Can. J. Math., Vol. XXII, No. 3, 1970, pp. 641–644

ON A CONJECTURE CONCERNING SEMIGROUP HOMOMORPHISMS

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1. Introduction. In this paper we settle (with a counterexample) the question raised by Clifford and Preston in [**2**, p. 275], concerning maximal group homomorphic images of semigroups. We also consider the question in a more general context and characterize all such examples. The notation and definitions follow [**1**; **2**].

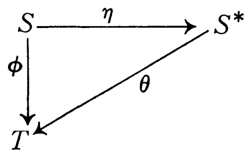
By a *type* of semigroups we mean a class $\mathscr{T}$ of semigroups, closed under isomorphisms and containing the one-element semigroup. If S is any semigroup and $\mathscr{T}$ is a type, then a semigroup S^* is defined, in [**1**, p. 18], to be a *maximal homomorphic image of S having type* $\mathscr{T}$ if

(i) $S^* \in \mathscr{T}$,

(ii) S^* is a homomorphic image of S, and

(iii) whenever $T \in \mathscr{T}$ and T is a homomorphic image of S, then there exists a homomorphism from S^* onto T.

In [**2**, p. 275], this definition was made somewhat more restrictive by requiring, in conjunction with (i), (ii), and (iii), that there exist a fixed homomorphism η of S onto S^* with the factorization property: if ϕ is a homomorphism of S onto $T \in \mathscr{T}$, then there exists a homomorphism θ of S^* onto T such that the diagram



commutes. Under these conditions we shall call S^* the *greatest homomorphic image of S having type* $\mathscr{T}$.

In general, a semigroup may have no maximal homomorphic image of type $\mathscr{T}$; it may have several non-isomorphic maximal homomorphic images of type $\mathscr{T}$; it may have a maximal but no greatest, and it may have both a maximal and a greatest homomorphic image of type $\mathscr{T}$, and the two not be isomorphic. Examples of semigroups and types satisfying each of these situations may be found in [**5**; **7**]. In [**7**], Tamura distinguished four kinds of maximal homomorphic images of given types. His greatest homomorphic image is our maximal and his greatest decomposition corresponds to our greatest homomorphic image.

Received April 7, 1969.

Now a greatest homomorphic image S^* of type $\mathscr{T}$ is unique to within isomorphisms; in fact, $S^* \cong S/\rho$, where ρ is the intersection of all congruences σ on S, such that S/σ has type $\mathscr{T}$ [**2**].

In this paper we shall be particularly interested in the case where $\mathscr{T}$ is the type "being a group".

2. Question and example. Let S be a semigroup with a completely simple kernel K, and let $e^2 = e \in K$. Let H_e denote the maximal subgroup eSe of K, containing e. Then if H_e is a homomorphic image of S, it is a maximal group homomorphic image [**3**]. The question raised in [**2**, p. 275] was as follows.

If H_e is a homomorphic image of S, is it always then the greatest group homomorphic image?

The answer to the question is no, as we shall show with an example. In particular, we shall exhibit a completely simple semigroup where H_e is a maximal, but not the greatest, group homomorphic image.

Now a completely simple semigroup is a rectangular band [**1**] of mutually isomorphic groups $H_e = eSe$, where $e^2 = e \in S$. Such a semigroup has the following structure: $S = \{(a; \alpha, \beta) | a \in G, \alpha \in A, \beta \in B\}$, where G is a group (called the structure group), A and B are index sets, and where, for some subset P of G (called the matrix for S), the binary operation is given by $(a; \alpha, \beta)(b; \gamma, \delta) = (ap_{\beta\gamma}b; \alpha, \delta)$ for $p_{\beta\gamma} \in P$. The matrix P can be normalized so that for $1 \in A \cap B$, $p_{\beta 1} = p_{1\alpha} = e$, the identity of G, for each $\beta \in B$ and $\alpha \in A$. Moreover, $eSe \cong G$ for each $e^2 = e \in S$.

In [**6**] it was shown that if ω is a homomorphism of G onto a group G^*, such that P is contained in the kernel of ω, then $(a; \alpha, \beta) \rightarrow a\omega$ defines a homomorphism of S onto G^*. Moreover, every homomorphism of S onto a group is obtained in this way. In particular then, if N is the normal subgroup of G generated by the matrix P, G/N is the greatest group homomorphic image of S.

We now exhibit a completely simple semigroup with structure group $G \cong H_e$, that is a homomorphic image of S (and is thus a maximal group homomorphic image), but where $G \ncong G/N$, the greatest group homomorphic image. Thus we seek the following type structure group G. It is to have normal subgroups $N \subset K \subset G$ such that $G/K \cong G$ while $G/N \ncong G$. Of course, such groups exist, for example, if we take G to be the direct product of infinitely many copies of C_4, the cyclic group of order 4, then C_4 is isomorphic to a normal subgroup K of G. Also, C_2, the cyclic group of order 2, is isomorphic to a normal subgroup $N \subset K$, and G/K and G are isomorphic while G/N and G are not isomorphic.

For our example then, we take S to be a completely simple semigroup with the structure group defined above. Let $a \in G$ generate the subgroup $N \cong C_2$. We let A and B be index sets of cardinality greater than one and define the matrix P for S as follows:

$$p_{\beta\alpha} = \begin{cases} a & \text{if } \alpha \neq 1 \text{ and } \beta \neq 1, \\ e & \text{otherwise,} \end{cases}$$

where $1 \in A \cap B$ and e is the identity of G. Then P generates N so that G/N is the greatest group homomorphic image of S. However, $G \cong G/K$, so that G is a homomorphic image of S. Thus G is a maximal, but not the greatest, group homomorphic image of S.

3. Concluding remarks. In a more general context, the question of Clifford and Preston might be rephrased as follows. *Under what conditions can a semigroup have two non-isomorphic, maximal, homomorphic images having type $\mathscr{T}$?* The answer is given in this section. We then determine when a completely simple semigroup can have a maximal group homomorphic image which is not isomorphic to the greatest group homomorphic image.

Let S be a semigroup and let $\mathscr{C}_S$ denote the set of all congruence relations on S. A family $I = \{F_\alpha | \alpha \in A\}$ of subsets of $\mathscr{C}_S$ will be called *independent* if for $\rho \in F_\alpha$ and $\sigma \in F_\beta$, $S/\rho \cong S/\sigma$ if $\alpha = \beta$, and $S/\rho \not\cong S/\sigma$ if $\alpha \neq \beta$.

THEOREM 1. *Let S be a semigroup and $\mathscr{T}$ a type. Then S has two non-isomorphic maximal homomorphic images of type $\mathscr{T}$, S/ρ_1, and S/σ_1, if and only if there exist two infinite, properly ascending chains of congruence relations on S:*

$$\rho_1 \subset \rho_2 \subset \rho_3 \subset \ldots \subset \rho_{2i} \subset \rho_{2i+1} \subset \ldots,$$

and

$$\sigma_1 \subset \sigma_2 \subset \sigma_3 \subset \ldots \subset \sigma_{2i} \subset \sigma_{2i+1} \subset \ldots,$$

such that

(i) *S/ρ_i and S/σ_i are maximal homomorphic images of type $\mathscr{T}$ for each i, and*

(ii) *$\{\rho_{2i+1}, \sigma_{2i}\}_{i=1}^{\infty}$ and $\{\sigma_{2i+1}, \rho_{2i}\}_{i=1}^{\infty}$ are independent.*

Proof. Suppose that S has two non-isomorphic maximal homomorphic images, S/ρ_1 and S/σ_1, having type $\mathscr{T}$. Then $\rho_1 \neq S \times S$ and $\rho_2 \neq S \times S$. Moreover, there exists a homomorphism α of S/ρ_1 onto S/σ_1 which determines a congruence relation ρ_2 on S, defined by $a\rho_2 b$ if and only if $(a\rho_1)\alpha = (b\rho_1)\alpha$. Since α is not one-to-one, $\rho_1 \subset \rho_2$, $\rho_1 \neq \rho_2$. Also $S/\rho_2 \cong S/\sigma_1$, and since there is a homomorphism from S/σ_1 onto S/ρ_1, there is a homomorphism from S/ρ_2 onto S/ρ_1. This homomorphism then determines a congruence $\rho_3 \supset \rho_2$, such that $S/\rho_3 \cong S/\rho_1$. Now there exists a homomorphism of S/ρ_3 onto S/ρ_2 which determines another congruence relation $\rho_4 \supset \rho_3$. By continuing in this way we obtain an infinite, properly ascending chain of congruences on S,

$$\rho_1 \subset \rho_2 \subset \rho_3 \subset \ldots \subset \rho_{2i} \subset \rho_{2i+1} \subset \ldots,$$

such that $S/\rho_{2i+1} \cong S/\rho_1$ and $S/\rho_{2i} \cong S/\sigma_1$ for each i. In particular, each S/ρ_i is a maximal homomorphic image of S having type $\mathscr{T}$.

Similarly, by starting with a homomorphism from S/σ_1 onto S/ρ_1, we can show the existence of a chain of congruence relations on S,

$$\sigma_1 \subset \sigma_2 \subset \sigma_3 \subset \ldots \subset \sigma_{2i} \subset \sigma_{2i+1} \subset \ldots,$$

such that $S/\sigma_{2i+1} \cong S/\sigma_1$ and $S/\sigma_{2i} \cong S/\rho_1$ for each i. Moreover, since

$S/\rho_1 \ncong S/\sigma_1$, $\{\rho_{2i+1}, \sigma_{2i}\}_{i=1}^{\infty}$ and $\{\sigma_{2i+1}, \rho_{2i}\}_{i=1}^{\infty}$ form independent sets of congruences on S. Thus (i) and (ii) hold.

The converse is immediate.

COROLLARY 2. *If a semigroup S has a collection $\{S_\alpha | \alpha \in A, |A| \geqq 2\}$ of maximal homomorphic images of type $\mathscr{T}$, no two of which are isomorphic, then*

(i) *for each $\alpha \in A$, there is an infinite properly ascending chain of congruence relations on S, and*

(ii) *there exists an independent family $\{F_\alpha | \alpha \in A\}$ of subsets of $\mathscr{C}_S$, where $|F_\alpha| = \infty$, for each $\alpha \in A$.*

Also, as an immediate consequence of the theorem, we have the following corollary which relates to the example given in § 2.

COROLLARY 3. *Let S be a completely simple semigroup with structure group G, and suppose that G/N is the greatest group homomorphic image of S. Then S has a maximal group homomorphic image which is not isomorphic to G/N if and only if there exists an infinite, properly ascending chain of normal subgroups of G,*

$$N = N_1 \subset N_2 \subset N_3 \subset \ldots \subset N_{2i} \subset N_{2i+1} \subset \ldots,$$

such that

(i) *$\{G/N_{2i+1}\}_{i=1}^{\infty}$ is a family of greatest group homomorphic images of S, and*

(ii) *$\{G/N_{2i}\}_{i=1}^{\infty}$ is a family of mutually isomorphic maximal, but not greatest, group homomorphic images of S.*

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