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FACTOR-CORRESPONDENCES IN REGULAR RINGS

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1. Introduction. Factor-correspondences are nothing more than a way of describing isomorphisms between principal ideals in a regular ring. However, due to a remarkable decomposition theorem of M. J. Wonenburger [**7**, Lemma 1], they have proved to be a highly effective tool in the study of completeness properties in matrix rings over regular rings [**7**, Theorem 1]. Factor-correspondences also figure in the proof of D. Handelman's theorem that an $\aleph_0$-continuous regular ring is unit-regular [**4**, Theorem 3.2].

The aim of the present article is to sharpen the main result in [**7**] and to re-examine its applications to matrix rings. The basic properties of factor-correspondences are reviewed briefly for the reader's convenience.

2. Factor-correspondences. Throughout, R denotes a regular ring (with unity).

Definition 1 (cf. [**5**, p. 209ff], [**7**, p. 212]). A *right factor-correspondence* in R is a right R-isomorphism $\varphi : J \to K$, where J and K are principal right ideals of R (left factor-correspondences are defined dually).

With notation as in Definition 1, write $J = eR$, $K = fR$ with e, f idempotent. Defining $y = \varphi(e)$, $x = \varphi^{-1}(f)$, one sees that φ (resp. φ^{-1}) is left-multiplication by y (resp. x) on J (resp. K). (For example, $\varphi(er) = \varphi(eer) = \varphi(e)er = yer$ for all $r \in R$.) In particular, $xyx = x(yx) = \varphi^{-1}(\varphi(x)) = x$ and similarly $y = yxy$. One has $J = xR$, $K = yR$. (For example, $x = \varphi^{-1}(f) \in J$, so $xR \subset J$, whereas $J = \varphi^{-1}(K) = xK \subset xR$, thus $J = xR$.)

Conversely, if x, y are elements of R such that $xyx = x$ and $yxy = y$, one sees that $xr \mapsto y(xr)$ defines a right factor-correspondence $\varphi : xR \to yR$ with $\varphi^{-1}(yr) = x(yr)$.

We denote by R_d (resp. R_s) the ring R regarded as a right (resp. left) R-module in the natural way. (Thus, in another notation, $R_d = R_R$ and $R_s = {}_RR$.) One writes $2R_d = R_d \oplus R_d$ for the right R-module of ordered pairs of elements of R (and nR_d for the module of n-tuples). If A is a finitely generated projective right module over the regular ring R, one writes $L(A)$ for the set of all finitely generated submodules B of A; $L(A)$ may also be described as the set of all direct summands of A [**2**, p. 6, Theorem 1.11]. Ordered by inclusion, $L(A)$ is a complemented modular

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lattice, with $B \vee C = B + C$ and $B \wedge C = B \cap C$ [**2**, p. 15, Theorem 2.3].

LEMMA 1 [**7**, p. 212, Lemma 1]. *If R is a regular ring and $M \in L(2R_d)$, one can write*

$$M = [M \cap (1, 0)R] \oplus (a_1, a_2)R \oplus [M \cap (0, 1)R]$$

with a_1, a_2 elements of R such that $Ra_1 = Ra_2$.

There is more to the statement of Wonenburger's lemma, as follows. Since $\mathrm{pr}_1[M \cap (1, 0)R]$ and $\mathrm{pr}_2[M \cap (0, 1)R]$ are principal right ideals of R [**2**, p. 1, Theorem 1.1], one can write

$$M \cap (1, 0)R = (e_1, 0)R, \quad M \cap (0, 1)R = (0, e_2)R$$

with e_1, e_2 idempotents of R, thus M is the direct sum of three cyclic submodules:

$$M = (e_1, 0)R \oplus (a_1, a_2)R \oplus (0, e_2)R.$$

The proof of [**7**, Lemma 1] shows, moreover, that the middle term $(a_1, a_2)R$ may be prescribed to be the set $\{(r, s) \in M : e_1 r = e_2 s = 0\}$ and one can suppose further that a_1 is idempotent (thus $a_2 a_1 = a_2$). Note that M is the graph of a function (necessarily R-linear) if and only if $M \cap (0, 1)R = 0$; it is the graph of a bijection if and only if

$$M \cap (0, 1)R = M \cap (1, 0)R = 0.$$

Since, in a regular ring R, $Ra = (Ra)^{rl} = \{a\}^{rl}$ (the exponents denote right and left annihilators), the condition $Ra_1 = Ra_2$ signifies that a_1 and a_2 have the same right annihilators; whence:

LEMMA 2 [**7**, p. 212, Lemma 2]. *If R is a regular ring and a, b are elements of R such that $Ra = Rb$, then $ar \mapsto br$ ($r \in R$) defines a right factor-correspondence $aR \to bR$.*

With notation as in Lemma 2, one writes $(a{:}b)$ for the right factor-correspondence $ar \mapsto br$; its graph is $(a, b)R \in L(2R_d)$. The action of the function $(a{:}b)$ is indicated by $(a{:}b)ar = br$, $r \in R$.

Conversely, suppose $\varphi : J \to K$ is any right factor-correspondence in R. Choose elements x, y of R such that $J = xR$ and $\varphi(s) = ys$ for all $s \in J$; then for all $r \in R$ one has $\varphi(xr) = y(xr)$, thus the graph of φ is the cyclic submodule

$$\{(x, yx)r : r \in R\} = (x, yx)R \in L(2R_d).$$

LEMMA 3. *Every right factor-correspondence in a regular ring R is of the form $(a_1 : a_2)$ for suitable elements a_1, a_2 of R with $Ra_1 = Ra_2$.*

Proof. Let $\varphi : J \to K$ be a right factor-correspondence in R, M its graph. Since $M \in L(2R_d)$ by the preceding remark, one may apply to it the decomposition of Lemma 1; since M is the graph of a bijection, one has

$$M \cap (1, 0)R = M \cap (0, 1)R = 0,$$

thus $M = (a_1, a_2)R$ with $Ra_1 = Ra_2$. By Lemma 2, the pair a_1, a_2 defines a right factor-correspondence $(a_1 : a_2)$ whose graph is $(a_1, a_2)R = M$; in other words, $(a_1 : a_2) = \varphi$.

Remarks. It follows from Lemmas 1 and 2 that if $M \in L(2R_d)$ is the graph of a bijection, then it must be the graph of a right factor-correspondence. More generally, if $M \in L(2R_d)$ is the graph of a function φ, then $M \cap (0, 1)R = 0$; writing $M \cap (1, 0)R = (e_1, 0)R$, e_1 idempotent, Lemma 1 gives a decomposition

$$M = (e_1, 0)R \oplus (a_1, a_2)R, \quad Ra_1 = Ra_2,$$

and one can arrange to have $e_1a_1 = 0$. The domain of φ is

$$\mathrm{pr}_1 M = e_1R + a_1R = e_1R \oplus a_1R$$

(the sum is direct because $e_1a_1 = 0$), and the graph of φ is

$$M = \{(e_1r + a_1s, a_2s) : r, s \in R\},$$

so that $\varphi(e_1r + a_1s) = a_2s$ for all r, s in R; thus $\varphi|e_1R = 0$ and $\varphi|a_1R = (a_1 : a_2)$. The gist of what is going on is that it means a great deal for a graph to be finitely generated. (For example, if A is a projective module over a regular ring and if M is a finitely generated submodule of $2A = A \oplus A$ such that M is the graph of a function φ, then the domain $\mathrm{pr}_1 M$ and range $\mathrm{pr}_2 M$ of φ are finitely generated, hence are direct summands of A [**2**, p. 6, Theorem 1.11], hence are projective; thus the epimorphism $\varphi : \mathrm{pr}_1 M \rightarrow \mathrm{pr}_2 M$ splits.) The message of Lemma 1 is that every finitely generated submodule of $2R_d$ is the direct sum of the graph of an isomorphism and two "defect" terms.

Definition 2. For right factor-correspondences φ, ψ in the regular ring R, one writes $\varphi \leqq \psi$ if ψ extends φ, that is, if the graph of φ is contained in the graph of ψ. This is a partial ordering in the set of all right factor-correspondences.

If φ, ψ are right factor-correspondences and one writes $\varphi = (a_1 : a_2)$, $\psi = (b_1 : b_2)$ via Lemma 3, then $\varphi \leqq \psi$ signifies that $(a_1, a_2)R \subset (b_1, b_2)R$; equivalently, $a_1R \subset b_1R$ and $\psi(a_1) = a_2$.

3. Right $\aleph$-continuous regular rings. Let $\aleph$ be an infinite cardinal. A lattice L is said to be *upper* $\aleph$*-complete* if every nonempty subset of L of cardinality $\leqq \aleph$ has a supremum in L; L is said to be *upper* $\aleph$*-continuous* if it is upper $\aleph$-complete and if

$$a \wedge (\vee \{b : b \in B\}) = \vee \{a \wedge b : b \in B\}$$

for every $a \in L$ and every nonempty, simply ordered subset B of L whose cardinality is $\leqq \aleph$. The terms "lower $\aleph$-complete lattice" and "lower $\aleph$-continuous lattice" are defined dually. A regular ring R is said to be *right $\aleph$-continuous* if $L(R_d)$ is upper $\aleph$-continuous (equivalently, the anti-isomorphic lattice $L(R_s)$ is lower $\aleph$-continuous); *left $\aleph$-continuous* if $L(R_s)$ is upper continuous; and *$\aleph$-continuous* if it is both left and right $\aleph$-continuous. A regular ring R is left $\aleph$-continuous if and only if the opposite ring R^0 is right $\aleph$-continuous. For $\aleph$ finite, all of these conditions are trivially fulfilled by every lattice (or regular ring).

The following lemma is contained in the proof of [**7**, Theorem 1]:

LEMMA 4. *Let $\aleph$ be an infinite cardinal, R a regular ring such that the lattice $L(2R_d)$ is upper $\aleph$-continuous. Let $\mathscr{G}$ be the set of graphs of the right factor-correspondences in R (thus $\mathscr{G} \subset L(2R_d)$). Then $\mathscr{G}$ is an $\aleph$-inductive subset of $L(2R_d)$, in the following sense: if $\mathscr{S}$ is an increasingly filtering subset of $\mathscr{G}$ of cardinality $\leqq \aleph$ and if $M = \vee \mathscr{S}$ in $L(2R_d)$, then $M \in \mathscr{G}$.*

Proof. Since $L(2R_d)$ is isomorphic to the lattice of principal right ideals of the matrix ring $M_2(R)$ [**2**, p. 15, Proposition 2.4], the hypothesis on R is that $M_2(R)$ is a right $\aleph$-continuous regular ring (hence so is its "corner" R, cf. [**2**, p. 175, Proposition 14.6]). To say that $\mathscr{S}$ is increasingly filtering means that for every pair G_1, G_2 in $\mathscr{S}$, there exists $G_3 \in \mathscr{S}$ containing both G_1 and G_2.

Since the modules in $\mathscr{S}$ are graphs of bijective functions, one has

$$G \cap (1, 0)R = G \cap (0, 1)R = 0 \text{ for all } G \in \mathscr{S},$$

hence $M \cap (1, 0)R = M \cap (0, 1)R = 0$ by the upper $\aleph$-continuity of $L(2R_d)$, cf. [**2**, p. 160, Proposition 13.1]. By Lemma 1, $M = (a_1, a_2)R$ with $Ra_1 = Ra_2$, thus $M \in \mathscr{G}$ by Lemma 2.

The following theorem sharpens a result in [**7**]:

THEOREM 1 [**7**, Theorem 1]. *Let $\aleph$ be an infinite cardinal, R a right $\aleph$-continuous regular ring, and let $\mathscr{G}$ be the set of graphs of the right factor-correspondences in R, ordered by inclusion. The following conditions are equivalent:*

(a) *the lattice $L(2R_d)$ of finitely generated submodules of $2R_d$ is upper $\aleph$-complete;*

(b) *every increasingly filtering subset of $\mathscr{G}$ of cardinality $\leqq \aleph$ has a supremum in $L(2R_d)$;*

(c) *every simply ordered subset of $\mathscr{G}$ of cardinality $\leqq \aleph$ has a supremum in $L(2R_d)$;*

(d) *every well-ordered subset of $\mathscr{G}$ of cardinality $\leqq \aleph$ has a supremum in $L(2R_d)$.*

If the above conditions are fulfilled, then so are the following:

(1) *$L(2R_d)$ is upper $\aleph$-continuous (that is, $M_2(R)$ is right $\aleph$-continuous);*

(2) *If $\mathscr{S}$ is an increasingly filtering subset of $\mathscr{G}$ of cardinality $\leqq \aleph$ and if $M = \vee \mathscr{S}$ in $L(2R_d)$, then $M \in \mathscr{G}$ (thus M is a supremum of $\mathscr{S}$ in $\mathscr{G}$).*

Proof. The implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) are trivial.

(d) $\Rightarrow$ (a): Let $\mathscr{M}$ be a nonempty subset of $L(2R_d)$ of cardinality $\leqq \aleph$; assuming (d), we are to show that $\mathscr{M}$ has a supremum in $L(2R_d)$. By a transfinite induction on the cardinality of $\mathscr{M}$, we can suppose that $\mathscr{M}$ is well-ordered. (Sketch: Assume all's well for cardinality $< \aleph$, and suppose $\mathscr{M}$ has cardinality $\aleph$. Let Ω be the least ordinal with cardinality $\aleph$, and index $\mathscr{M}$ by Ω, say $\mathscr{M} = \{M_\alpha : \alpha < \Omega\}$. For every $\alpha < \Omega$ define

$$M^\alpha = \vee \{M_\beta : \beta < \alpha\},$$

which exists by the induction hypothesis. If one can show that $\vee M^\alpha$ exists, then it will serve as $\vee M$.) If $\mathscr{M}$ has a largest element, we are through; otherwise, we can suppose $\mathscr{M} = \{M^\alpha : \alpha < \Lambda\}$, where Λ is a limit ordinal of cardinality $\leqq \aleph$ and where $\alpha \leqq \beta$ implies $M^\alpha \leqq M^\beta$. Assuming (d), one shows, as in the proof of [**7**, Theorem 1], that the family (M^α) has a supremum in $L(2R_d)$ (for this, it is not necessary to know that the supremum hypothesized in (d) is an element of $\mathscr{G}$). (The idea of the proof is to use Lemma 1 to replace $\mathscr{M}$ by a well-ordered subset of $\mathscr{G}$.)

(a) $\Rightarrow$ (1), (2): Since $L(R_d)$ is, by hypothesis, upper $\aleph$-continuous, the upper $\aleph$-completeness of $L(2R_d)$ implies that $L(2R_d)$ is also upper $\aleph$-continuous, by [**1**, Theorem 4.3]; thus (a) implies (1), and then (2) follows from Lemma 4.

LEMMA 5. *Let $\aleph$ be an infinite cardinal, R a regular ring such that $L(R_d)$ is upper $\aleph$-complete.*

(i) *If $(J_i)_{i \in I}$ is any family in $L(R_d)$ with* card $I \leqq \aleph$, *then*

$$\vee J_i = (\cup J_i)^{lr} = (\textstyle\sum J_i)^{lr}.$$

(ii) *If $(K_i)_{i \in I}$ is any family in $L(R_s)$ (note the subscript) with* card $I \leqq \aleph$, *then $\cap K_i \in L(R_s)$.*

Proof. (ii) Since the principal left ideal lattice $L(R_s)$ is anti-isomorphic to $L(R_d)$, it is lower $\aleph$-complete; thus $\wedge K_i$ exists in $L(R_s)$. Then $\wedge K_i = \cap K_i$, cf. [**2**, p. 161, proof of Proposition 13.2].

(i) Write $J = \vee J_i$ and set $K_i = J_i^{\,l}$. Thus $K_i \in L(R_s)$, so by (ii) one has $\cap K_i = K$ for some $K \in L(R_s)$. Then $K = \cap J_i^{\,l} = (\cup J_i)^l$, so

$$(\cup J_i)^{lr} = K^r = (\wedge K_i)^r = \vee (K_i^{\,r}) = \vee (J_i^{\,lr}) = \vee J_i = J.$$

Definition 3. Let $\aleph$ be an infinite cardinal, R a regular ring, and write X for the set of right ideals J of R such that J is generated (as a right ideal) by a set of cardinality $\leqq \aleph$. One says that R is *right $\aleph$-injective*, cf.

[**2**, p. 105] if every R-linear mapping $\varphi : J \to R_d$, where $J \in X$, is extendible to an R-linear mapping $R_d \to R_d$ (equivalently, there exists $x \in R$ such that $\varphi(s) = xs$ for all $s \in J$). Left $\aleph$-injective rings are defined dually.

The following lemma is a reworking (and extension to arbitrary cardinals) of [**4**, proof of Theorem 3.2]:

LEMMA 6 [**4**, pp. 188–189]. *Let $\aleph$ be an infinite cardinal. If R is a right $\aleph$-continuous, right $\aleph$-injective regular ring, then the matrix ring $M_2(R)$ is right $\aleph$-continuous.*

Proof. Let us verify criterion (b) of Theorem 1. Let $\mathscr{S}$ be an increasingly filtering subset of $\mathscr{G}$ (the set of graphs of the right factor-correspondences in R) with card $\mathscr{S} \leqq \aleph$. Write $\mathscr{S} = \{G_i : i \in I\}$, card $I \leqq \aleph$. For each i, we know from Lemma 3 that G_i is the graph of some $(a_i : b_i)$, where $Ra_i = Rb_i$, thus $G_i = (a_i, b_i)R$. Let $G = \cup G_i$ be the set-theoretic union of the G_i; since $\mathscr{S}$ is increasingly filtering, G is the graph of a right R-isomorphism $\alpha : J \to K$, where $J = \cup a_iR$, $K = \cup b_iR$ (set-theoretic unions, both right ideals), and since $G_i \subset G$ we know that α extends $(a_i : b_i)$ for all i. Since $L(R_d)$ is upper $\aleph$-complete, there exist idempotents e, f in R such that $eR = \vee a_iR$ and $fR = \vee b_iR$ in $L(R_d)$.

The sum $J + (1 - e)R$ is direct; define $\beta : J + (1 - e)R \to R_d$ by

$$\beta|J = \alpha \text{ and } \beta|(1 - e)R = 0.$$

Since R is right $\aleph$-injective, there exists $y \in R$ such that left-multiplication by y coincides with β on $J + (1 - e)R$; thus $y(1 - e) = 0$ and

$$ya_i = \alpha(a_i) = (a_i : b_i)a_i = b_i \text{ for all } i \in I.$$

Briefly, $ye = y$ and $ya_i = b_i$ for all i. Similarly, there exists $x \in R$ such that $xf = x$ and $xb_i = a_i$ for all i. Then $yxya_i = yxb_i = ya_i$, $(yxy - y)a_i = 0$ for all i, therefore $(yxy - y)J = 0$; by Lemma 5, $(yxy - y)eR = 0$, and since $ye = y$ this means $yxy - y = 0$. Similarly $xyx - x = 0$. Therefore the mapping $\varphi : xR \to yR$ defined by $\varphi(xr) = yxr$ $(r \in R)$ is a right factor-correspondence with $\varphi^{-1}(yr) = xyr$. One has $xR = eR$ and $yR = fR$. (For example, $b_iR = ya_iR \subset yR$ for all i, hence $fR \subset yR$ because $fR = \vee b_iR$. On the other hand,

$$yJ = \beta(J) = \alpha(J) = K \subset fR,$$

so $(1 - f)yJ = 0$; by Lemma 5, $(1 - f)yeR = 0$, thus

$$(1 - f)ye = 0,\ (1 - f)y = 0,\ y = fy,\ yR \subset fR.$$

Thus $yR = fR$.) The domain $xR = eR$ of φ contains every a_iR, and $\varphi(a_i) = ya_i = b_i$ shows that $(a_i : b_i) \leqq \varphi$.

The graph of φ is $(x, yx)R \in L(2R_d)$; let us show that $(x, yx)R$ serves as sup G_i in $L(2R_d)$. On the one hand, $(a_i : b_i) \leqq \varphi$ shows that $G_i = (a_i, b_i)R \subset (x, yx)R$. On the other hand, suppose $G_i \subset M \in L(2R_d)$ for all i; it is to be shown that $(x, yx)R \subset M$, in other words that $(x, yx) \in M$. Define $\sigma : R_d \rightarrow 2R_d$ by

$$\sigma(r) = (x, yx)r, \; r \in R;$$

σ is right R-linear and

$$\sigma(b_i) = (xb_i, yxb_i) = (a_i, ya_i) = (a_i, b_i) \in G_i \subset M,$$

thus $b_iR \subset \sigma^{-1}(M)$ for all i. Since $\sigma^{-1}(M)$ is a principal right ideal of R [**2**, p. 14, Lemma 2.1], it follows that $fR \subset \sigma^{-1}(M)$, thus $(x, yx)f \in M$; since $xf = x$ this means $(x, yx) \in M$ and the proof is complete.

THEOREM 2. *If R is a right $\aleph$-continuous, right $\aleph$-injective regular ring, then every matrix ring $M_n(R)$ is right $\aleph$-continuous.*

Proof. The case for $n = 2$ (Lemma 6) implies the case of general n by [**3**, Theorem 3.1 and its Corollary 3].

Problem. In Theorem 2, is $M_n(R)$ also right $\aleph$-injective? The answer is yes for $\aleph = \aleph_0$:

COROLLARY 1 [**2**, p. 183, Proposition 14.19]. *If R is a right $\aleph_0$-continuous, right $\aleph_0$-injective regular ring, then so is every matrix ring $M_n(R)$.*

Proof. Let $S = M_n(R)$, which is right $\aleph_0$-continuous by Theorem 2; since $M_2(S) = M_{2n}(R)$ is also right $\aleph_0$-continuous, S is right $\aleph_0$-injective [**2**, p. 180, Corollary 14.14]. (The basic reason that things go well for $\aleph_0$ is that, over a regular ring, every countably generated submodule of a projective module is projective [**2**, p. 20, Corollary 2.15].)

COROLLARY 2 [**2**, Proposition 14.19]. *Let $\aleph$ be an infinite cardinal, R a right $\aleph$-continuous and right $\aleph$-injective regular ring, A a finitely generated projective right R-module, and $T = \text{End}_R(A)$ the endomorphism ring of A. Then T is right $\aleph$-continuous and right $\aleph_0$-injective.*

Proof. If A is generated by n elements, one has $nR_d = A \oplus B$ for a suitable right R-module B; then $T = \text{End}_R(A)$ is a corner of $M_n(R)$, that is, $T = eM_n(R)e$ for a suitable idempotent e. Since $M_n(R)$ is right $\aleph$-continuous (Theorem 2) so is its corner T [**2**, p. 163, proof of Proposition 13.7]. Also, $2nR_d = 2A \oplus 2B$, so $M_2(T) = \text{End}_R(2A)$ [**6**, p. 34, Corollary 8] is a corner of $M_{2n}(R)$; since $\aleph \geqq \aleph_0$, R is a fortiori right $\aleph_0$-continuous and right $\aleph_0$-injective, therefore so is $M_{2n}(R)$ (Corollary 1), hence its corner $M_2(T)$ is right $\aleph_0$-continuous [**2**, p. 175, Proposition 14.6]; therefore T is right $\aleph_0$-injective [**2**, p. 180, Corollary 14.14].

Added in proof. The question following Theorem 2 has been answered in the affirmative by K. R. Goodearl.

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