact set of positive tight measures is β_0-equicontinuous (equivalently, a tight set of measures).

Spaces for which (1) holds have been investigated by Knowles [**14**] and Moran [**18**]; the latter refers to such spaces as "strongly measure-compact". To their results we add only the observations that the class of strongly measure-compact spaces is invariant under the formation of closed subspaces and countable intersections of subspaces of a fixed space and contains all the σ-compact spaces.

Spaces satisfying (2) have been called T-spaces [**24**] or Prohorov spaces. Not every space is a T-space, but locally compact spaces and complete metric spaces are known to have this property. Relying on an interesting characterization of weak*-compactness due to Topsøe [**27**], we obtain a number of permanence properties. For example, the class of T-spaces is preserved by taking closed subspaces, countable products, and countable intersections of subspaces of a fixed space. An open subset of a T-space is a T-space; conversely, if X is a union of open T-subspaces, then X is a T-space. The last result holds for a finite union of closed T-subspaces, but may fail for a countable union. A result concerning preservation of the T-space property under continuous maps is given, and a number of relevant counter-examples are recorded.

Next we prove an extension of the Conway-LeCam Theorem for β_0. A space is hemicompact if every compact subset is contained in some member of a fixed countable family of compact subsets; a k-space if the topology is determined by the compact subsets [**6**, p. 248]. Then we have: every hemicompact k-space is a T-space (and is, indeed, β-simple). More generally, if X is a topological sum of hemicompact k-spaces, then $(C^*(X), \beta_0)$ is a strong Mackey space.

Recent results of Preiss [**22**] show that the analogous assertions for σ-compact metric spaces fail; indeed such a space is a T-space if and only if it admits a complete metric. Here it is shown that when X is σ-compact metric, then $(C^*(X), \beta_0)$ is a Mackey (or strong Mackey) space if and only if X admits a certain decomposition into σ-compact locally compact subspaces.

Combining the results on strongly measure-compact spaces and T-spaces, we state a general theorem on preservation of β-simplicity. Also, an example is given of a β-simple space for which $\beta_0 = \beta = \beta_1$ is not complete (or even sequentially complete).

We wish to thank Euline Green, Judy McKinney, and Dennis Sentilles for stimulating conversations in connection with the preparation of this paper.

2. Notation and preliminary results. Basic references for topology and topological vector spaces are, respectively, Dugundji (**6**) and Schaefer [**23**]. We recall a few basic concepts and results from topological measure theory as given by Varadarajan [**30**]. For each completely regular space X, let $M(X)$ be the space of continuous linear functionals on $C^*(X)$ (endowed with the usual sup-norm topology).

Definition 2.1. If $\Phi \in M(X)$, then Φ is

(a) *σ-additive*, if for any sequence (f_n) in $C^*(X)$ which is monotone decreasing and pointwise convergent to $0 (f_n \downarrow 0)$, $\Phi(f_n) \to 0$.

(b) *τ-additive*, if for any net (f_α) in $C^*(X)$ which is monotone decreasing ($\alpha \leqq \beta$ implies $f_\beta(x) \leqq f_\alpha(x)$ for all $x \in X$) and pointwise convergent to $0 (f_\alpha \downarrow 0)$, $\Phi(f_\alpha) \to 0$.

(c) *tight*, if for any uniformly bounded net (f_α) in $C^*(X)$ which converges to 0 uniformly on compact subsets of X, $\Phi(f_\alpha) \to 0$.

The spaces of σ-additive, τ-additive, and tight functionals are written $M_\sigma(X)$, $M_\tau(X)$, and $M_t(X)$, respectively; the positive functionals in these classes are $M_\sigma^+(X)$, $M_\tau^+(X)$, and $M_t^+(X)$. We have $M_t(X) \subset M_\tau(X) \subset M_\sigma(X) \subset M(X)$. Conditions for equality of these classes have been investigated by Varadarajan, Kirk [**12**; **13**], Knowles [**14**], Moran [**17**; **18**; **19**], and others; we adopt Moran's terminology.

Definition 2.2. A completely regular space X is *measure-compact* if $M_\sigma(X) = M_\tau(X)$; *strongly measure-compact* if $M_\sigma(X) = M_t(X)$.

Every Lindelöf space is measure-compact [**30**, p. 175].

Now $M_\sigma(X)$, $M_\tau(X)$, and $M_t(X)$ are closed subspaces of the Banach space $M(X)$ (with the natural dual norm). More often we shall be interested in the weak*-topology on $M(X)$ or its subspaces with respect to $C^*(X)$. An assertion such as "A is relatively weak*-compact in $M_t^+(X)$" means "the weak*-closure of A *in* $M_t^+(X)$ is weak*-compact".

A fundamental result of the theory is the natural 1-1 correspondence between functionals in $M(X)$ and signed Baire measures on X [**30**, p. 165]. The family $\mathrm{Ba}(X)$ of Baire sets is here defined to be the least σ-algebra containing all zero-sets (sets of the form $f^{-1}(0)$, where $f \in C^*(X)$). A cozero set is the complement of a zero set. A countably additive positive Baire measure μ is zero-set regular: i.e. for $A \in \mathrm{Ba}(X)$,

$$\mu A = \sup\{\mu Z : Z \text{ a zero-set}, Z \subset A\} = \inf\{\mu U : U \text{ a cozero set}, A \subset U\}.$$

A Baire measure μ may be classified as σ-additive, τ-additive, or tight according as the functional

$$\Phi(f) = \int_X f d\mu \quad (f \in C^*(X))$$

has the properties. We shall not distinguish between the functional and the associated measure. There is one further point to consider. It can be deduced from [**14**, p. 144] and [**30**, p. 180] that every tight Baire measure μ has a unique extension to a compact-regular Borel measure ν (the Borel sets are the least σ-algebra containing the open sets). Then μ and ν integrate continuous functions the same, and since we are only concerned with the duality between $C^*(X)$ and $M(X)$, we shall frequently identify $M_t(X)$ with the space of such Borel measures.

Definition 2.3. A subset A of $M_t(X)$ is *tight* if

(a) A is norm-bounded $(\sup\{|\nu|(X):\nu \in A\} < \infty)$.

(b) For every $\epsilon > 0$, there is a compact subset K of X such that $|\nu|(X\backslash K) < \epsilon$ for all $\nu \in A$.

Here $|\nu|$ is, as usual, the total variation of the Borel measure ν.

If $X \subset Y$, then X is absolutely Borel measurable in Y if, for each positive compact-regular Borel measure ν on Y, there are Borel sets A and B in Y with $A \subset X \subset B$ and $\nu(B\backslash A) = 0$.

PROPOSITION 2.4 [**14**, p. 148]. *$M_\tau(X) = M_t(X)$ if and only if X is absolutely Borel measurable in βX.*

PROPOSITION 2.5 [**30**, p. 224]. *If X is a separable metric space, then $M_\sigma(X) = M_t(X)$ if and only if X is absolutely Borel measurable in its completion.*

Metric spaces such that $M_\tau(X) = M_t(X)$ are called inner regular by Dudley [**5**]. By a complete metric space we mean a completely regular space which admits a compatible complete metric. It is well known that such a space is a G_δ set in its Stone-Čech compactification [**32**, p. 180].

COROLLARY 2.6. *Locally compact spaces and complete metric spaces satisfy $M_\tau(X) = M_t(X)$.*

Next we summarize the basic properties of β_0, β, and β_1.

Definition 2.7. The topology β_0 on $C^*(X)$ is the finest locally convex topology which coincides with the compact-open topology on norm-bounded sets.

This concept was mentioned briefly in the remarkable article of LeCam [**15**, p. 217]; it has subsequently been formulated in various equivalent ways by other authors [**8**; **9**; **11**; **25**; **29**]. A synthesis of these efforts is given by Sentilles [**24**]. One useful consequence: a linear map from $C^*(X)$ to a locally convex space E is β_0-continuous if and only if its restriction to each norm-bounded set is continuous with respect to the compact-open topology. Some additional known results about β_0 are contained in the next theorem.

PROPOSITION 2.8. (a) *The dual space of $(C^*(X), \beta_0)$ is $M_t(X)$. A subset of $M_t(X)$ is β_0-equicontinuous if and only if it is tight. Moreover, β_0 is the topology of uniform convergence on tight subsets of $M_t^+(X)$.*

(b) *If X is locally compact, then β_0 coincides with the usual strict topology.*

We refer the reader to Sentilles' paper for the definition of β and β_1 in the completely regular case, and the following results.

PROPOSITION 2.9. (a) *The dual space of* $(C^*(X), \beta_1)$ *(respectively* $(C^*(X), \beta)$*) is* $M_\sigma(X)$ *(respectively* $M_\tau(X)$*).*

(b) β_1 *(respectively* β*) is the topology of uniform convergence on weak*-compact subsets of* $M_\sigma^+(X)$ *(respectively* $M_\tau^+(X)$*).*

(c) $(C^*(X), \beta_1)$ *is a strong Mackey space for any* X.

(d) $\beta = \beta_1$ *if and only if* X *is measure-compact.*

(e) *In general,* $\beta_0 \leqq \beta \leqq \beta_1$.

As pointed out by Sentilles, the topologies β and β_1 are equivalent to the topologies T_τ and T_σ introduced by Fremlin, Garling, and Haydon [**8**]. A subset H of $M_\tau(X)$ is *uniformly* τ*-additive* if, whenever $f_\alpha \downarrow 0$ in $C^*(X)$, then $|\mu|(f_\alpha) \to 0$ uniformly with respect to $\mu \in H$; replacing nets by sequences, we have the notion of *uniform* σ*-additivity*. Then from results in [**24**] we have the characterizations of the β- and β_1-equicontinuous sets as the uniformly τ-additive and uniformly σ-additive subsets of M_τ and M_σ, respectively.

Definition 2.10. A space X is a T*-space* if every weak*-compact subset of $M_t^+(X)$ is tight.

It is well-known that locally compact spaces and complete metric spaces are T-spaces.

PROPOSITION 2.11. *If* X *is strongly measure-compact, then the following are equivalent*:

(a) X *is a* T*-space.*

(b) β_0 *is strong Mackey.*

(c) β_0 *is Mackey.*

Proof. (a) $\Rightarrow$ (b): We have $M_\sigma(X) = M_t(X)$, and from 2.8 and 2.10, β_0 is the topology of uniform convergence on weak*-compact subsets of $M_t^+(X)$. Thus $\beta_0 = \beta_1$ (2.9 (b)); hence β_0 is strong Mackey (2.9 (c)).

(b) $\Rightarrow$ (c): This is obvious.

(c) $\Rightarrow$ (a): β_0 and β_1 have the same dual, β_0 is Mackey, and $\beta_0 \leqq \beta_1$. Thus $\beta_0 = \beta_1$ is the topology of uniform convergence on weak*-compact subsets of $M_\sigma^+(X) = M_t^+(X)$. These sets are consequently β_0-equicontinuous and so are tight (2.8 (a)).

Now we can make our central definition.

Definition 2.12. A space X is β*-simple* if $\beta_0 = \beta = \beta_1$.

Sentilles notes that complete separable metric spaces and σ-compact locally compact spaces have this property.

THEOREM 2.13. *The following conditions on a space* X *are equivalent*:

(a) X *is* β*-simple.*

(b) X *is a strongly measure-compact* T*-space.*

(c) $M_\sigma(X) = M_t(X)$, *and* $(C^*(X), \beta_0)$ *is a Mackey space.*

https://doi.org/10.4153/CJM-1972-087-2 Published online by Cambridge University Press

(d) $M_\sigma(X) = M_t(X)$, *and* $(C^*(X), \beta_0)$ *is a strong Mackey space.*
(e) *Every weak*-compact subset of* $M_\sigma(X)$ *is a tight subset of* $M_t(X)$.
(f) *Every weak*-compact subset of* $M_\sigma^+(X)$ *is a tight subset of* $M_t^+(X)$.

Proof. (a) $\Rightarrow$ (b): $\beta_0 = \beta_1$, so the dual spaces coincide. Moreover β_0 is strong Mackey, so X is a T-space (2.11).

(b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d): These follow from 2.11.

(d) $\Rightarrow$ (e): See 2.8 (a). (e) $\Rightarrow$ (f): The proof is trivial. (f) $\Rightarrow$ (a): We always have $\beta_0 \leqq \beta_1$; but the hypothesis and 2.9 (b) imply that $\beta_1 \leqq \beta_0$.

In view of this result, it seems desirable to investigate the nature of β-simple spaces by considering the problems of characterizing strongly measure-compact spaces and T-spaces separately. However, it is important to note that the conclusion, "β_0 is Mackey", need not follow from either of these properties alone. The space of ordinals less than the first uncountable ordinal is a T-space (since locally compact), but, as Conway [**4**] has shown, $\beta = \beta_0$ is not Mackey. An example of a countable, strongly measure-compact space such that β_0 is not Mackey will be given later (5.6).

3. Measure-compact and strongly measure-compact spaces. We begin the section with a result on induced mappings of functions and measures; its measure-theoretic content is well-known (e.g., [**18**, p. 495]). If $T: (C^*(Y), \beta)) \to (C^*(X), \beta)$ is continuous, then T is said to be β-β continuous. The observations concerning β_1-β_1 and β-β continuity are due to Judy McKinney.

THEOREM 3.1. *Let* $\varphi: X \to Y$ *be continuous,* $T : C^*(Y) \to C^*(X)$ *be the induced linear mapping defined by* $T(f) = f \circ \varphi$, *and* T^* *be its algebraic adjoint. Then*

(a) $T^*(M'(X)) \subset M'(Y)$, *where* $M' = M, M_\sigma, M_\tau$, *or* M_t.
(b) T is β_1-β_1, β-β, *and* β_0-β_0 *continuous.*
(c) *If* $\mu \in M_\sigma(X)$ *and* A *is a Baire set in* Y, *then* $(T^*\mu)(A) = \mu(\varphi^{-1}(A))$.

Proof. (a): The map T is norm-decreasing, hence T^* must map the norm dual of $C^*(X)$ into the norm dual of $C^*(Y)$; moreover, $T^* : M(X) \to M(Y)$ is weak*-continuous. If $\mu \in M_\sigma(X)$ and (f_n) is a sequence in $C^*(Y)$ which is monotone decreasing and pointwise convergent to 0, then $(T(f_n))$ has the same properties. Hence $(T^*\mu)(f_n) = \mu(T(f_n)) \to 0$, and so $T^*\mu \in M_\sigma(Y)$. The proof that $T^*(M_\tau(X)) \subset M_\tau(Y)$ is similar. If (f_α) is uniformly bounded in $C^*(Y)$ and converges to 0 uniformly on compact sets, then (Tf_α) has the same properties. It follows that $T^*(M_t(X)) \subset M_t(Y)$.

(b): Obviously T is a positive linear map; thus if A is a weak*-compact subset of $M_\sigma^+(X)$, then T^*A is a weak*-compact subset of $M_\sigma^+(Y)$. An application of 2.9(b) and standard arguments reveal that T is β_1-β_1 continuous. The proof of β-β continuity is similar. The argument in (a) shows that T, when restricted to norm-bounded sets, is continuous with respect to the compact-open topology. The remark following 2.7 establishes the β_0-β_0 continuity of T.

https://doi.org/10.4153/CJM-1972-087-2 Published online by Cambridge University Press

(c) The proof is omitted, since it follows from known results on measure-preserving transformations; see for example, p. 163 of Halmos' text on measure theory.

In connection with (c), we remark that the stated result is also valid for compact-regular Borel measures and Borel sets.

Moran [**18**] has shown that a Baire set in a measure-compact (respectively strongly measure-compact) space is measure-compact (respectively strongly measure-compact). Using similar arguments, we establish an analogous result for closed subspaces. Recall that the support of a Baire measure μ on X is $\{x \in X\colon$ if U is a cozero set and $x \in U$, then $\mu(U) > 0\}$. A space X is measure-compact if and only if every non-zero member of $M_\sigma^+(X)$ has non-empty support [**17**, p. 634].

PROPOSITION 3.2. *A closed subspace of a measure-compact* (*strongly measure-compact*) *space is measure-compact* (*strongly measure-compact*).

Proof. Let F be a closed subspace of X, with $\varphi : F \to X$ the natural embedding, and define T and T^* as in 3.1. If $\mu \in M_\sigma^+(F)$, then $T^*\mu \in M_\sigma^+(X)$ and, for any Baire set B in X, $(T^*\mu)(B) = \mu(B \cap F)$.

(a) If X is measure-compact and μ is a measure in $M_\sigma^+(F)$ with empty support, then $T^*\mu \in M_\sigma^+(X)$ has empty support. Indeed if $x \in X\backslash F$, then there is a cozero set V in X with $x \in V \subset X\backslash F$, and $(T^*\mu)(V) = 0$. If $x \in F$, then there is a cozero set U in F with $x \in U$ and $\mu(U) = 0$. Thus there is a cozero set W in X with $x \in W \cap F \subset U$, so that $(T^*\mu)(W) = 0$. Since X is measure-compact, $T^*\mu = 0$, and so $\mu = 0$; thus F is measure-compact.

(b) If X is strongly measure-compact, and μ is a non-zero member of $M_\sigma^+(F)$, then there is a compact subset K of X with $\inf\{(T^*\mu)(U) : U$ a cozero set in $X, K \subset U\} = \delta > 0$ [**18**, p. 499]. Let $K_1 = K \cap F$, and suppose V is a cozero set in F with $K_1 \subset V$. Then the closed set $F\backslash V$ and the compact set K are disjoint, hence there is a cozero set W in X with $K \subset W$ and $W \cap (F\backslash V) = \emptyset$. Then $\mu V \geqq \mu(W \cap F) = (T^*\mu)(W) \geqq \delta$. It now follows from Definition 4.1 and Theorem 4.4 of Moran [**18**] that F is strongly measure-compact.

We need one additional result of Moran [**18**, p. 503] : a countable product of strongly measure-compact spaces is strongly measure-compact.

PROPOSITION 3.3. *The intersection of a countable family of strongly measure-compact subspaces of a fixed space is strongly measure-compact.*

Proof. This can be shown directly, but instead we appeal to a simple observation of Negrepontis [**20**, p. 604] : a topological property which is preserved by closed subspaces and countable products is also preserved by countable intersections.

A proof of this result in the locally compact case has been given by Kirk [**12**, p. 340].

Any σ-compact space X is Lindelöf (so $M_\sigma(X) = M_\tau(X)$) and a Borel set in βX (so $M_\tau(X) = M_t(X)$). Thus we have

PROPOSITION 3.4. *A σ-compact space is strongly measure-compact.*

Example 3.5. An arbitrary intersection of strongly measure-compact subspaces of a fixed space need not even be measure-compact. Moran [**17**, p. 638] has given an example of a realcompact space X which is not measure-compact. Such a space is the intersection of a family of σ-compact (locally compact) subspaces of βX [**10**, 8B].

4. T-spaces. A useful criterion for weak*-compactness of subsets of $M_t^+(X)$ has been obtained by Topsøe [**27**, p. 203]. A family $\mathscr{G}$ of open subsets of X is said to dominate the family $\mathscr{C}$ of compact subsets (in symbols $\mathscr{G} > \mathscr{C}$) provided that each member of $\mathscr{C}$ is a subset of some member of $\mathscr{G}$. We now restate two of Topsøe's results in a form suitable to our needs.

PROPOSITION 4.1. *A subset H of $M_t^+(X)$ is relatively weak*-compact if and only if*

(a) *H is norm-bounded and*

(b) *whenever $\mathscr{G}$ is a family of open subsets of X such that $\mathscr{G} > \mathscr{C}$, for each $\epsilon > 0$ there is a finite subfamily $\mathscr{G}_0$ of $\mathscr{G}$ such that* $\inf\{\mu(X\backslash G) : G \in \mathscr{G}_0\} < \epsilon$ for all $\mu \in H$.

Again we are interpreting members of $M_t^+(X)$ as Borel measures.

COROLLARY 4.2. *If H is relatively weak*-compact in $M_t^+(X)$ and $\mathscr{G}$ is an open cover of X such that $G_1, G_2 \in \mathscr{G}$ implies the existence of $G_3 \in \mathscr{G}$ with $G_1 \cup G_2 \subset G_3$, then for each $\epsilon > 0$ there is a member G of $\mathscr{G}$ with $\mu(X\backslash G) < \epsilon$* for all $\mu \in H$.

Now we give the main results of the section.

PROPOSITION 4.3. *Let $\varphi : X \to Y$ be a continuous map such that the inverse image of a compact set is compact. If Y is a T-space, then so is X.*

Proof. Let H be weak*-compact in $M_t^+(X)$. Using the notation and results of 3.1, $T^*(H)$ is weak*-compact in $M_t^+(Y)$. Given $\epsilon > 0$, find a compact subset K of Y such that $(T^*\mu)(Y\backslash K) < \epsilon$ for all $\mu \in H$. Then $\mu(X\backslash\varphi^{-1}(K)) < \epsilon$ for $\mu \in H$, and so H is tight.

COROLLARY 4.4. *A closed subset of a T-space is a T-space.*

Proof. If X is a closed subset of a T-space Y, then the identity map of X into Y satisfies the conditions of the previous theorem.

If C is a Borel set in X, and μ is a compact-regular Borel measure on X, then μ_C, the restriction of μ to the Borel sets of C, is a compact-regular Borel measure on C.

THEOREM 4.5. *Let C be a closed subset of X, H a relatively weak*-compact subset of $M_t^+(X)$. Then $H_1 = \{\mu_C : \mu \in H\}$ is a relatively weak*-compact subset of $M_t^+(C)$.*

Proof. Let $\mathscr{G}$ be a family of open subsets of C which dominates the family of compact subsets of C. For each $G \in \mathscr{G}$, choose an open set O_G in X such that $O_G \cap C = G$. Now let $G' = O_G \cup (X - C)$; then $\mathscr{G}' = \{G' : G \in \mathscr{G}\}$ is a family of open sets which dominates the compact subsets of X.

Given $\epsilon > 0$, Proposition 4.1 lets us choose $G_1', \ldots, G_n'$ in $\mathscr{G}'$ such that, for each $\mu \in H$, $\inf\{\mu(X - G_i')\} < \epsilon$. Thus for each $\mu_C \in H_1$, $\inf\{\mu_C(C \backslash G_i)\} < \epsilon$. The result follows from 4.1.

COROLLARY 4.6. *A space which is a finite union of closed T-subspaces is a T-space.*

Proof. Let $X = \bigcup_1^n C_i$, with each C_i closed in X and a T-space. If H is weak*-compact in $M_t^+(X)$, then $H_i = \{\mu_{C_i} : \mu \in H\}$ is relatively weak*-compact in $M_t^+(C_i)$. Given $\epsilon > 0$, choose a compact subset K_i of C_i with $\mu_{C_i}(C_i \backslash K_i) = \mu(C_i \backslash K_i) < \epsilon/n$ for each i and μ. Let $K = \bigcup_1^n K_i$; then $\mu(X \backslash K) \leqq \sum_1^n \mu(C_i \backslash K_i) < \epsilon$ for all $\mu \in H$.

THEOREM 4.7. *A space which is covered by a family of open T-subspaces is a T-space.*

Proof. Let $\{O_\lambda\}$ be an open cover of X, and assume that each O_λ is a T-space. Then $\{O_\lambda\}$ has an open refinement $\{U_\beta\}$ such that each $\mathrm{Cl}_X U_\beta$ (the closure of U_β in X) is contained in some O_λ; thus each $\mathrm{Cl}_X U_\beta$ is a T-space, by 4.4. Now let $\{V_\alpha\}$ be the family of all finite unions of sets U_β; $\{V_\alpha\}$ is directed upward by inclusion and covers X.

If H is weak*-compact in $M_t^+(X)$, then given $\epsilon > 0$ there is an index α_0 such that $\mu(X \backslash V_{\alpha_0}) < \epsilon/2$ for all $\mu \in H$, by 4.2. Hence $\mu(X \backslash \mathrm{Cl}_X V_{\alpha_0}) < \epsilon/2$ for $\mu \in H$. Now if $V_{\alpha_0} = \bigcup_1^n U_{\beta_i}$, then $\mathrm{Cl}_X V_{\alpha_0} = \bigcup_1^n \mathrm{Cl}_X U_{\beta_i}$ is a T-space, by 4.6. Let $C = \mathrm{Cl}_X V_{\alpha_0}$. Then $H_1 = \{\mu_C : \mu \in H\}$ is relatively weak*-compact in $M_t^+(C)$, by 4.5, and so there is a compact subset K of C with $\mu(C \backslash K) = \mu_C(C \backslash K) < \epsilon/2$ for $\mu \in H$. Then $\mu(X \backslash K) < \epsilon$ for $\mu \in H$. This completes the proof.

COROLLARY 4.8. *A space which is covered by the interiors of a family of closed T-subspaces is a T-space.*

COROLLARY 4.9. *An open subset of a T-space is a T-space.*

Proof. This is immediate from 4.4 and 4.8.

A space is a local T-space if each point has a neighbourhood (not necessarily open) which is a T-space.

COROLLARY 4.10. *A local T-space is a T-space.*

Proof. Let X be a local T-space, x a point in X, N its T-space neighbourhood. Then there is an open subset U of X with $x \in U \subset N$. Now U is a T-space by 4.9; hence X is a T-space by 4.7.

As a trivial consequence, any locally compact space is a T-space (of course this is easy to show directly).

A space which is a countable union of closed T-subspaces need not be a T-space (Example 4.18).

PROPOSITION 4.11. *A space with a locally finite cover of closed T-subspaces is a T-space.*

Proof. Let (C_α) be such a cover of a space X. If $x \in X$, then there is an open set U and sets $C_{\alpha_1}, \ldots, C_{\alpha_n}$ with $x \in U \subset \bigcup_1^n C_{\alpha_i}$. Then U is a T-space by 4.6 and 4.9. Hence X is a T-space by 4.7.

THEOREM 4.12. *A countable product of T-spaces is a T-space.*

Proof. Let (X_n) be a countable family of T-spaces, and let $X = \Pi X_n$, with projections $\varphi_n : X \to X_n$ and induced maps $T_n^* : M_t(X) \to M_t(X_n)$. Let H be weak*-compact in $M_t^+(X_n)$. Given $\epsilon > 0$, for each n there is a compact subset K_n of X_n with $T^*\nu(X_n \backslash K_n) < \epsilon/2^n$ for all $\nu \in H$. Then if $K = \Pi K_n \subset X$, $\nu(X \backslash K) < \epsilon$ for $\nu \in H$.

THEOREM 4.13. *If (X_n) is a countable family of T-subspaces of a fixed space X, then $\bigcap_1^\infty X_n$ is a T-space.*

Proof. The class of T-spaces is preserved under the formation of closed subspaces and countable products. As in Proposition 3.3, the result is immediate.

COROLLARY 4.14. *A G_δ set in a T-space is a T-space.*

Thus Čech complete spaces (spaces which are G_δ's in their Stone-Čech compactifications) are T-spaces. In particular, we have the well-known result (Prohorov's theorem) that every complete metric space is a T-space. Also any space X such that $\beta X \backslash X$ is countable is a T-space.

It has come to our attention that proofs of 4.4, 4.12, and 4.13 are given in [**11**], while other results in this section are improvements on theorems in that paper.

The easiest way to produce a non-trivial weak*-compact set of measures is to find a weak*-convergent sequence. Thus it is of interest to determine if the ranges of weak*-convergent sequences must be tight.

Definition 4.16. A space X is a *sequential T-space* if the range of every weak*-convergent sequence in $M_t^+(X)$ is tight.

LeCam has shown that every metric space is a sequential T-space [**15**, p. 222]. LeCam's result can be extended to the spaces of countable type introduced by Arhangel'skii [**1**]. A space X is of countable type if there are collections (K_α) of compact subsets and $(G_{n,\alpha})$ of open subsets of X such that: (1) any compact subset of X is contained in some K_α and (2) if U is open and contains K_α, then, for some n, $K_\alpha \subset G_{n,\alpha} \subset U$. The sets $G_{n,\alpha}$ may be assumed to be cozero sets. The proof of the next result is very similar to LeCam's argument (see, for example, the monograph of Topsøe [**28**, p. 45]).

THEOREM 4.17. *Every space of countable type is a sequential T-space.*

It is easily seen that 4.3, 4.4, 4.12, and 4.13 remain valid for sequential T-spaces. Also, since any open subset O of a space X can be written as $O = O_1 \cap X$, where O_1 is open in βX (hence locally compact), it follows from 4.13 that 4.9 and consequently 4.14 hold for sequential T-spaces.

We close this section with a series of examples related to the preceding results.

Example 4.18. A σ-compact space need not be a (sequential) T-space. We refer to Varadarajan [**30**, p. 225], Fernique [**7**, p. 24] and Choquet [**3**, p. 14] for three distinct constructions. The first two are hemicompact, and Varadarajan's space, which we refer to below as V, is countable; none is a k-space. Thus an F_σ in a T-space (βV) need not be a T-space.

Example 4.19. If $\varphi : X \to Y$ is a continuous bijection and a Baire isomorphism (i.e., images and inverse images of Baire sets are Baire sets), then neither φ nor φ^{-1} preserves the T-space property in general. Indeed if $X = V$ and Y is the one-point compactification of the positive integers, the natural one-to-one correspondence φ which leaves the integers fixed serves as one example. On the other hand, Fernique's example of a non-T-space is l^2 (separable Hilbert space) with the weak topology. If we call this space Y, let X be l^2 with the usual norm, and take φ to be the identity map, then X is a T-space (complete metric) but $\varphi(X)$ is not (that φ is a Baire isomorphism follows easily from the fact that every norm-closed ball in l^2 is a zero-set in the weak topology). We remark that measure-compactness and strong measure-compactness are much better behaved under continuous Baire isomorphisms [**18**, 4.6 and 5.2].

Example 4.20. An uncountable product of T-spaces need not be a T-space; an uncountable intersection of T-subspaces of a fixed space need not be a T-space. Indeed the real line $\mathbf{R}$ is a T-space, and V is Lindelöf, hence realcompact [**10**, p. 115], so that V is homeomorphic to a closed subspace of $\mathbf{R}^{C^*(V)}$ [**10**, p. 160]. Thus $\mathbf{R}^{C^*(V)}$ is not a T-space, by 4.4 (note that card$C^*(V) = c$). On the other hand, any space X is an intersection of locally compact (hence T-) subspaces of βX.

Example 4.21. The two most prominent classes of T-spaces, locally compact spaces, and complete metric spaces are Baire spaces [**6**, p. 249], but in general the notions of T-space and Baire space are unrelated. Indeed V is a Baire space which is not a T-space. On the other hand, a linear space E of countably infinite dimension, endowed with the finest locally convex topology [**23**, p. 56] is not a Baire space (it is a countable union of finite-dimensional subspaces, which are closed and nowhere dense). However, E is linearly homeomorphic to the strong dual of the Fréchet-Montel space $\mathbf{R}^N$; hence, by a result of Fernique [**7**, p. 28], E is a T-space.

Some additional examples are given in the next section.

5. T-spaces and k-spaces: an extension of the Conway-LeCam theorem. Before stating the main result, we consider some topological preliminaries.

A completely regular space X is a k_R-space if the continuity of a real-valued function is implied by its continuity on compact subsets. A k-space is a k_R-space, but the converse is not true (see [**21**] for additional information). We do have the following; it can be deduced from results in [**31**], but we include a proof.

LEMMA 5.1. *A hemicompact k_R-space is a k-space.*

Proof. Let $X = \bigcup_1^\infty K_n$ where (K_n) is an increasing sequence of compact subsets and every compact set is contained in some K_n. Let $D \subset X$, and suppose $D \cap K_n$ is closed for every n. It suffices to show that if $x_0 \in X \backslash D$, then there is an f in $C^*(X)$ with $f|D = 1$, $f(x_0) = 0$. We may assume that $x_0 \in K_1$. There is a function $f_1 \in C^*(K_1)$, $0 \leqq f_1 \leqq 1$, $f_1|(D \cap K_1) = 1$, $f_1(x_0) = 0$. Define f_2' on $K_1 \cup (D \cap K_2)$ by $f_2'|K_1 \equiv f_1$, $f_2'|(D \cap K_2) \equiv 1$. Then f_2' is continuous on the compact subset $K_1 \cup (D \cap K_2)$ of K_2, hence has an extension $f_2 \in C^*(K_2)$, with $0 \leqq f_2 \leqq 1$, $f_2|(D \cap K_2) = 1$. By an obvious induction we can find for each n, $f_n \in C^*(K_n)$, $f_n|(D \cap K_n) \equiv 1$, $f_n|K_{n-1} = f_{n-1}$. The function f defined by $f|K_n = f_n$ is then continuous (by the k_R-property) and separates x_0 and D.

The next result has been essentially obtained by Fremlin, Garling and Haydon [**8**]; the proof given here makes use of Topsøe's criterion (4.1).

THEOREM 5.2. *If X is a hemicompact k_R-space, then $(C^*(X), \beta_0)$ is a strong Mackey space, and X is β-simple.*

Proof. According to 2.11, 2.13, and 3.4, it suffices to show that X is a T-space. Let $X = \bigcup_1^\infty K_n$, where (K_n) is an increasing fundamental family of compact subsets, and let H be weak*-compact in $M_t^+(X)$. If H is not tight, then there exists $\epsilon > 0$, a sequence (μ_n) in H, and compact sets $D_n \subset X \backslash K_n$ such that $\mu_n(D_n) > \epsilon$ for all n. Let $T_n = \bigcup_{i=n}^\infty D_i$; it follows from 5.1 that T_n is closed. If $U_n = X \backslash T_n$, then $K_n \subset U_n$, and so (U_n) dominates the family of compact subsets of X. Thus there is an integer n_0 such that $\inf \{\mu(X \backslash U_i) : 1 \leqq i \leqq n_0\} < \epsilon$ for all $\mu \in H$, using 4.1. Since, for any $n \geqq n_0$, $D_n \subset T_i$ for all $i \leqq n_0$, we have a contradiction.

In order to extend this result to topological sums of spaces, we need a lemma, involving a technique employed by Taylor [**26**] in his study of C^*-algebras. We wish to thank R. A. Fontenot for calling Taylor's result to our attention.

LEMMA 5.3. *Let $(X_\alpha)_{\alpha\in A}$ be a family of completely regular spaces with $X = \sum\{X_\alpha : \alpha \in A\}$ their topological sum. If H is a weak*-compact subset of $M_\tau(X)$, then for any $\epsilon > 0$ there is a finite subset F of A such that $|\mu|(\sum\{X_\alpha : \alpha \in A\backslash F\}) < \epsilon$* for all $\mu \in H$.

Proof. Let $A_0 = \{\alpha \in A$: there exists $\mu \in H$ with $|\mu|(X_\alpha) > 0\}$. For each $\alpha \in A_0$, choose $f_\alpha \in C^*(X_\alpha)$, $0 \leqq f_\alpha \leqq 1$, and $\mu_\alpha \in H$ such that

$$\int_{X_\alpha} f_\alpha d\mu_\alpha \neq 0.$$

Interpret A as a topological space with the discrete topology, and define $T : C^*(A) \to C^*(X)$ as follows: if $\gamma = (\gamma_\alpha) \in C^*(A)$, then $T(\gamma)|X_\alpha = \gamma_\alpha f_\alpha$. It is easy to see that $T^*(M_\tau(X)) \subset M_\tau(A) = M_t(A) = l_1(A)$; consequently $T^*(H)$ is weakly compact, and therefore norm-compact, in $l^1(A)$. Thus there is a countable subset B_0 of A such that $|T^*\mu|(\{\alpha\}) = 0$ for all $\alpha \in A\backslash B_0$. Since $A_0 \subset B_0$, A_0 is countable.

It follows from τ-additivity that $|\mu|(\sum\{X_\alpha : \alpha \in A\backslash A_0\}) = 0$ for all $\mu \in H$. Moreover, H is weak*-compact in $M_\sigma(X)$; hence if $(f_n) \downarrow 0$ in $C^*(X)$, then $|\mu|(f_n) \to 0$ uniformly with respect to $\mu \in H$ [**8**; **30**]. If A_0 is enumerated as (α_n) and f_n is the characteristic function of $\sum\{X_{\alpha_i} : i \geqq n\}$, the desired result follows.

THEOREM 5.4. *Let $(X_\alpha)_{\alpha\in R}$ be a family of completely regular spaces, and suppose that any of the following conditions holds*:

(1) *$(C^*(X_\alpha), \beta_0)$ is Mackey for all α*;

(2) *$(C^*(X_\alpha), \beta_0)$ is strong Mackey for all α*,

(3) *$(C^*(X), \beta)$ is Mackey for all α*;

(4) *$(C^*(X), \beta)$ is strong Mackey for all α.*

If $X = \sum\{X_\alpha : \alpha \in A\}$, then $C^*(X)$ with the corresponding topology has the same property.

Proof. If H is weak*-compact (convex and circled) in $M_\tau(X)$ or $M_t(x)$, and for any α, we let $H_\alpha = \{\mu|X_\alpha\} \subset M_\tau(X_\alpha)$ (or $M_t(X_\alpha)$), then H_α is weak*-compact (convex and circled). All four results now follow easily upon applying 5.3 and the characterizations of the β- and β_0-equicontinuous sets as the uniformly τ-additive and tight sets, respectively.

COROLLARY 5.5. *The topological sum X of a family $(X_\alpha)_{\alpha\in A}$ of hemicompact k-spaces is a T-space, and $(C^*(X), \beta_0)$ is a strong Mackey space. However, X is β-simple if and only if the index set A has cardinal of measure zero.*

Proof. The first assertion is an immediate consequence of 5.2 and 5.4. Now each X_α is σ-compact, hence $M_\tau(X_\alpha) = M_t(X_\alpha)$. If 5.3 is applied to a single

τ-additive measure, it is easy to see that $M_\tau(X) = M_t(X)$. Thus from 2.13, X is β-simple if and only if $M_\sigma(X) = M_\tau(X)$. If A is countable, then X is a hemicompact k-space, and therefore is β-simple. If A is uncountable, then the largest cardinal of a closed discrete subset of X is precisely the cardinal of A (any closed discrete subset of X_α is countable). Since X is paracompact, the second assertion is now a consequence of Katetov's theorem [**30**, p. 177].

Example 5.6. The spaces of Varadarajan and Fernique (4.18) are hemicompact spaces, but not T-spaces, and β_0 is not Mackey.

Example 5.7. Every completely regular space can be embedded as a closed subspace of a pseudocompact k_R-space [**21**, p. 56]. Thus a space of this type need not be a T-space.

It is natural to inquire if the preceding results admit a generalization to σ-compact k-spaces, in particular to σ-compact metric spaces. Some remarkable results of D. Preiss [**22**] show that the answer is "no". Indeed we can use 2.11 and the work of Preiss to state:

THEOREM 5.8. *If X is a σ-compact metric space, then the following conditions on X are equivalent*:

(1) *$(C^*(X), \beta_0)$ is Mackey*;
(2) *$(C^*(X), \beta_0)$ is strong Mackey*;
(3) *X is a T-space*;
(4) *X admits a compatible complete metric*;
(5) *X contains no G_δ subspace homeomorphic to the rationals.*

This result is precise, but perhaps slightly deficient in that none of the equivalent properties may be readily determinable for a given space X. The analysis presented here is an attempt to provide a somewhat more computable criterion.

LEMMA 5.9. *If (Y, d) is a complete σ-compact metric space, then Y contains a dense open σ-compact locally compact subspace.*

Proof. Let $y_0 \in Y$, and let $\epsilon > 0$ be given. Let $N(y_0, \epsilon) = \{y : d(y_0, y) < \epsilon\}$. Then the closure D of $N(y_0, \epsilon)$ is a complete σ-compact metric space, and so by the Baire category theorem there is an open set V in Y such that $V \cap D$ is a non-empty, relatively compact subset of D. There is a point $z_0 \in V \cap N(y_0, \epsilon)$, and z_0 has a compact neighbourhood in Y. Thus $Y_0 = \{y \in Y : y$ has a compact neighbourhood$\}$ is the desired subspace (Y_0 is σ-compact because it is separable metric, hence Lindelöf).

THEOREM 5.10. *Let X be a complete σ-compact metric space. Then X has a decomposition into non-empty pairwise disjoint sets $(X_\alpha)_{\alpha<\alpha_0}$, where α_0 is a countable ordinal, such that*

(a) *each X_α is σ-compact locally compact,*
(b) *for any $\beta \leqq \alpha_0$, $\bigcup_{\alpha<\beta} X_\alpha$ is open in X.*

Proof. Let $X_0 = \{x \in X : x$ has a compact neighbourhood in $X\}$, and, inductively, let $X_\alpha = \{x \in X \backslash \bigcup_{\gamma<\alpha} X_\gamma : x$ has a compact neighbourhood in $X \backslash \bigcup_{\gamma<\alpha} X_\gamma\}$, for each ordinal $\alpha < \omega_1$ (possibly $X_\alpha = \emptyset$ for large α).

According to 5.9, X_0 is a dense open σ-compact locally compact subspace of X, and if $X_0 = X$ we are done. Otherwise, $X \backslash X_0$ is a complete σ-compact metric space, and X_1 is a dense open σ-compact locally compact subspace of $X \backslash X_0$ such that $X_0 \cup X_1$ is open in X. Continuing this process inductively, we obtain a family $(X_\alpha)_{\alpha<\omega_1}$ satisfying (a) and (b).

We claim that, for some $a_0 < \omega_1$, $\bigcup_{\alpha<\alpha_0} X_\alpha = X$. If not, then $X_\alpha \neq \emptyset$ for all $\alpha < \omega_1$. Let $Y = \bigcup_{\alpha<\omega_1} X_\alpha \subset X$. Then Y is open in X, and each $p \in Y$ has an open neighbourhood whose closure in X is a (σ-compact) subset of Y. Thus since Y is Lindelöf, Y is σ-compact. It follows from (b) that, for some $\alpha_0 < \omega_1$, $Y \subset \bigcup_{\beta<\alpha_0} X_\beta$, contradicting the fact that X_{α_0} is non-empty. This completes the proof.

It can be shown that the decomposition given here is maximal in the following sense: if $(Y_\alpha)_{\alpha<\gamma_0}$ is any other decomposition satisfying (a) and (b), then, for any $\beta \leqq \alpha_0$, $\bigcup_{\alpha<\beta} Y_\alpha \subset \bigcup_{\alpha<\beta} X_\alpha$.

Now we prove the converse to 5.10.

LEMMA 5.11. *If X is a metric space, B is a closed locally compact subspace, and $X \backslash B$ is a T-space, then X is a T-space.*

Proof. Let us first assume that B is compact. Let d be a metric on X, and define $F_1 = \{x : d(x, B) \geqq 1\}$, $F_n = \{x : 1/n \leqq d(x, B) \leqq 1/n - 1\}$ for $n > 1$.

If H is a weak*-compact subset of $M_t^+(X)$, and if H_n is the subset of $M_t^+(F_n)$ consisting of restrictions of members of H, then H_n is relatively weak*-compact (4.5). Since F_n is closed in the T-space $X \backslash B$, it follows from 4.4 that given $\epsilon > 0$, there is a compact set $K_n \subset F_n$ such that $\mu(F_n \backslash K_n) < \epsilon/2^n$ for all n. Now $K_0 = B \cup \bigcup_1^\infty K_n$ is compact (see, for example [**15**, p. 222]), and $\mu(X \backslash K_0) < \epsilon$ for all $\mu \in H$.

Taking up the general case in which B is closed and locally compact, fix $x_0 \in B$, and let V be an open subset of B such that $x_0 \in V$ and $\bar{V}$ is compact. Then $(X \backslash B) \cup \bar{V}$ is a T-space, and so is its open subspace $(X \backslash B) \cup V$, by 4.9. Now $(X \backslash B) \cup V$ is open in X, and it follows from 4.7 that X is a T-space.

THEOREM 5.12. *Let X be a metric space, and suppose X is a disjoint union of non-empty subspaces $(X_\alpha)_{\alpha<\alpha_0}$, where α_0 is an ordinal and*

(a) *each X_α is locally compact,*

(b) *for any $\beta \leqq \alpha_0$, $\bigcup_{\alpha<\beta} X_\alpha$ is open in X.*

Then X is a T-space.

Proof. Note that X need not be σ-compact, and α_0 need not be a countable ordinal. We use transfinite induction on the proposition $P(\gamma)$: every metric space Y with a decomposition $(Y_\alpha)_{\alpha<\gamma}$ into non-empty disjoint subsets satisfying (a) and (b) is a T-space.

https://doi.org/10.4153/CJM-1972-087-2 Published online by Cambridge University Press

Now $P(1)$ is clear, since Y is then locally compact. Suppose $P(\gamma)$ holds for all $\gamma < \alpha_0$, and let $X = \bigcup_{\alpha<\alpha_0} X_\alpha$ satisfy (a) and (b).

Case 1. If α_0 is a limit ordinal, then for each $\gamma < \alpha_0$, $(X_\alpha)_{\alpha<\gamma}$ is a decomposition of $W_\gamma = \bigcup_{\alpha<\gamma} X_\alpha$ which satisfies (a) and (b). Thus W_γ is a T-subspace of X. Moreover, each W_γ is open in X, and so $X = \bigcup_{\gamma<\alpha} W_\gamma$ is a T-space, by 4.7.

Case 2. If $\alpha_0 = \alpha_1 + 1$, then, since $(X_\alpha)_{\alpha<\alpha_1}$ is a decomposition of $W = \bigcup_{\alpha<\alpha_1} X_\alpha$ satisfying (a) and (b), W is a T-space. It now follows from 5.11 that X is a T-space.

COROLLARY 5.13. *If X is a σ-compact metric space, then $(C^*(X), \beta_0)$ is Mackey (or strong Mackey) if and only if X is the union of a pairwise disjoint family $(X_\alpha)_{\alpha<\alpha_0}$ of non-empty σ-compact locally compact subspaces, indexed by some countable ordinal α_0, such that $\bigcup_{\alpha<\beta} X_\alpha$ is open for all $\beta \leqq \alpha_0$.*

Any σ-compact metric space X may be subjected to the canonical decomposition of 5.10. If, at any stage, X_α fails to be dense in

$$X \backslash \bigcup_{\beta<\alpha} X_\beta,$$

then $(C^*(X), \beta_0)$ is not Mackey. In any case, a decision will be reached after countably many iterations.

According to 5.2 and 5.8, any hemicompact metric space admits a complete metric. There is a simpler way to see this, however. It is not difficult to show that a hemicompact metric space is locally compact, hence an open set in its Stone-Čech compactification, and therefore is completely metrizable [**32**, p. 180].

Finally let us note that for any $\alpha < \omega_1$, there is a countable, complete metric space X of rank α: i.e., such that $X_\beta \neq \emptyset$ for $\beta < \alpha$ (notation as in 5.10), while $X_\alpha = \emptyset$. Indeed, this is trivial for $\alpha = 1$. Suppose that, for each $\beta < \alpha$, there is a countable complete metric space $X^{(\beta)}$ of rank β. If α is a limit ordinal, with $\alpha_1 < \alpha_2 < \ldots \to \alpha$, let

$$X^{(\alpha)} = \sum_{n=1}^{\infty} X^{(\alpha_n)}.$$

If $\alpha = \alpha' + 1$, let Y be a countable topological sum of copies of $X^{(\alpha')}$, and let $X^{(\alpha)}$ be a countable topological sum of copies (Y_i) of Y, together with an exceptional point p, having a base of neighbourhoods consisting of sets $\{p\} \cup (\sum_{i=n}^{\infty} Y_i)$. It is not difficult to show that $X^{(\alpha)}$ has the desired properties.

6. Some remarks on the strict topology. Combining 2.13, 3.2, 3.3, 3.4, 4.4, 4.12, 4.13, and 5.2 we have

THEOREM 6.1. *The class of β-simple spaces contains the hemi-compact k-spaces and is preserved by taking closed subspaces, countable products, and countable intersections of subspaces of a fixed space.*

https://doi.org/10.4153/CJM-1972-087-2 Published online by Cambridge University Press

A complete separable metric space (Polish space) is a G_δ set in a compact metric space [**32**, 24.13 and 24B]; but an open set in a compact metric space is a hemicompact k-space.

COROLLARY 6.2. *A Polish space is β-simple.*

If X is locally compact, Buck [**2**] showed that $(C^*(X), \beta_0)$ is a complete locally convex space. Some results on completeness in the completely regular case were given by Sentilles [**24**]. In particular, β_0 is complete if and only if X is a k_R-space.

Example 6.3. A β-simple space such that $\beta_0 = \beta = \beta_1$ is not complete or even sequentially complete: This construction is based on an argument due to Michael [**16**, p. 282]. Let X be the irrationals, and let Y be the space obtained by taking a countable number of copies of the one-point compactification of the positive integers and identifying the points at infinity. Then X is a Polish space and Y is a hemicompact k-space, so $X \times Y$ is β-simple by 6.1 and 6.2. Y may be visualized as a countably infinite collection of "spokes" emerging from a central point p; denote the mth point on the nth spoke by $y(m, n)$, and the union of the first n spokes (including p) by F_n. Let (d_n) be a sequence of rationals increasing monotonically to $\sqrt{2}$, and for each n let $(c_{k,n})$ be a sequence of rationals which satisfies $d_n < c_{k,n} < \sqrt{2}$ for all k and decreases monotonically to d_n.

Let $A = \{(x, y(m, n)) : x$ is irrational, and $d_n < x < c_{m,n}\}$ and let $A_n = A \cap (X \times F_n)$. Since all d_n and $c_{k,n}$ are rational, it is easy to see that each A_n is open and closed in $X \times Y$; hence the characteristic function ψ_n of A_n is continuous. But $(\sqrt{2}, p)$ is a limit point of A, hence the characteristic function ψ_0 of A is not continuous. Now if K is a compact subset of $X \times Y$, then its projection on Y is compact, hence contained in some F_n. It follows that the sequence (ψ_n) is β_0-Cauchy in $C^*(X)$ and converges pointwise to the discontinuous function ψ_0. Thus $(C^*(X), \beta_0)$ is not sequentially complete.

7. An open question. We do not know any example of a T-space for which $M_\iota \neq M_\tau$. In this regard it would be of particular interest to determine whether or not the real line with the right half-open interval topology is a T-space.

REFERENCES

1. A. V. Arhangel'skii, *Bicompact sets and the topology of spaces*, Trans. Moscow Math. Soc. *13* (1965), 1–62.

2. R. C. Buck, *Bounded continuous functions on a locally compact space*, Michigan Math. J. *5* (1958), 95–104.

3. G. Choquet, *Sur les ensembles uniformément négligeables*, Sém. Choquet (1969/70), No. 6.

4. J. B. Conway, *The strict topology and compactness in the space of measures*, Trans. Amer. Math. Soc. *126* (1967), 474–486.

5. R. M. Dudley, *Convergence of Baire measures*, Studia Math. *27* (1966), 251–268.

6. J. Dugundji, *Topology* (Allyn and Bacon, Boston, 1966).

7. X. Fernique, *Processus linéaires, processus généralises*, Ann. Inst. Fourier (Grenoble) *17* (1967), 1–92.
8. D. H. Fremlin, D. J. H. Garling, and R. G. Haydon, *On measures on topological spaces*, Proc. Internat. Colloquium, Bordeaux, 1971 (to appear).
9. R. Giles, *A generalization of the strict topology*, Trans. Amer. Math. Soc. *161* (1971), 467–474.
10. L. Gillman and M. Jerison, *Rings of Continuous Functions* (Van Nostrand, Princeton, 1960).
11. J. Hoffmann-Jørgensen, *A generalization of the strict topology*, Aarhus Universitet Preprint Series 1969/70, No. 32.
12. R. B. Kirk, *Locally compact, B-compact spaces*, Nederl. Akad. Wetensch. Proc. Ser. A. *72* (1969), 333–344.
13. ——— *Measures in topological spaces and B-compactness*, Nederl. Akad. Wetensch. Proc. Ser. A. *72* (1969), 172–183.
14. J. D. Knowles, *Measures on topological spaces*, Proc. London Math. Soc. *17* (1967), 139–156.
15. L. LeCam, *Convergence in distribution of stochastic processes*, Univ. of California, Publ. in Statistics *2* (1953–8), 207–236.
16. E. A. Michael, *Local compactness and cartesian products of quotient maps and k-spaces*, Ann. Inst. Fourier (Grenoble) *18* (1968), 281–286.
17. W. Moran, *The additivity of measures on completely regular spaces*, J. London Math. Soc. *43* (1968), 633–639.
18. ——— *Measures and mappings on topological spaces*, Proc. London Math. Soc. *19* (1969), 493–508.
19. ——— *Measures on metacompact spaces*, Proc. London Math. Soc. *20* (1970), 507–524.
20. S. Negrepontis, *Baire sets in topological spaces*, Arch. Math. (Basel) *18* (1969), 603–608.
21. N. Noble, *k-spaces and some generalizations*, Doctoral Dissertation, University of Rochester, 1967.
22. D. Preiss, *Metric spaces in which Prohorov's theorem is not valid*, Proc. Prague Symposium on General Topology, 1971 (to appear).
23. H. H. Schaefer, *Topological vector spaces* (Macmillan, New York, 1966).
24. F. D. Sentilles, *Bounded continuous functions on a completely regular space* (to appear).
25. W. H. Summers, *The general complex bounded case of the strict weighted approximation property*, Math. Ann. *192* (1971), 90–98.
26. D. C. Taylor, *A general Phillips Theorem for C*-algebras and some applications*, Pacific J. Math. *40* (1972), 477–488.
27. F. Topsøe, *Compactness in spaces of measures*, Studia Math. *36* (1970), 195–212.
28. ——— *Topology and measure* (Springer-Verlag lecture notes, vol. 133, 1970).
29. A. C. M. Van Rooj, *Tight functionals and the strict topology*, Kyungpook Math. J. *7* (1967), 41–43.
30. V. S. Varadarajan, *Measures on topological spaces*, Amer. Math. Soc. Transl. *48* (1965), 161–228.
31. S. Warner, *The topology of compact convergence on continuous function spaces*, Duke Math. J. *25* (1958), 265–282.
32. S. Willard, *General topology* (Addison-Wesley, Reading, Mass., 1970).

University of Missouri,
Columbia, Missouri;
Louisiana State University,
Baton Rouge, Louisiana