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SOME OPEN SETS FOR WHICH THE HEAT EQUATION IS SIMPLICIAL

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1. Introduction. Let us associate to each open set $U \subset \mathbf{R}^{n+1}$ the space H_U of real functions f which are twice continuously differentiable in $x_1 \ldots x_n$ and once continuously differentiable in x_{n+1} and which satisfy the heat equation: $\Delta f = \partial f/\partial x_{n+1}$ where $\Delta = \sum_{i=1}^{n} \partial^2/\partial x_i^2$. Then we have what in the axiomatic of Bauer is called a *strong harmonic space* [**2**, p. 61]. We will call functions of H_U *harmonic in* U.

If ω is a bounded open subset of $\mathbf{R}^{n+1}$ we will denote by $A(\omega)$ the space of continuous functions on $\bar{\omega}$ which are harmonic in ω. We ask whether $A(\omega)$ is simplicial (i.e. a simplex space). That is to say, is the state space of $A(\omega)$ a simplex? Our intuition tells us that the answer is always yes, but we have been unable to answer the question in general. In this paper we prove that the answer is yes if irregular points of $\partial\omega$ can only occur at finitely many values of the time coordinate x_{n+1}.

In § 2 we prove that $A(\omega)$ is simplicial if the irregular points are all at the top of ω and all have half neighbourhoods below them in ω. In this case the set of regular boundary points is closed and $A(\omega)$ is just the space of continuous functions on this compact set. In particular the regular points and the Choquet boundary coincide.

In § 3 we remove the "half neighbourhood" restriction. We do this by approximating ω by open sets of the type in § 2 using the idea of inverse limit of compact convex sets in Jellett [**5**]. The set of regular points is no longer closed but is still equal to the Choquet boundary.

In § 4 we allow ω to have irregular points at a finite number of values of the time coordinate. We do this by cutting ω up into horizontal slices, each of which is of the type in § 3. We use some results of split faces and quotients of compact convex sets to put the pieces back together as a simplex.

Now a word about notation and terminology. If $x \in \mathbf{R}^{n+1}$ we will let $\lambda(x) = (x_1, x_2, \ldots, x_n)$ and $t(x) = x_{n+1}$. We call the last coordinate axis the time axis and think of it as vertical. Harmonic space notation follows Bauer [**2**]. ω will always denote a bounded open subset of $\mathbf{R}^{n+1}$. We denote by $\partial\omega$ the topological boundary of ω and by $\partial_r\omega$ the subset of regular boundary points [**2**, p. 128]. If $x \in \bar{\omega}$ we will write μ_x^{ω} (or simply μ_x if there is no ambiguity) for *harmonic measure* on $\partial\omega$. This is defined in [**2**, § 4.1] to be the balayaged measure $\epsilon_x^{c\omega}$ [**2**, § 3.5]. ($c\omega$ denotes the complement of ω.) If ω is a regular set then

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this definition gives us the usual harmonic measure [**2**, p. 12]. The map $x \to \mu_x^\omega$ is called balayage and is a dilation for $A(\omega)$ [**3**, Theorem 3.2].

Finally if K is a compact convex set we let $A(K)$ denote the Banach space of continuous affine (real) functions on K. We always assume that K is embedded in a locally convex topological vector space, so that $A(K)$ separates points of K. If X is a compact Hausdorff space and A is subspace of $C(X)$ which contains the constants then the state space of A is the set

$$K = \{k \in A^*:k \geqq 0, k(1) = 1\},$$

endowed with the weak*-topology. Then K is a compact convex subset of the unit ball of A^* and the closure of A is isometrically isomorphic to $A(K)$. If $x \in X$, let $\pi(x)$ denote the evaluation functional, $f \to f(x)$, on A. Then π is a continuous map from X into K. The *Choquet boundary* of A is the set of $x \in X$ such that $\pi(x)$ is in $E(K)$, the set of extreme points of K. Phelps' book [**6**, § 6] gives a discussion of these ideas.

I am indebted to Professor Effros for bringing the general problem to my attention and for several valuable discussions about the heat equation. His paper with Kazdan [**3**] provides a very readable introduction to the problem.

2. The simplest case. As always ω denotes a bounded open subset of $\mathbf{R}^{n+1}$. We let

$$T = T(\omega) = \sup \{t(x):x \in \bar{\omega}\}.$$

Suppose that $x \in \partial\omega$ and $t(x) = T$. Suppose there is a neighbourhood N of x such that

$$(2.1) \quad \{y \in N:t(y) < t(x)\} \subset \omega.$$

Then x is the midpoint of the top of some box contained in ω and it follows from [**3**, § 7, Observation 2.] that x is an irregular point of $\partial\omega$. The following Proposition assumes that all irregular points are of this type.

PROPOSITION 1. *Suppose that whenever x is an irregular point of $\partial\omega$ then $t(x) = T(\omega) = T$ and there is a neighbourhood N of x satisfying* (2.1). *Then $\partial_r\omega$ is compact and the restriction map $A(\omega) \to C(\partial_r\omega)$ is an isometric isomorphism onto $C(\partial_r\omega)$.*

Proof. First of all $\partial_r\omega$ is compact. Indeed if $x \in \partial\omega$ is irregular, then the interior of the neighbourhood N of (2.1) cannot contain any regular point of $\partial\omega$ (by the remark immediately preceeding the proposition). So $\partial_r\omega$ is a closed subset of the compact set $\partial\omega$.

Now we show that if $x \in \bar{\omega}$ then μ_x lives on $\partial_r\omega$. First we note that μ_x always lives on $\{y:t(y) \leqq t(x)\}$. Indeed if we define $s(y)$ to be 0 if $t(y) \leqq t(x)$ and 1 if $t(y) > t(x)$ then s is superharmonic and

$$0 = s(x) \geqq \hat{R}_s^{c\omega}(x) = \mu_x(s),$$

[**2**, Kor. 3.4.2], so that μ_x lives on the set $\{s = 0\}$. Thus if $t(x) < T$, then μ_x lives on $\partial\omega \cap \{y:t(y) < T\}$ which is contained in $\partial_r\omega$ by assumption. If $t(x) = T$ the result follows from [**3**, Lemma 3.1].

If $f \in A(\omega)$ then $f(x) = \mu_x(f)$ for any $x \in \bar{\omega}$ by [**3**, Theorem 3.2] and with what we have just shown, we deduce that the restriction map $A(\omega) \to C(\partial_r\omega)$ is an isometry. It remains to show that it is onto.

If $g \in C(\partial_r\omega)$ we define f on $\bar{\omega}$ by $f(x) = \mu_x(g)$. Then g is the restriction of f to $\partial_r\omega$ by [**2**, Satz 4.3.1]. We will show that $f \in A(\omega)$. By [**2**, Satz 4.1.1] f is harmonic in ω. We must show that f is continuous at points of $\partial\omega$. The continuity of f at irregular points of $\partial\omega$ follows from Lemma 2 below. Suppose x is a regular point of $\partial\omega$ and take any $\epsilon > 0$. Choose $\delta > 0$ so that if $z \in \omega$ and $|z - x| < \delta$ then $|f(z) - f(x)| < \epsilon$ (use the regularity of x). We claim that if $y \in \partial\omega$, $|y - x| < \delta$ then $|f(y) - f(x)| \leqq \epsilon$. Indeed there are points of ω arbitrarily close to y. If y is regular our claim follows from the definition of regularity; if y is irregular it follows from the continuity of f at irregular points just proven. Thus f is continuous at x.

LEMMA 2. *Suppose $x \in \partial\omega$ and for some open neighbourhood N of x*

$$\{y \in N : t(y) < t(x)\} \subset \omega.$$

Then for any continuous function g on $\partial\omega$,

$$\mu_x(g) = \lim_{y \to x} \{\mu_y(g) : y \in \bar{\omega}, t(y) \leqslant t(x)\}.$$

Proof. By [**2**, Satz 2.7.4] it is enough to show this when g is the restriction of a finite continuous potential p. In this case our requirement becomes

$$\hat{R}_p{}^E(x) = \lim_{y \to x} \{R_p{}^E(y) : y \in \bar{\omega}, t(y) \leqslant t(x)\},$$

where $E = \mathbf{R}^{n+1} - \omega$.

Let $\omega' = \omega \cup N$ and $E' = \mathbf{R}^{n+1} - \omega'$. We will show that $\hat{R}_p{}^E(y) = \hat{R}_p{}^{E'}(y)$ for every $y \in N$ with $t(y) \leqq t(x)$. Since $E' \subset E$ we have at least that $\hat{R}_p{}^{E'} \leqq \hat{R}_p{}^E$. We will first show that $R_p{}^E(y) \leqq R_p{}^{E'}(y)$ if $t(y) < t(x)$. Indeed if $t(y) < t(x)$ and u is hyperharmonic with $u \geqq p$ on E', define u' as follows:

$$u'(z) = \begin{cases} u(z), & \text{if } t(z) \leqq t(y) \\ \infty, & \text{if } t(z) > t(y). \end{cases}$$

Then u' is hyperharmonic and $u' \geqq p$ on E. Since $u'(y) = u(y)$ we deduce $R_p{}^E(y) \leqq R_p{}^{E'}(y)$. Now suppose $y \in N$ and $t(y) \leqq t(x)$. Then

$$\begin{aligned} R_p{}^{E'}(y) &= \liminf_{z \to y} \{R_p{}^{E'}(z) : z \in N, t(z) < t(x)\} \\ &\qquad \text{(since } R_p{}^{E'} \text{ is continuous in } N) \\ &\geqq \liminf_{z \to y} \{R_p{}^E(z) : z \in N, t(z) < t(x)\} \\ &\qquad \text{(by what we have just proven)} \\ &\geqq \hat{R}_p{}^E(y). \end{aligned}$$

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Thus $\hat{R}_p{}^E$ coincides with $\hat{R}_p{}^{E'}$ at points $y \in N$ with $t(y) \leqq t(x)$. Since $x \in \omega'$, $\hat{R}_p{}^{E'}$ is continuous at x and our result for $\hat{R}_p{}^E$ is immediate.

COROLLARY 3. *If ω satisfies the hypotheses of Proposition* 1, *then $A(\omega)$ is simplicial and the Choquet boundary of $A(\omega)$ is the set of regular boundary points of ω.*

3. An intermediate case. Our objective in this section is to strengthen Corollary 3 by removing the hypothesis (2.1) in Proposition 1. This is to say, we will prove that $A(\omega)$ is simplicial if the irregular points of $\partial\omega$ all live on a horizontal line above ω. Our method is to approximate an open set ω with this property by sets which also contain half neighbourhoods of every irregular point (condition 2.1). The approximation argument uses the notion of inverse limit found in [**5**] and [**13**].

Suppose that Λ is a directed set. Suppose that for each $\alpha \in \Lambda$ we are given a compact convex set K_α and for $\alpha, \beta \in \Lambda$, $\alpha \leqq \beta$, we are given a continuous affine map $\pi_{\alpha\beta}: K_\beta \to K_\alpha$, so that the following conditions are satisfied:

(i) for each α, $\pi_{\alpha\alpha}$ is the identity map;

(ii) if $\alpha < \beta < \gamma$ then $\pi_{\alpha\gamma} = \pi_{\alpha\beta} \circ \pi_{\beta\gamma}$.

Then we say that $(K_\alpha, \pi_{\alpha\beta})_{\alpha,\beta\in\Lambda}$ is an *inverse family* of compact convex sets. The *inverse limit* K_∞ of the family is defined as

$$K_\infty = \{\{k_\alpha\} \in \textstyle\prod_\Lambda K_\alpha : \text{if } \alpha < \beta \text{ then } \pi_{\alpha\beta}(k_\beta) = k_\alpha\}.$$

K_∞ is a closed subset of $\prod K_\alpha$ (product topology) and so is a compact convex set (coordinatewise operations) and the coordinate projection $\pi_{\alpha\infty}: K_\infty \to K_\alpha$ is continuous and affine.

Let us denote by $T_{\alpha\beta}: A(K_\alpha) \to A(K_\beta)$ for $\alpha \leqq \beta$, and $T_{\alpha\infty}: A(K_\alpha) \to A(K_\infty)$ the operators induced by the maps $\pi_{\alpha\beta}$ and $\pi_{\alpha\infty}$. These are positive linear operators of norm 1 mapping 1 into 1. From (ii) we have the property $T_{\alpha\gamma} = T_{\beta\gamma} \circ T_{\alpha\beta}$ if $\alpha < \beta < \gamma$.

We remark that K_∞ has the expected universal property with regard to $\{K_\alpha\}$. Suppose K is a compact convex set and for every $\alpha \in \Lambda$ there is a continuous affine map $\pi_\alpha: K \to K_\alpha$, such that if $\alpha < \beta$ then $\pi_{\alpha\beta} \circ \pi_\beta = \pi_\alpha$. Then there is a continuous affine map $\pi: K \to K_\infty$ such that for every $\alpha \in \Lambda$ $\pi_\alpha = \pi_{\alpha\infty} \circ \pi$. Indeed we simply let $\pi(k) = \{\pi_\alpha(k)\}$ and do the necessary verifications.

The following proposition is found in [**13**, Proposition 7.1]. It also occurs in [**5**, Theorem 2] with a different proof, but with the additional assumption that the maps $\pi_{\alpha\beta}$ are onto. Our proof follows [**5**], except that we need a lemma to make up for the fact that the operators $T_{\alpha\infty}$ need not be injective.

PROPOSITION 4. *If K_∞ is the inverse limit of a family $\{K_\alpha\}$ of simplexes then K_∞ is a simplex.*

Proof. Following [**5**] we will use the fact that K is a simplex if and only if $A(K)$ has the weak Riesz separation property (R.S.P.) i.e. if $a_1 \ldots a_4 \in A(K)$

and $a_1 \wedge a_2 > a_3 \vee a_4$ then there is $a_5 \in A(K)$ such that $a_1 \wedge a_2 > a_5 > a_3 \vee a_4$. It is enough to check this for a dense subspace of $A(K_\infty)$ containing the constants and $L = \bigcup_\alpha T_{\alpha\infty}(A(K_\alpha))$ is such a subspace since it separates points of K_∞ [**5**, Lemma].

Let $T_{\alpha_i\infty}(f_i)(i = 1, \ldots 4)$ be four elements of L such that $T_{\alpha_1\infty}(f_i) \wedge T_{\alpha_2\infty}(f_2) > T_{\alpha_3\infty}(f_3) \vee T_{\alpha_4\infty}(f_4)$. Choose $\beta > \alpha_i$ for $i = 1, \ldots 4$ and set $a_i = T_{\alpha_i\beta}(f_i)$. Then our inequality reads

$$T_{\beta\infty}(a_1) \wedge T_{\beta\infty}(a_2) > T_{\beta\infty}(a_3) \vee T_{\beta\infty}(a_4).$$

LEMMA. *Suppose $a \in A(K_\beta)$ and $T_{\beta\infty}(a) > 0$. Then there exists $\alpha_0 > \beta$ such that $T_{\beta\alpha}(a) > 0$ for all $\alpha > \alpha_0$.*

Proof. Suppose not. Then $\Lambda' = \{\alpha \in \Lambda : \alpha \geqq \beta$ and $T_{\beta\alpha}(a)$ is not strictly positive$\}$ is cofinal in Λ (and hence is directed). For each $\alpha \in \Lambda'$ choose $k_\alpha \in K_\alpha$ such that $T_{\beta\alpha}(a)(k_\alpha) \leqq 0$. Choose a directed set Γ and a map $\tau : \Gamma \to \Lambda'$ so that $\{k_{\tau(\gamma)} : \gamma \in \Gamma\}$ is a universal subnet of $\{k_\alpha : \alpha \in \Lambda'\}$ [**11**, Chapter 2, Exercise J]. Then for every $\alpha \in \Lambda$ the net

$$\{\pi_{\alpha\tau(\gamma)}(k_{\tau(\gamma)}) : \gamma \in \Gamma, \tau(\gamma) \geqq \alpha\}$$

converges in K_α (since τ is universal and K_α is compact). Let $h_\alpha \in K_\alpha$ be the limit of this net. We assert first of all that $\{h_\alpha\} \in K_\infty$. Indeed if $\lambda < \alpha$ then

$$\begin{aligned} \pi_{\lambda\alpha}(h_\alpha) &= \pi_{\lambda\alpha}\Big(\lim_{\gamma\in\Gamma} \pi_{\alpha\tau(\gamma)}(k_{\tau(\gamma)})\Big) \\ &= \lim_{\gamma} \pi_{\lambda\alpha} \circ \pi_{\alpha\tau(\gamma)}(k_{\tau(\gamma)}) \quad \text{(continuity of } \pi_{\lambda\alpha}) \\ &= \lim_{\gamma} \pi_{\lambda\tau(\gamma)}(k_{\tau(\gamma)}) = h_\lambda. \end{aligned}$$

Next we assert that $T_{\beta\infty}(a)(\{h_\alpha\}) \leqq 0$, a contradiction. Indeed,

$$\begin{aligned} T_{\beta\infty}(a)(\{h_\alpha\}) = a(\pi_{\beta\infty}(\{h_\alpha\})) = a(h_\beta) &= \lim_{\gamma} a(\pi_{\beta\tau(\gamma)}(k_{\tau(\gamma)})) \\ &= \lim_{\gamma} T_{\beta\tau(\gamma)}(a)(k_{\tau(\gamma)}) \leqslant 0. \end{aligned}$$

Now we return to the proposition. From our inequality $T_{\beta\infty}$ is strictly positive on the four functions $a_1 - a_3$, $a_1 - a_4$, $a_2 - a_3$, $a_2 - a_4$. By the lemma there is a large enough $\alpha > \beta$ such that $T_{\beta\alpha}$ is strictly positive on these four functions:

$$T_{\beta\alpha}(a_1) \wedge T_{\beta\alpha}(a_2) > T_{\beta\alpha}(a_3) \vee T_{\beta\alpha}(a_4).$$

By the weak R.S.P. for $A(K_\alpha)$ there is $a_5 \in A(K_\alpha)$ such that

$$T_{\beta\alpha}(a_1) \wedge T_{\beta\alpha}(a_2) > a_5 > T_{\beta\alpha}(a_3) \vee T_{\beta\alpha}(a_4).$$

Apply $T_{\alpha\infty}$ and use the fact that $T_{\alpha\infty} \circ T_{\beta\alpha} = T_{\beta\infty}$ to get

$$T_{\beta\infty}(a_1) \wedge T_{\beta\infty}(a_2) > T_{\alpha\infty}(a_5) > T_{\beta\infty}(a_3) \vee T_{\beta\infty}(a_4).$$

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Now we return to ω and prove a few lemmas.

LEMMA 5. *Suppose* $x_0 \in \omega$ *and* $\omega \subset \{x : t(x) \geqq t(x_0) - 2/m^3\}$ *for some integer* m. *Then all but at most* $(m^n - (m-1)^n)/(m^n - 1)$ *of the mass of the measure* μ_{x_0} *is concentrated in the cylinder*

$$\{x : |\lambda(x - x_0)|^2 \leqq (4n \log m)/m^2, \ t(x) \leqq t(x_0)\}.$$

Proof. Choose coordinates so $x_0 = (0, 1/(m-1)^2)$. Then

$$\omega \subset \{x : t(x) \geqq 1/m^2\}.$$

Indeed if $x \in \omega$ then

$$\begin{aligned} t(x) &\geqq \frac{1}{(m-1)^2} - \frac{2}{m^3} \geqq \frac{1}{(m-1)^2} - \frac{2(m-1)}{(m-1)^2 m^2} \\ &\geqq \frac{1}{(m-1)^2} - \frac{(2m-1)}{(m-1)^2 m^2} \\ &= \frac{1}{(m-1)^2} - \frac{1}{(m-1)^2} + \frac{1}{m^2} = \frac{1}{m^2}. \end{aligned}$$

Define $f(\lambda, t) = t^{-n/2} \exp(-|\lambda|^2/4t)$ for $t > 0$. Then $f|\bar{\omega} \in A(\omega)$. If $(\lambda, t) \in \omega$ but does not lie in the above cylinder, then $|\lambda|^2 > 4n \log m/m^2$ and

$$\begin{aligned} f(\lambda, t) &\leqq t^{-n/2} \exp(-n \log m/m^2 t) \\ &\leqq (1/m^2)^{-n/2} \exp(-n \log m/m^2(1/m^2)) = 1 \\ &\quad (\text{since } t \geqq 1/m^2 \text{ and } f \text{ is decreasing in } t). \end{aligned}$$

Also if $(\lambda, t) \in \omega$ then $f(\lambda, t) \leqq f(0, 1/m^2) = m^n$. Suppose the μ_{x_0}-measure of the cylinder is $1 - \epsilon$. Then

$$\mu_{x_0}(f) \leqq \epsilon + (1 - \epsilon)m^n.$$

Since $f \in A(\omega)$, $\mu_{x_0}(f) = f(x_0)$ [**3**, Theorem 3.2], and so

$$\epsilon + (1 - \epsilon)m^n \geqq f(x_0) = f(0, 1/(m-1)^2) = (m-1)^n.$$

Solve for ϵ to get $\epsilon \leqq (m^n - (m-1)^n)/(m^n - 1)$.

LEMMA 6. *Suppose* f *is continuous on the open set* ω. *For some fixed* T *let* $\omega_1 = \{x \in \omega : t(x) < T\}$ *and* $\omega_2 = \{x \in \omega : t(x) > T\}$. *Suppose* f *is harmonic in* ω_1 *and* ω_2. *Then* f *is harmonic in* ω.

Proof. By [**2**, Satz 1.1.3] it is enough to take an arbitrary regular open set u such that $\bar{u} \subset \omega$ and to show that for each $x \in u$, $\mu_x^u(f) = f(x)$. If u is such a set let $u_i = u \cap \omega_i$ for $i = 1, 2$. Since u is regular and $f|\partial u$ is continuous there is a unique function g which is harmonic in u, continuous on $\bar{u}$ and coincides with f on ∂u.

Let ξ denote the space of functions continuous on $\bar{u}$ and harmonic in u_1 and u_2. Then ξ separates points of $\bar{u}$ and $f - g \in \xi$ so that we can apply the minimum principle [**2**, p. 7] to deduce that $f - g$ attains its minimum on $\bar{u}$ at a point $y_0 \in \bar{u}$ for which ϵ_{y_0} is the only ξ-representing measure. We will show that $y_0 \in \partial u$. If not then $y_0 \in u$ and since clearly y_0 is not in u_1 or u_2 we must have $t(y_0) = T$ and $y_0 \in \partial u_1$. Since $y_0 \in u$, y_0 is the midpoint of the top of a box contained in u_1 and by [**3**, § 7, Observation 2.] is an irregular point of ∂u_1. Hence $\epsilon_{y_0}{}^{cu_1}$ is an ξ-representing measure for y_0 different from ϵ_{y_0} [**2**, Satz 4.3.1], a contradiction.

Thus $f - g$ attains its minimum on $\bar{u}$ in ∂u and since $f = g$ on ∂u we deduce $f \geqq g$. The same argument applied to $g - f$ tells us $g \geqq f$. Hence for $x \in u$

$$f(x) = g(x) = \mu_x^u(g) = \mu_x^u(f),$$

the middle equality a consequence of the definition of g.

The following lemma is a basic tool which we shall use a number of times. Let us fix some notation. If ω is an open set in $\mathbf{R}^{n+1}$, fix an open ball $S(\omega)$ in $\mathbf{R}^n$ which contains all spatial coordinates of points in ω, i.e., $\lambda(\omega) \subset S(\omega)$.

Now let $T = T(\omega)$ and if $t_0 < T$ let

$$\omega_{t_0} = \omega \cup [S(\omega) \times (t_0, T)].$$

Since $S(\omega)$ is open, ω_{t_0} is open.

LEMMA 7. Let $T = T(\omega)$ *and suppose* f *is a function continuous on* $\bar{\omega} \cap \{t < T\}$ *and harmonic in* ω. *Suppose* X *is a compact set containing* $\bar{\omega}$, *and* h *and* k *are continuous functions on* X. *Suppose* $h \leqq f \leqq k$ *in* $\bar{\omega} \cap \{t < T\}$. *If* $\epsilon > 0$ *then there is* $t' < T$ *such that if* $t' \leqq t_0 < T$ *then we can find* $a \in A(\omega_{t_0})$ *with* $a = f$ *at points of* $\bar{\omega} \cap \{t \leqq t_0\}$, *and* $h - \epsilon < a < k + \epsilon$ *at points of* $\bar{\omega}_{t_0} \cap X$.

Proof. We may suppose $||k|| \leqq 1$ and $||h|| \leqq 1$, and $\epsilon > 1$. Let $\epsilon_1 = \epsilon/3$. Since k and h are uniformly continuous on X we may choose $\delta > 0$ so that if $x, y \in X$ then

$$|x - y| \leqq \delta => |h(x) - h(y)| < \epsilon_1 \text{ and } |k(x) - k(y)| < \epsilon_1.$$

Choose an integer m with the properties:

$$\frac{m^n - (m-1)^n}{m^n - 1} \leqq \epsilon_1 \quad \text{and} \quad \left[\frac{4n \log m}{m^2}\right]^{\frac{1}{2}} + \frac{2}{m^3} \leqq \delta.$$

Let $T > t_0 \geqq T - 2/m^3$. Let u be the open set $S(\omega) \times (t_0, T)$. We now define two functions on $\bar{u}$. If $y \in \bar{u}$ let

$$N(y) = \{x \in X : |x - y| \leqq \delta\}.$$

For $y \in \bar{u}$ define

$$k_1(y) = \begin{cases} \inf \{k(x) + \epsilon_1 : x \in N(y)\}, & \text{if } N(y) \neq \emptyset \\ 1 + \epsilon_1, & \text{if } N(y) = \emptyset \end{cases}$$

$$h_1(y) = \begin{cases} \sup \{h(x) - \epsilon_1 : x \in N(y)\}, & \text{if } N(y) \neq \emptyset \\ -1 - \epsilon_1, & \text{if } N(y) = \emptyset. \end{cases}$$

Then k_1 and $-h_1$ are lower semicontinuous on $\bar{u}$ and if $y \in \bar{u} \cap X$ then

$$k_1(y) \geqq k(y) \geqq h(y) \geqq h_1(y).$$

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Indeed for such a y, $N(y) \neq \emptyset$, so that there are points x and z in $N(y)$ such that $k_1(y) = k(x) + \epsilon_1 \geqq k(y)$ and $h_1(y) = h(z) - \epsilon_1 \leqq h(y)$, the inequalities following from the choice of δ. In particular we have that $k_1 \geqq f \geqq h_1$ at points of $\bar{u} \cap \bar{\omega} \cap \{t < T\}$. We now define functions k_2 and h_2 on $\bar{u}$:

$$k_2(x) = h_2(x) = f(x), \quad \text{if } x \in \bar{\omega} \text{ and } t(x) = t_0;$$

$$k_2(x) = k_1(x) \text{ and } h_2(x) = h_1(x) \quad \text{otherwise.}$$

Since f is continuous and $\bar{\omega} \cap \{t = t_0\}$ is closed, k_2 is lower semicontinuous and h_2 is upper semicontinuous. Also $k_1 \geqq k_2 \geqq h_2 \geqq h_1$. It follows from a theorem of Hahn and Tong [**12**, § 6.4.4] that there is a continuous function g on $\bar{u}$ such that $k_2 \geqq g \geqq h_2$. Clearly $g = f$ on $\bar{\omega} \cap \{t = t_0\}$ and $||g|| \leqq 1 + \epsilon_1$.

Now define a on $\bar{\omega}_{t0}$ as follows:

$$a(x) = \begin{cases} f(x), & \text{if } t(x) < t_0. \\ \mu_x^u(g), & \text{if } t(x) \geqq t_0 \end{cases}$$

We assert that a is continuous on $\bar{\omega}_{t0}$ and harmonic in ω_{t0}. Indeed, f is harmonic in $\{x \in \omega_{t0}: t(x) < t_0\}$ and continuous on the closure of this set. By Proposition 1, $\mu_x^u(g)$ is harmonic in u and continuous on $\bar{u}$. Since $f(x) = g(x) = \mu_x^u(g)$ whenever $x \in \bar{\omega}$ and $t(x) = t_0$ (such an x is a regular point of ∂u), we deduce that a is continuous. It follows from Lemma 6 that a is harmonic in ω_{t0}.

Now we show that in $\bar{\omega}_{t0} \cap X$, a is between $h - \epsilon$ and $k + \epsilon$. This is true by hypothesis if $t(x) < t_0$. Suppose $x \in \bar{\omega}_{t0} \cap X$ and $t(x) \geqq t_0$. Let

$$C_x = \{y \in \bar{u}: t(y) \leqq t(x) \text{ and } |\lambda(y - x)|^2 \leqq 4n \log m/m^2\}.$$

It follows from Lemma 5 that $\mu_x(C_x) \geqq 1 - (m^n - (m - 1)^n)/(m^n - 1)$. If $y \in C_x$ then

$$|x - y| \leqq |\lambda(x - y)| + |t(x - y)| \leqq \left[\frac{4n \log m}{m^2}\right]^{\frac{1}{2}} + \frac{2}{m^3} \leqq \delta$$

and therefore $x \in N(y)$ and

$$g(y) \leqq k_2(y) \leqq k_1(y) \leqq k(x) + \epsilon_1,$$

and

$$g(y) \geqq h_2(y) \geqq h_1(y) \geqq h(x) - \epsilon_1.$$

Thus

$$\begin{aligned} a(x) = \mu_x^u(g) &= \int_{C_x} g d\mu_x^u + \int_{C_x^c} g d\mu_x^u \\ &\leqq (k(x) + \epsilon_1)\mu_x^u(C_x) + ||g||\mu_x^u(C_x^c) \\ &\leqq k(x) + \epsilon_1 + (1 + \epsilon_1)(m^n - (m - 1)^n)/(m^n - 1) \\ &\leqq k(x) + \epsilon_1 + (1 + \epsilon_1)\epsilon_1 \\ &< k(x) + 3\epsilon_1 = k(x) + \epsilon. \end{aligned}$$

(C_x^c denotes the complement of C_x.) By a similar calculation $a(x) > h(x) - \epsilon$.

In some sense, as t approaches $T = T(\omega)$, the sets ω_t approximate the set ω. The notion of inverse limit makes this precise.

If $t \leqq s < T$ let us denote by τ_{ts} the restriction map

$$\tau_{ts}:A(\omega_t) \to A(\omega_s).$$

This is a positive linear operator which is norm decreasing and maps 1 into 1. Let K_t denote the state space of $A(\omega_t)$:

$$K_t = \{k \in A(\omega_t)^*:k \geqq 0, k(1) = 1\}.$$

K_t is a weak*-compact convex subset of $A(\omega_t)^*$ and the adjoint map τ_{ts}^* maps K_s into K_t. Denote by π_{ts} the map from K_s to K_t obtained by restricting τ_{ts}^*. Then π_{ts} is continuous and affine, π_{tt} is the identity and if $t \leqq s \leqq r$ then $\pi_{ts} \circ \pi_{sr} = \pi_{tr}$ (since $\tau_{sr} \circ \tau_{ts} = \tau_{tr}$). Thus $(K_t, \pi_{ts})_{t \leqq s < T}$ is an inverse family of compact convex sets. Let K_T denote the inverse limit of this family.

Now let A denote the space of functions continuous on $\bar{\omega} \cup [\overline{S(\omega)} \times T]$ and harmonic on ω. Let K be the state space of A. We have the restriction maps

$$\tau_t:A(\omega_t) \to A \quad (\text{for } t < T),$$

and these induce continuous affine maps

$$\pi_t:K \to K_t.$$

If $t \leqq s$ we have $\tau_s \circ \tau_{ts} = \tau_t$ and hence $\pi_{ts} \circ \pi_s = \pi_t$. By the universal property of K_T there is a continuous affine map $\pi:K \to K_T$ such that $\pi_{tT} \circ \pi = \pi_t$.

PROPOSITION 8. *The map $\pi:K \to K_T$ is a bijection.*

Proof. (i) π is surjective. Suppose $k = \{k_t\}$ is in K_T. For each $t < T$ choose a probability measure μ_t on $\bar{\omega}_t$ such that for each $f \in A(\omega_t)$, $k_t(f) = \mu_t(f)$. Fix $s < T$ and choose a subnet $\{\mu_{\sigma(\gamma)}:\gamma \in \Gamma)$ of $\{\mu_t:s \leqq t < T\}$ which converges to a probability measure μ on $\bar{\omega}_s$. Since μ_t lives on $\bar{\omega}_r$ for every $r \leqq t$, μ lives on

$$\bigcap_{\gamma \in \Gamma} \bar{\omega}_{\sigma(\gamma)} = \bar{\omega} \cup [\overline{S(\omega)} \times \{T\}].$$

Denote by h the member of K given by the functional $f \to \mu(f)$ for $f \in A$. We will show that $\pi(h) = k$.

It will be enough to show that $\pi_t(h) = k_t$ for $t < T$ (since $\pi_{tT} \circ \pi = \pi_t$). For $t < T$ take any $f \in A(\omega_t)$. If $r \geqq t$ then we can integrate f with respect to μ_r.

$$\begin{aligned}\mu_r(f) &= \mu_r(\tau_{tr}(f)) = \tau_{tr}(f)(k_r)\\ &= f(\pi_{tr}(k_r)) = f(k_t).\end{aligned}$$

On the other hand for $\gamma \in \Gamma$ the numbers $\mu_{\sigma(\gamma)}(f)$ converge to

$$\mu(f) = \mu(\tau_t(f)) = \tau_t(f)(h) = f(\pi_t(h)).$$

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We deduce that $f(\pi_t(h)) = f(k_t)$ and since f is arbitrary we have that $\pi_t(h) = k_t$.

(ii) π is injective. Suppose $x, y \in K$ and $x \neq y$. Choose $g \in A$ and $\epsilon > 0$ so that $||g|| = 1$ and $|x(g) - y(g)| > 2\epsilon$. It follows from Lemma 7 (with f, h and k all equal to g and X equal to $\bar{\omega} \cup \overline{S(\omega)} \times \{T\}$) that there is $a \in A(\omega_t)$ for some $t < T$ such that in $\bar{\omega} \cap \overline{S(\omega)} \times \{T\}$ we have $g - \epsilon \leqq a \leqq g + \epsilon$. It follows that $||\tau_t(a) - g|| \leqq \epsilon$ and hence that

$$\begin{aligned}|\pi_t(x)(a) - \pi_t(y)(a)| &= |x(\tau_t(a)) - y(\tau_t(a))| \\ &\geqq |x(g) - y(g)| - |x(\tau_t(a) - g)| - |y(\tau_t(a) - g)| \\ &> 2\epsilon - \epsilon - \epsilon = 0 \quad \text{since } ||x|| = ||y|| = 1.\end{aligned}$$

Thus $\pi_t(x) \neq \pi_t(y)$ and hence $\pi(x) \neq \pi(y)$.

THEOREM 9. *Suppose that whenever x is an irregular point of $\partial\omega$ then $t(x) = T(\omega)$. Then $A(\omega)$ is simplicial.*

Proof. First we show that $A(\omega_{t_0})$ is simplicial for $t_0 < T(\omega)$, by showing that ω_{t_0} satisfies the hypotheses of Proposition 1. Clearly any x in $S(\omega) \times \{T(\omega)\}$ satisfies the neighbourhood condition (2.1) for ω_{t_0}, so that it is enough to show that all other boundary points of ω_{t_0} are regular. Any such point is either on the side or base of the cylinder $S(\omega) \times (t_0, T)$, or is in $\partial\omega$. If x is on the side of the cylinder then there is an upward pointing cone in $\mathbf{R}^{n+1}\backslash\omega_{t_0}$ with vertex at x, so that, by [**3**, 7.1], x is regular. If $t(x) = t_0$ and $x \notin \partial\omega$ then we have again such a cone. Finally suppose $t(x) \leqq t_0$ and $x \in \partial\omega$. It will be enough, by [**2**, 4.3.1] to show that $\mathbf{R}^{n+1}\backslash\omega_{t_0}$ is not thin at x (see [**2**, p. 107] for the definition of thinness). First we remark that $S(\omega) \times (t_0, T)$ is thin at x: use the superharmonic function $f(y) = 0$ if $t(y) \leqq t_0$, $f(y) = 1$ if $t(y) > t_0$. It follows that the smaller set $\omega_{t_0}\backslash\omega$ is thin at x and hence if $\mathbf{R}^{n+1}\backslash\omega_{t_0}$ were thin at x then $\mathbf{R}^{n+1}\backslash\omega \subset (\mathbf{R}^{n+1}\backslash\omega_{t_0}) \cap (\omega_{t_0}\backslash\omega)$ would be thin at x (union of thin sets is thin [**2**, 3.1, Exercise 5 and 3.3.3]) which would contradict the regularity of x in $\partial\omega$ [**2**, 4.3.1].

Thus K_t is a simplex for every $t < T$ and Proposition 4 tells us that K_T is a simplex. Proposition 8 tells us that K is a simplex. Let K' be the state space of $A(\omega)$. We shall show that K' can be identified as a closed face of K, and hence is a simplex. Let τ' be the restriction map $\tau' : A \rightarrow A(\omega)$. Then τ' is surjective and thus induces continuous affine injection on the state space $\pi' : K' \rightarrow K$. It is a general result that the annihilator in K of any subset of $A(K)^+$ is a closed face of K. Since $\pi'(K')$ is clearly the annihilator in K of

$$\ker (\tau')^+ = \{f \in A : f \geqq 0 \text{ and } f|\bar{\omega} = 0\},$$

we deduce $\pi'(K')$ is a closed face of K, hence is a simplex. Since π' is injective, K' is a simplex.

Now let us identify the Choquet boundary of $A(\omega)$ in case ω satisfies the hypothesis of Theorem 9. It turns out that this is again the set of regular

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points of $\partial\omega$. To see this we use the following criterion for an extreme point of a simplex, which is a generalization of [**5**, § 2, Proposition].

PROPOSITION 10. *Suppose K is a simplex and suppose $x \in K$ has the following property: whenever we are given f_1 and f_2 in $A(K)$ such that $f_i(x) \leqq 0$ for each i, and $\epsilon > 0$, then we can find $g \in A(K)^+$ such that $g \geqq f_i$ for each i and $g(x) < \epsilon$. Then x is an extreme point of K.*

Proof. We will use the criterion of the proposition of [**5**, § 2]. Suppose $f \in A(K)$ and $f(x) = 0$. We must find $g \in A(K)^+$ such that $g \geqq f$ and $g(x) = 0$. We will construct a sequence $\{g_n\} \subset A(K)^+$ such that for $n \geqq 1$, $g_n \geqq f$, $g_n(x) < 1/2^n$ and $||g_{n+1} - g_n|| \leqq 1/2^n$ if $n > 1$. We get g_1 immediately from the hypotheses of the proposition. Suppose we have constructed $g_1, \ldots, g_n$ with these properties. Then f and $g_n - 1/2^n$ are both $\leqq 0$ at x, so by hypothesis we can find $h \in A(K)^+$ such that $h \geqq f$, $h \geqq g_n - 1/2^n$, and $h(x) < 1/2^{n+1}$. Then $a = h \wedge g_n$ is a concave continuous function on K and $b = f \vee (g_n - 1/2^n) \vee 0$ is a convex continuous function on K and $a \geqq b$, so by the standard separation theorem for simplexes [**4**], we can find $g_{n+1} \in A(K)$ such that $a \geqq g_{n+1} \geqq b$. Then $g_{n+1} \geqq 0$, $g_{n+1} \geqq f$, $g_{n+1}(x) \leqq h(x) < 1/2^{n+1}$ and $g_n - 1/2^n \leqq g_{n+1} \leqq g_n$ from which $||g_{n+1} - g_n|| \leqq 1/2^n$.

The sequence $\{g_n\}$ is Cauchy and thus converges uniformly to $g \in A(K)^+$. Clearly $g \geqq f$ and $g(x) = 0$.

PROPOSITION 11. *Suppose that if $x \in \partial\omega$ and $t(x) < T(\omega)$ then x is a regular point of $\partial\omega$. Then the Choquet boundary of $A(\omega)$ is the set of regular points of $\partial\omega$.*

Proof. It follows from [**3**, Theorem 3.2] that Choquet boundary points of $A(\omega)$ are always regular points of $\partial\omega$. Suppose x is a regular point of $\partial\omega$. We will use the criterion of Proposition 10 to prove that x is the Choquet boundary.

Suppose f_1 and f_2 are in $A(\omega)$ and $f_i(x) \leqq 0$ for each i. Choose $\epsilon > 0$. For $z \in \bar{\omega}$ define

$$h(z) = \epsilon/3 \vee (f_1(z) + \epsilon/3) \vee (f_2(z) + \epsilon/3),$$

and $f(z) = \mu_z^\omega(h)$. By [**2**, Satz 4.1.1] f is harmonic in ω and by the hypothesis of the proposition, f is continuous in $\bar{\omega} \cap \{t < T(\omega)\}$. Also for $z \in \bar{\omega}$, $f(z) \geqq h(z)$. Indeed $f(z) = \mu_z^\omega(h)$ and this is no smaller than the three numbers $\mu_z^\omega(\epsilon/3)$, $\mu_z^\omega(f_1 + \epsilon/3)$ and $\mu_z^\omega(f_2 + \epsilon/3)$. But these are equal to $\epsilon/3, f_1(z) + \epsilon/3$ and $f_2(z) + \epsilon/3$ and $h(z)$ is the largest of these. Define $\bar{f}$ on $\bar{\omega}$ as follows:

$$\bar{f}(z) = \begin{cases} f(z), & \text{if } t(z) < T(\omega) \\ \underline{\lim}_{\substack{y \to z \\ t(y) < T(\omega), y \in \omega}} f(y), & \text{if } t(z) = T(\omega). \end{cases}$$

Then

$$\bar{f}(x) = \overline{\lim_{y \to x}} \{\mu_y^\omega(h) : y \in \omega, t(y) < T(\omega)\} = h(x) = \epsilon/3,$$

since x is regular and h is continuous, and a simple argument shows that $\bar{f}$ is upper semicontinuous on $\bar{\omega}$. Thus $\bar{f}$ is the pointwise infimum of the continuous functions which majorize it and hence we can choose k continuous on $\bar{\omega}$ such that $k \geqq \bar{f}$ and $k(x) < \bar{f}(x) + \epsilon/3 = 2\epsilon/3$. Clearly $k \geqq f$ on $\bar{\omega} \cap \{t < T(\omega)\}$ and so Lemma 7 (with $X = \bar{\omega}$) gives us $t_0 < T$ and $a \in A(\omega_{t_0})$ such that, in $\bar{\omega}$,

$$h - \epsilon/3 \leqq a \leqq k + \epsilon/3.$$

Then $g = a|\bar{\omega}$ is in $A(\omega)$, $g \geqq 0 \vee f_1 \vee f_2$ and

$$g(x) \leqq k(x) + \epsilon/3 < \epsilon.$$

By Proposition 10, x is an extreme point of the state space of $A(\omega)$, hence is in the Choquet boundary.

4. The main theorem. In this section we extend the results of Theorem 9 and Proposition 11 to the case where ω has irregular points only at a finite number of values of $t(x)$. First some general concepts.

Suppose K is a compact convex set and F is a closed convex subset of K. Let $A_F(K)$ denote the space of functions in $A(K)$ which are constant on F. Let K/F denote the state space of $A_F(K)$. Denote by $\tau: A_F(K) \to A(K)$ the natural injection. Then τ is a positive isometry mapping 1 into 1, hence τ induces a continuous affine surjection $\pi: K \to K/F$ of the state spaces.

Let us denote by F' the complementary set of F, i.e. the union of all faces of K disjoint from F. In the following lemmas we will assume that F is a split face of K [**1**, § 2]. In this case F' is a convex face.

LEMMA 12. *Suppose F is a closed split face of K. Then the restriction of π to F' is injective, $\pi(F)$ and $\pi(F')$ are faces of K/F, and $\pi(E(K)) = E(K/F)$.*

Proof. By [**1**, Corollary 3.8] $A_F(K)$ separates points of F', so that π is injective on F'. Let χ_F denote the characteristic function of F and let $\Gamma = \{a \in A(K): a > \chi_F\}$. Let $\bar{\chi}_F = \inf \Gamma$. By [**1**, Theorem 3.3], for any $a \in \Gamma$, there is $c \in \Gamma \cap A_F(K)$ such that $c \leqq a$. Hence there is a function Φ on K/F ($\Phi = \inf \Gamma \cap A_F(K)$) such that $\bar{\chi}_F = \Phi \circ \pi$. By [**1**, Theorem 3.5] Γ is directed down; hence $\Gamma \cap A_F(K)$ is directed down and Φ is affine. Since $0 \leqq \Phi \leqq 1$, $\Phi^{-1}(0)$ and $\Phi^{-1}(1)$ are faces of K/F. But $\Phi^{-1}(0) = \pi(\bar{\chi}_F{}^{-1}(0)) = \pi(F')$ and $\Phi^{-1}(1) = \pi(\bar{\chi}_F{}^{-1}(1)) = \pi(F)$ by [**1**, Corollary 1.2].

Finally we show $\pi(E(K)) = E(K/F)$. Since π is surjective, the inverse image of an extreme point of K/F is a non-empty closed face of K, hence contains an extreme point of K. Hence $\pi(E(K)) \supset E(K/F)$. For the converse suppose $x \in E(K)$. Then either $x \in F$ or $x \in F'$. In the first case $\pi(x) \in E(K/F)$ since $\pi(F)$ is a one-point face, hence an extreme point of K/F. In the second case $\pi(x)$ is an extreme point of $\pi(F')$ (since π is injective), hence an extreme point of K/F (since $\pi(F')$ is a face).

PROPOSITION 13. *Suppose F is a closed split face of a compact convex set K. Suppose F and K/F are simplexes. Then K is a simplex.*

Proof. We will use the criterion that a convex set X is a simplex if and only if it is the base of a lattice cone. (This is actually the definition of simplex in [**6**, § 9].) Let P be the positive cone of $A(K)^*$. Then K is a base of P and F and F' are bases of subcones Q and Q' of P. Since F is a face of K, Q is a face of P, and since F is split, Q' is a face of P and P is the direct sum of Q and Q'. Since F is a simplex, Q is a lattice cone. Since K/F is a simplex and $\pi(F')$ is a face of K/F (Lemma 12), $\pi(F')$ is a simplex (a face of a lattice cone is a lattice cone) and since $\pi|F'$ is injective (Lemma 12), F' is a simplex. Hence Q' is a lattice cone, and $P = Q \oplus Q'$ is a lattice. (It is easy to verify that the direct sum of lattice cones is a lattice cone.) Hence K is a simplex.

Now for some more general concepts. A *function space* on a compact convex set X is a closed subspace of $C(X)$ which contains the constants. If A is a function space on X and Y is a subset of X we denote by A_Y the space of functions in A which are constant on Y. Then A_Y is also a function space on X. If K is the state space of A, then we define

$$Y^{\perp} = \{a \in A : a|Y = 0\}$$

and

$$Y^{\perp\perp} = \{k \in K : k(a) = 0 \text{ for all } a \in Y^{\perp}\}.$$

We remark that $K/Y^{\perp\perp}$ is (naturally isomorphic to) the state space of A_Y. Indeed if we identify A with $A(K)$, it suffices to observe that $a \in A$ is constant on Y if and only if a is constant on $Y^{\perp\perp}$. For this we need only remark that a is zero on Y if and only if a is zero on $Y^{\perp\perp}$ and use the fact that A contains the constants.

Let $A|Y = \{a|_Y : a \in A\}$. Then $A|Y$ is a space of continuous functions on Y which contains the constants and the restriction map $\rho : A \to A|Y$ is a positive, norm one, surjection mapping 1 into 1. Hence ρ induces an injection $\sigma : F \to K$ of the state space F of $A|Y$ into K. Clearly $\sigma(F) \subset Y^{\perp\perp}$ but the reverse inclusion need not hold.

LEMMA 14. *Suppose that whenever $a \in A|Y$ and $a \geqq 0$, there exists $c \in A$ such that $c|Y = a$ and $c \geqq 0$. Then $Y^{\perp\perp}$ is (naturally isomorphic to) the state space of $A|Y$.*

Proof. We will show that $Y^{\perp\perp} \subset \sigma(F)$. Then $Y^{\perp\perp} = \sigma(F)$ and since σ is injective, we are done. Suppose $k \in Y^{\perp\perp}$. Define a functional k_1 on $A|Y$ by setting $k_1(\rho(f)) = k(f)$ for any $f \in A$. Since $k \in \ker(\rho)^{\perp}$, k_1 is well-defined. Clearly k_1 is linear and $k_1(1) = 1$. By the hypothesis of the lemma $k_1 \geqq 0$ (since $k \geqq 0$). It follows that $k_1 \in F$ and hence $k = \sigma(k_1) \in \sigma(F)$.

Now let us take an example. Suppose ω is a bounded open set. Let us fix a time $t_1 < T(\omega)$. Let ω_1 denote the open set $\omega \cap \{t < t_1\}$, and suppose $\omega_1 \neq \phi$. Let K be the state space of $A(\omega)$, and $\omega_1^{\perp\perp}$, as usual, the double annihilator of ω_1 in K.

LEMMA 15. *$\omega_1^{\perp\perp}$ is a closed face of K and is (naturally isomorphic to) the state space of $A(\omega_1)$.*

Proof. Let B be a closed rectangular box with its base in $\{t = t_1\}$ which contains $\bar{\omega} \cap \{t \geqq t_1\}$. We remark that any continuous function defined on the base and sides of B extends to a unique continuous function on B which is harmonic in the interior of B. If the initial function is positive, so is the extension.

First we show that $\omega_1^{\perp\perp}$ is the state space of $A(\omega_1)$. Since $A(\omega_1) = A(\omega)|\omega_1$, it is enough, by Lemma 14, to show that any $f \in A(\omega_1)^+$ can be extended to a member of $A(\omega)^+$. If $f \in A(\omega_1)^+$ let g be any positive continuous function on the base and sides of B which agrees with f in $\bar{\omega}_1 \cap \{t = t_1\}$. Let h be the harmonic extension of g inside B. Then $h \geqq 0$ and the function $\bar{f}$ on $\bar{\omega}$ defined as f on $\bar{\omega}_1$ and as h on $\bar{\omega} \cap \{t \geqq t_1\}$ is positive and continuous on $\bar{\omega}$ and, by Lemma 6, is harmonic in ω. So $\bar{f}$ is an extension of f in $A(\omega)^+$.

Finally we use [**8**, Proposition 1] to show that $\omega_1^{\perp\perp}$ is a closed face of K. Suppose $f \in \omega_1^{\perp}$ and $\epsilon > 0$. We must find $g \in \omega_1^{\perp}$ such that $g \geqq |f| - \epsilon$. Let p be a continuous function on the base and sides of B with $p = |f|$ on $\bar{\omega} \cap \{t = t_1\}$ and $p > 0$ otherwise. Let s be the harmonic extension of p inside B. Then s is strictly positive in $B \cap \{t > t_1\}$. Now $s > |f| - \epsilon$ in $\bar{\omega} \cap \{t = t_1\}$ so that there is $t_2 > t_1$ such that $s > |f| - \epsilon$ in $\bar{\omega} \cap \{t_1 \leqq t \leqq t_2\}$. Since $s > 0$ in $B \cap \{t \geqq t_2\}$ we can choose a number $M \geqq 1$ so that $Ms > \|f\|$ in this set. Then $Ms > |f| - \epsilon$ in $\bar{\omega} \cap \{t \geqq t_1\}$ and $Ms = 0$ on $\bar{\omega}_1 \cap \{t = t_1\}$ (since $s = p = |f| = 0$ on this set). Then if g is defined as 0 on ω_1 and as Ms on $\bar{\omega} \cap \{t \geqq t_1\}$, then $g \in A(\omega)$ by Lemma 6, so that $g \in \omega_1^{\perp}$ and $g \geqq |f| - \epsilon$.

With an additional assumption we can prove that $\omega_1^{\perp\perp}$ is a split face of K.

LEMMA 16. *With ω and ω_1 as in Lemma 15 suppose that if $x \in \partial\omega$ and $t_1 < t(x) < T(\omega)$, then $x \in \partial_r(\omega)$. Then $\omega_1^{\perp\perp}$ is a closed split face of K.*

Proof. By Lemma 15, $\omega_1^{\perp\perp}$ is a closed face of K. According to [**1**, Theorem 3.5], a closed face F of K is split if for every $a_0 \in A(F)^+$, the set $\{a \in A(K): a > 0 \text{ and } a|_F > a_0\}$ is directed downwards. We will show that this condition holds for $F = \omega_1^{\perp\perp}$.

First let us restate the condition. We remark that we can identity $A(\omega_1^{\perp\perp})$ and $A(\omega_1)$, since $A(\omega_1)$ is a function space and $\omega_1^{\perp\perp}$ is its state space by Lemma 15. Similarly, we identify $A(K)$ and $A(\omega)$. Secondly we point out that if $c \in A(\omega)$ and $a_0 \in A(\omega_1)$, then $c|\bar{\omega}_1 > a_0$ if and only if $c|\omega_1^{\perp\perp} > a_0$. This follows since both $c|\bar{\omega}_1$ and a_0 are in $A(\omega_1)$ and $\omega_1^{\perp\perp}$ is the state space of $A(\omega_1)$. Thus we can restate the condition as follows: for every $a_0 \in A(\omega_1)^+$ the set $\{a \in A(\omega): a > 0 \text{ and } a|\bar{\omega}_1 > a_0\}$ is directed downwards.

Now let us prove this. Suppose $a_0 \in A(\omega_1)^+$ and $a_1, a_2 \in A(\omega)$ with $a_1 > 0$, $a_2 > 0$ and $a_1 \wedge a_2 > a_0$ on $\bar{\omega}_1$. Choose $\epsilon > 0$ so that $a_1 \wedge a_2 > 2\epsilon$ and $a_0 + 2\epsilon < a_1 \wedge a_2$ on $\bar{\omega}_1$. Let $G = \omega \cap \{t > t_1\}$. Choose a continuous function

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g on ∂G such that $g = a_0 + \epsilon$ on $\bar{\omega}_1$ and $\epsilon \leqq g \leqq (a_1 \wedge a_2) - \epsilon$. Define a function f on $\bar{\omega}$ as follows:

$$f(x) = \begin{cases} a_0(x) + \epsilon, \text{ if } x \in \bar{\omega}_1 \\ \mu_x{}^G(g), \text{ if } x \in \bar{G}. \end{cases}$$

The two formulas for $f(x)$ agree at points $x \in \bar{\omega}_1 \cap \bar{G}$ since such points are regular points of G, hence $\mu_x{}^G(g) = g(x)$. Clearly f is continuous on $\bar{\omega}_1$ and f is continuous on $\bar{G} \cap \{t < T(\omega)\}$ by our regularity hypothesis (x regular in $\partial\omega$ implies x regular in ∂G [**2**, Satz, 4.2.6]). Thus f is continuous on $\bar{\omega} \cap \{t < T(\omega)\}$. By Lemma 6 and [**2**, Satz 4.1.1], f is harmonic in ω.

Let $k = (a_1 \wedge a_2) - \epsilon$. Then for $x \in \bar{G}$,

$$f(x) = \mu_x{}^G(g) \leqq \mu_x{}^G((a_1 \wedge a_2) - \epsilon) \leqq \mu_x{}^G(a_i - \epsilon) = a_i - \epsilon$$

for $i = 1, 2$, the last equality holding since $\mu_x{}^G$ is an $A(G)$-representing measure for x [**3**, Theorem 3.2]. Hence $f \leqq k$ in $\bar{\omega}$. Also $f \geqq \epsilon$ since this is true for g and $a_0 \geqq 0$. So by Lemma 7 (with $X = \bar{\omega}$ and $h = \epsilon$) we can find $t_0 > t_1$ and $a \in A(\omega_{t_0})$ such that $a(x) = f(x) > a_0(x)$ if $x \in \bar{\omega}_1$, $(x \in \bar{\omega}_1 \Rightarrow t(x) \leqq t_1 < t_0)$, and

$$0 = \epsilon - \epsilon < a < k + \epsilon = a_1 \wedge a_2$$

in $\bar{\omega}$. Then $c = a|\bar{\omega}$ has the required properties.

THEOREM 17. *Suppose ω is a bounded open set and S is a finite set of numbers such that if x is an irregular boundary point of ω then $t(x) \in S$. Then $A(\omega)$ is a simplex space and the Choquet boundary of $A(\omega)$ is $\partial_r\omega$, the set of regular boundary points of ω.*

Proof. The proof is by induction on the number n of elements in $S \cup \{T(\omega)\}$. If $n = 1$ the theorem follows from Theorem 9 and Proposition 11. Suppose the result proved for n and suppose $S \cup \{T(\omega)\}$ contains $n + 1$ elements. We may assume that $t \leqq T(\omega)$ for all $t \in s$. Denote by t_1 the largest element of $S \cup \{T(\omega)\}$ which is strictly less than $T(\omega)$. Let $\omega_1 = \omega \cap \{t < t_1\}$ and $\omega_2 = \omega \cap \{t > t_1\}$. We may assume $\omega_1 \neq \phi$ (or we are finished by Theorem 9 and Proposition 11).

First of all the induction hypothesis applies to ω_1. Indeed if x is an irregular boundary point of ω_1 then $t(x) = t_1$ or $x \in \partial\omega$; in the second case x is an irregular boundary point of ω by [**2**, Satz 4.2.6]. In either case $t(x) \in S\backslash\{T(\omega)\}$ which has n elements (and contains $t_1 = T(\omega_1)$), so that $A(\omega_1)$ is a simplex space and the Choquet boundary of $A(\omega_1)$ is the set of regular boundary points of ω_1.

Secondly we show that the case $n = 1$ applies to ω_2. Suppose x is an irregular boundary point of ω_2. Then $t(x) \neq t_1$ (since ω_2 lies above t_1) and so $x \in \partial\omega$. It follows from [**2**, Satz 4.2.6] that x is irregular for ω. Hence $t(x) \in S$ and since $t(x) \neq t_1$ we deduce that $t(x) = T(\omega) = T(\omega_2)$. Thus $A(\omega_2)$ is a simplex space, and the Choquet boundary of $A(\omega_2)$ is the set of regular boundary points of ω_2.

If $\bar{\omega}_1 \cap \bar{\omega}_2 = \phi$ then there is a natural isomorphism $A(\omega) \cong A(\omega_1) \oplus A(\omega_2)$ and since the direct sum of simplex spaces is a simplex space (use the R.S.P. [**5**]), $A(\omega)$ is a simplex space. Also $\partial_c A(\omega) = \partial_c A(\omega_1) \cup \partial_c A(\omega_2)$, where ∂_c denotes Choquet boundary. Since $\partial_r \omega = \partial_r \omega_1 \cup \partial_r \omega_2$ and we have the theorem for ω_1 and ω_2, we have $\partial_c A(\omega) = \partial_r \omega$.

Now assume $\bar{\omega}_1 \cap \bar{\omega}_2 \neq \phi$. Let K be the state space of $A(\omega)$. We will show that K is a simplex. Let $F = \bar{\omega}_1^{\perp\perp}$ be the double annihilator in K of $\bar{\omega}_1$. Then F is a closed split face of K by Lemma 8, and is the state space of $A(\omega_1)$ by Lemma 15. Since $A(\omega_1)$ is a simplex space, F is a simplex. To show that K is a simplex, it is enough, by Proposition 13, to show that K/F is a simplex.

Let $A_1 = A(\omega)_{\bar{\omega}_1}$, the space of functions in $A(\omega)$ constant on $\bar{\omega}_1$, and let $A_2 = A(\omega_2)_{\bar{\omega}_1 \cap \bar{\omega}_2}$. The restriction map $A_1 \rightarrow A_2$ is clearly an isomorphism, hence the state spaces of A_1 and A_2 are isomorphic. By the remarks before Lemma 14, A_1 has state space K/F and A_2 has state space $K_2/(\bar{\omega}_1 \cap \bar{\omega}_2)^{\perp\perp}$, where K_2 is the state space of $A(\omega_2)$. Thus, to show K/F is a simplex, it is enough to show that $K_2/(\bar{\omega}_1 \cap \bar{\omega}_2)^{\perp\perp}$ is a simplex. This follows from three facts: (1) K_2 is a simplex (since $A(\omega_2)$ is a simplex space), (2) $(\bar{\omega}_1 \cap \bar{\omega}_2)^{\perp\perp}$ is a closed face of K_2 (the proof of this follows exactly the last paragraph of the proof of Lemma 15 with ω_2 instead of ω and $\bar{\omega}_1 \cap \bar{\omega}_2$ instead of ω_1) and (3) the next lemma.

LEMMA 18. *Suppose that K is a simplex and F is a closed face of K. Then $K|F$ is a simplex.*

Proof. It is enough to show that $A_F(K)$ has the Riesz separation property [**4**, condition (iii)]. Suppose f_1, f_2, g_1 and g_2 are in $A_F(K)$ and $f_1 \wedge f_2 \geqq g_1 \vee g_2$. Since $f_1 \wedge f_2$ and $g_1 \vee g_2$ are constant on F we can choose a real number r such that if $x \in F$ then $f_1(x) \wedge f_2(x) \geqq r \geqq g_1(x) \vee g_2(x)$. Define functions a and b on K as follows: if $x \in F$ then $a(x) = b(x) = r$, if $x \notin F$ then $a(x) \geqq f_1(x) \wedge f_2(x)$ and $b(x) = g_1(x) \vee g_2(x)$. Then a is convex, lower semicontinuous, b is concave, upper semicontinuous and $a \geqq b$, so that by a standard separation theorem [**4**] (K is a simplex), there is $c \in A(K)$ such that $a \geqq c \geqq b$. Then $c = r$ on F so that $c \in A_F(K)$ and $f_1 \wedge f_2 \geqq c \geqq g_1 \vee g_2$.

It remains to show that $\partial_r \omega = E(K)$ (by definition $\partial_c A(\omega) = E(K)$). We always have $E(K) \subset \partial_r \omega$ [**3**, Theorem 3.2], so suppose $x \in \partial_r \omega$. Suppose $x \in \partial \omega_1$. Then $x \in \partial_r \omega_1$ [**2**, Satz 4.2.6], so that $x \in E(F)$ since we have the theorem for ω_1, hence $x \in E(K)$ since F is a closed face of K.

If $x \notin \partial \omega_1$ then $x \in \partial \omega_2 \backslash \bar{\omega}_1$. Then $x \in \partial_r \omega_2$ [**2**, Satz 4.2.6], so that $x \in E(K_2)$ since we have the theorem for ω_2. An argument identical to the proof of Lemma 16 (with ω, ω_1 and $A(\omega_1)$ replaced by ω_2, $\bar{\omega}_1 \cap \bar{\omega}_2$ and $C(\bar{\omega}_1 \cap \bar{\omega}_2)$ respectively) shows that $(\bar{\omega}_1 \cap \bar{\omega}_2)^{\perp\perp}$ is a closed split face of K_2. Thus, by Lemma 12, $\pi_2(x)$ is in $E(K_2/(\bar{\omega}_1 \cap \bar{\omega}_2)^{\perp\perp})$ where π_2 is the natural surjection of K_2 onto $K_2/(\bar{\omega}_1 \cap \bar{\omega}_2)^{\perp\perp}$. Since this last set is naturally isomorphic to K/F, $\pi(x) \in E(K/F)$ where π is the natural surjection of K onto K/F. Hence the inverse image Q of $\pi(x)$ under π is a closed face of K. Since $\pi(x) \neq \pi(F)$

(shown below), $Q \cap F = \phi$. Since Q is a face, $Q \subset F'$, the complementary face of F. Since π is injective on F' (Lemma 12), $Q = \{x\}$. Since Q is a face, $x \in E(K)$.

Now we must show $\pi(x) \neq \pi(F)$. Since $x \notin \bar{\omega}_1$, we can find $f \in A(\omega)$ such that $f|\bar{\omega}_1 = 0$ and $f(x) > 0$. (Use the box B of Lemma 15; start with a function on the base and sides that is zero on $\partial B \cap \bar{\omega}_1$ and strictly positive elsewhere.) Then $f \in A(\omega)_{\bar{\omega}_1}$ and $\pi(x)(f) > 0$. But since $F = \bar{\omega}_1^{\perp\perp}$, $\pi(F)(f) = 0$, so that $\pi(F) \neq \pi(x)$.

We remark that Lemma 18 is a reformulation of the general result that a closed ideal in a simplex space is a simplex space [**14**, 3.4], at least for simplex spaces with unit. Indeed if K is a simplex then $A(K)$ is a simplex space and if F is a closed face of K then

$$J = \{a \in A(K) : a|F = 0\}$$

is a closed ideal [**14**, p. 111] in $A(K)$. Letting P_1 denote the intersection of the closed unit ball of J^* and the positive cone, we observe that P_1 is a copy of K/F and is a simplex since J is a simplex space [**14**, 2.2].

Remarks. 1. The proof of Lemma 12 was provided by B. Hirsberg and T. B. Andersen. The proof of Proposition 13 was suggested to me by M. Rogalski and A. Goullet de Rugy.

2. The proof of Lemma 15 was provided by D. Gregory. Is it true that the closed face $Y^{\perp\perp}$ is always split? Certainly the additional hypothesis of Lemma 16 is unnatural.

3. If ω is a subset of the plane whose boundary consists of a finite number of straight line segments, then ω satisfies the hypothesis of Theorem 17. Hence $A(\omega)$ is a simplex space and $\partial_c A(\omega) = \partial_r \omega$. Can we use this result to get at other subsets of the plane? For example, what about the inside of a Jordon curve or of a C^1 Jordon curve?

4. We conjecture that $A(\omega)$ is always a simplex. This has been proved for the Laplace equation by Boboc and Cornea [**9**, p. 521]. Essentially this proof can be found in [**3**, Theorem 3.3]. There is no hope that the same method could apply to our problem since balayage need not be an affine dilation for the heat equation.

5. The equation $\partial_c A(\omega) = \partial_r \omega$ is known to be false in general. A counterexample was given by Kohn and Sieveking [**10**, § 5]. Essentially, this is their example: let ω be the set of $x = (\lambda(x), t(x))$ in the plane such that $0 < \lambda(x) < 1$, $-1 < t(x) < 0$, and $t(x) \neq -1/n$ for all integers $n \geqq 1$. Then the point $(1/2, 0)$ is regular but not Choquet.

6. I am grateful to the referee for correcting a flaw in the proof of Theorem 9.

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