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ADDITIVITY OF THE P^n-INTEGRAL

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1. Introduction. It is known that the P^n-integral as originally defined is not additive on abutting intervals. This paper offers a slight modification in the definition of the integral and develops necessary and sufficient conditions for the integral to be additive.

The following example is given in [**2**]:

If n is odd, let

$$F(x) = \begin{cases} x \cos 1/x, & x \neq 0 \\ 0, & x = 0, \end{cases}$$

and if n is even, let

$$F(x) = \begin{cases} x \sin 1/x, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Define a function f by

$$f(x) = \begin{cases} F^{(n+1)}(x), & \text{for } x \in (0, i/\pi] \\ 0, & \text{for } x \in [-i/\pi, 0], \end{cases}$$

where $i = 2$ if n is odd and 1 if n is even.

It is easy to see that f is P^{n+1}-integrable over each of the intervals $[-i/\pi, 0]$ and $[0, i/\pi]$ but not over $[-i/\pi, i/\pi]$. The function f fails to be P^{n+1}-integrable over $[-i/\pi, i/\pi]$ essentially because $F(x)$ is not n-smooth at 0.

In the case $n = 2$ Skvorcov [**6**] obtained necessary and sufficient conditions for the P^2-integral of a given function to exist on an interval $[a, b]$ where it is known that the P^2-integral of that function exists on the two abutting intervals $[a, c]$ and $[c, b]$:

THEOREM [**6**, Theorem 2]. *Let the function $f(x)$ be P^2-integrable on the closed intervals $[a, c]$ and $[c, d]$ and have $F_1(x)$ and $F_2(x)$, respectively, for its P^2-integral on these intervals. Then $f(x)$ is P^2-integrable on $[a, b]$ if and only if there exists a number α such that the function*

$$F(x) = \begin{cases} F_1(x) + \dfrac{\alpha}{c-a}(x-a), & x \in [a, c] \\ F_2(x) + \dfrac{\alpha}{c-b}(x-b), & x \in [c, b] \end{cases}$$

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is smooth at the point c. If such a number α exists, then the function $F(x)$ is the P^2-integral of $f(x)$ on $[a, b]$.

Smoothness of F at c, of course, imposes certain constraints on F (and on f) in a neighbourhood of c. The proof of Skvorcov's result depends on the following:

LEMMA [**6**, Lemma 3]. *Let $f(x)$ be P^2-integrable on $[a, b]$ and have $F(x)$ for its P^2-integral. Then for any $\epsilon > 0$ there exists a majorant $M(x)$ and a minorant $m(x)$ such that if $R(x) = M(x) - F(x)$, $r(x) = F(x) - m(x)$, we have*

$$|R(x)| < \epsilon, \quad |r(x)| < \epsilon, \quad |R_+'(a)| < \epsilon, \quad |R_-'(b)| < \epsilon,$$
$$|r_+'(a)| < \epsilon, \quad |r_-'(b)| < \epsilon.$$

It is not known how to prove the lemma that would be required to obtain the corresponding additivity result for the P^n-integral [see the remark at the end of this paper]. In the following we obtain necessary and sufficient conditions that a function f be P^n-integrable on an interval $[a, b]$ phrased in terms of a different kind of neighbourhood property of $f(x)$.

2. Definitions. In the original definition of the P^n-integral there is a difficulty with the condition B_{n-2} [**4**, p. 150] since it is not linear on the set of major and minor functions. As a result, the proof of Lemma 5.1 [**4**] fails since the difference $Q(x) - q(x)$ need not satisfy the conditions of Theorem 4.2 [**4**].

It was shown in [**3**] that a simple modification of the definition of major and minor functions avoids this difficulty and leads to a definition of an integral which is strong enough to solve the coefficient problem in trigonometric series under the conditions imposed by James [**5**].

Let $F(x)$ be a real-valued function defined on the bounded interval $[a, b]$. If there exist constants $\alpha_1, \alpha_2, \ldots, \alpha_r$ which depend on x_0 only and not on h, such that

$$F(x_0 + h) - F(x_0) = \sum_{k=1}^{r} \alpha_k \frac{h^k}{k!} + o(h^r), \quad \text{as } h \to 0, \tag{2.1}$$

then α_k, $1 \leqq k \leqq r$, is called *the Peano derivative of order k* of F at x_0 and is denoted by $F_{(k)}(x_0)$. If F possesses derivatives $F_{(k)}(x_0)$, $1 \leqq k \leqq r - 1$, we write

$$\frac{h^r}{r!}\gamma_r(F; x_0, h) = F(x_0 + h) - F(x_0) - \sum_{k=1}^{r-1} \frac{h^k}{k!} F_{(k)}(x_0). \tag{2.2}$$

By restricting h to be positive (or negative) in (2.1) we can define *one-sided Peano derivatives*, which we write as $F_{(k)}(x_{0+})$ (or $F_{(k)}(x_{0-})$).

If there exist constants $\beta_0, \beta_2, \ldots, \beta_{2r}$ which depend on x_0, and not on h, such that

$$\frac{F(x_0 + h) + F(x_0 - h)}{2} = \sum_{k=0}^{r} \beta_{2k} \frac{h^{2k}}{(2k)!} + o(h^{2r}), \quad \text{as } h \to 0,$$

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then β_{2k}, $0 \leqq k \leqq r$ is called *the de la Vallée Poussin derivative of order* $2k$ of F at x_0 and is denoted by $D^{2k} F(x_0)$.

If F has derivatives $D^{2k} F(x_0)$, $0 \leqq k \leqq r - 1$, we write

$$\frac{h^{2r}}{(2r)!}\theta_{2r}(F; x_0, h) = \frac{F(x_0 + h) + F(x_0 - h)}{2} - \sum_{k=0}^{r-1} \frac{h^{2k}}{(2k)!} D^{2k} F(x_0)$$

and define

$$\bar{D}^{2r} F(x_0) = \limsup_{h\to 0} \theta_{2r}(F; x_0, h)$$

$$\underline{D}^{2r} F(x_0) = \liminf_{h\to 0} \theta_{2r}(F; x_0, h).$$

All the above symbols are defined similarly for odd-numbered indices (see, for example, [**4**, pp. 163–164].

We denote the ordinary derivative of $F(x)$ at x_0 of order k by $F^{(k)}(x_0)$.

The function F will be said to satisfy condition $A_n{}^*$ ($n \geqq 3$) in $[a, b]$ if it is continuous in $[a, b]$, if, for $1 \leqq k \leqq n - 2$, each $F_{(k)}(x)$ exists and is finite in (a, b) and if

(2.3) $$\lim_{h\to 0} h\theta_n(F; x, h) = 0,$$

for all $x \in (a, b) - E$ where E is countable.

When a function F satisfies condition (2.3) at a point x, F is said to be *n-smooth at x.*

THEOREM 2.1. *If F satisfies condition $A_{2m}{}^*(A_{2m+1}{}^*)$ in $[a, b]$, then $F_{(2k)}(x) = D^{2k}F(x)$ $(F_{(2k+1)}(x) = D^{2k+1}(x))$ does not have an ordinary discontinuity in (a, b) for $0 \leqq k \leqq m - 1$.*

Proof. This is Lemma 8.1 [**4**].

Note. Condition $A_{2m}{}^*$ is a stronger form of James' condition A_{2m}, [**4**], in that it replaces the requirement that $D^{2k} F(x)$ exist and be finite for $1 \leqq k \leqq m - 1$ by the same condition on the Peano derivatives. Theorem 2.1 then shows that $A_{2m}{}^*$ also implies James' condition B_{2m-2}, [**4**].

We shall make extensive use of the theory of n-convex functions in the following. For the definition and properties of n-convex functions we refer the reader to [**1**].

THEOREM 2.2. *If F satisfies $A_n{}^*$, $n \geqq 3$, in $[a, b]$ and*

(a) $\bar{D}^n F(x) \geqq 0$, $x \in (a, b) - E$, $|E| = 0$,
(b) $\bar{D}^n F(x) > -\infty$, $x \in (a, b) - S$, *S a scattered set,*
(c) $\limsup_{h\to 0} h\theta_n(F; x, h) \geqq 0 \geqq \liminf_{h\to 0} h\theta_n(F; x, h)$, $x \in S$,

then F is n-convex.

Proof. In [**1**, Theorem 16] Bullen proves a similar result which implies this theorem. In place of condition A_n^* he uses a condition C_n which is just A_n together with B_{n-2}, but as was noted above these are implied by A_n^*.

Definition 2.1. Let $f(x)$ be a function defined in $[a, b]$ and let $A \equiv \{a_i, i = 1, 2, \ldots, n\}$ be fixed points such that $a = a_1 < a_2 < \ldots < a_n = b$. The functions $Q(x)$ and $q(x)$ are called *P^n-major and minor functions* (respectively) of $f(x)$ over $(a_i) = (a_1, a_2, \ldots, a_n)$, *or with respect to the basis* A, if

(2.4.1) $Q(x)$ and $q(x)$ satisfy condition A_n^* in $[a, b]$;

(2.4.2) $Q(a_i) = q(a_i) = 0, \quad i = 1, 2, \ldots, n;$

(2.4.3) $\underline{D}^n Q(x) \geqq f(x) \geqq \bar{D}^n q(x), \quad x \in [a, b] - E, |E| = 0;$

(2.4.4) $\underline{D}^n Q(x) \neq -\infty, \bar{D}^n q(x) \neq +\infty, \quad x \in [a, b] - S$, S a scattered set;

(2.4.5) (i) $\limsup_{h \to 0} h\theta_n(Q; x, h) \geqq 0 \geqq \liminf_{h \to 0} h\theta_n(Q; x, h), \quad x \in S$

(ii) $\limsup_{h \to 0} h\theta_n(q; x, h) \geqq 0 \geqq \liminf_{h \to 0} h\theta_n(q; x, h), \quad x \in S.$

LEMMA 2.1. *For every pair $Q(x)$ and $q(x)$ the difference $Q(x) - q(x)$ is n-convex in $[a, b]$.*

Proof. The proof follows from Theorem 2.2 above.

Definition 2.2. For each major and minor function of $f(x)$ over $(a_i)_{i=1}^n = A$ the functions defined by

$$Q^*(x) = (-1)^r Q(x), \quad q^*(x) = (-1)^r q(x), \quad a_r \leqq x < a_{r+1}$$

are called *associated major and minor functions*, respectively, of $f(x)$ over (a_i) *or on $[a, b]$ with respect to the basis A.*

The proofs of the following lemmas and theorems are given in [**3**] and [**4**].

LEMMA 2.2. *For every pair of associated major and minor functions of $f(x)$ over (a_i),*

$$Q^*(x) - q^*(x) \geqq 0$$

for all x in $[a, b]$.

Definition 2.3. Let c be a point in (a_1, a_n) such that $c \neq a_i$, $i = 1, \ldots, n$. If for every $\epsilon > 0$ there is a pair $Q(x)$, $q(x)$ such that

(2.5) $|Q(c) - q(c)| < \epsilon,$

then $f(x)$ is said to be *P^n-integrable over (a_i, c).*

LEMMA 2.3. *If the inequality* (2.5) *holds, then*

$$|Q(x) - q(x)| < \epsilon k$$

for all x in $[a_1, a_n]$ where k is independent of x.

THEOREM 2.3. *If $f(x)$ is P^n-integrable over $(a_i;\, c)$, there is a function $F^*(x)$ which is the inf of all associated major functions of $f(x)$ over (a_i) and the sup of all associated minor functions.*

Definition 2.4. If $f(x)$ is P^n-integrable over $(a_i; c)$ and if $F^*(x)$ is the function of Theorem 2.3, define $F(x)$ by

$$F^*(x) = (-1)^r F(x), \quad a_r \leqq x < a_{r+1}.$$

If $a_s < c < a_{s+1}$, the P^n-integral of $f(x)$ over $(a_i; c)$ is defined to be $(-1)^s F(c)$. Since $(-1)^s F(a_i) = F(a_i) = 0$, the integral is defined to be zero if $c = a_i$, $i = 1, 2, \ldots, n$. We write

$$(-1)^s F(c) = \int_{(a_i)}^{c} f(t) d_n t.$$

THEOREM 2.4. *If $f(x)$ is P^n-integrable over $(a_i;\, c)$ it is also P^n-integrable over $(a_i;\, x)$ for every x in $[a_1, a_n]$. If $F(x)$ is the function of Definition* 2.4 *then for $a_r \leqq x < a_{r+1}$,*

$$(-1)^r F(x) = \int_{(a_i)}^{x} f(t) d_n t.$$

In view of Theorem 2.4, if $f(x)$ is integrable over $(a_i;\, c)$ we shall say it is integrable on $[a, b]$ with respect to the basis A. We shall refer to the function $F(x)$ of Definition 2.4 as the associated (P^n-)integral of f over $(a_i; x)$ (or with respect to the basis A).

THEOREM 2.5. *If $f(x)$ is P^n-integrable over $(a_i;\, x)$, it is also P^n-integrable over $(b_j;\, x)$, where $a_1 \leqq b_1 < \ldots < b_n \leqq a_n$. In addition if $F(x)$ is the associated P^n-integral of f over $(a_i;\, x)$, and $b_s \leqq x < b_{s+1}$ then*

$$(2.6) \qquad (-1)^s \int_{(b_j)}^{x} f(x) d_n x = F(x) - \sum_{j=1}^{n} \lambda(x; b_j) F(b_j),$$

where

$$\lambda(x; b_j) = \prod_{k \neq j} (x - b_k)/(b_j - b_k)$$

is a polynomial of degree $n - 1$ at most.

Because of Theorem 2.5 we shall sometimes use the phrase "$f(x)$ is P^n-integrable over $[a, b]$" without explicit reference to a basis (a_i).

COROLLARY. *If $f(x)$ is P^n-integrable over $[a, b]$, $Q(x)$, $q(x)$ are P^n-major and minor functions of $f(x)$ and $F(x)$ is the associated P^n-integral of $f(x)$, then $Q(x) - F(x)$ and $F(x) - q(x)$ are n-convex.*

3. Some preliminary considerations. We assume throughout the remainder of the paper that n is even; obvious modifications must be made in the notation to cover the case when n is odd.

THEOREM 3.1. *The function* $F(x)$ *of Definition* 2.4 *possesses derivatives* $F_{(k)}(x)$, $1 \leqq k \leqq n-2$, $x \in (a, b)$.

Proof. If $Q(x)$ denotes a P^n-major function of $f(x)$ over (a_i) then $Q(x) - F(x)$ is n-convex in $[a, b]$. By Theorem 7 [**1**], we have $(Q(x) - F(x))^{(k)} = (Q(x) - F(x))_{(k)}$ exists $(1 \leqq k \leqq n-2,\ x \in [a, b])$ and since by definition $Q_{(k)}(x)$ exists $(1 \leqq k \leqq n-2,\ x \in (a, b))$, the statement in the theorem follows.

THEOREM 3.2. [**1**, Corollary 8]. *If* F *is* n-*convex in* $[a, b]$, $|F| \leqq K$ *then*

$$|F_{(k)}(x)| \leqq \frac{AK}{\min\{(b-x)^k, (x-a)^k\}}, \quad 0 \leqq k \leqq n-1,$$

$x \in (a, b)$, *where* A *is a constant independent of* k, F *and* x, *and where, if* $k = n-1$, *the derivative is to be interpreted as* $\max(|F_{(n-1)}(x+)|, |F_{(n-1)}(x-)|)$.

THEOREM 3.3. *The function* $F(x)$ *of Definition* 2.4 *has the property that*

$$\limsup_{h\to 0} h\theta_n(F; x, h) \geqq 0 \geqq \liminf_{h\to 0} h\theta_n(F; x, h), \quad x \in (a, b). \tag{3.1}$$

Proof. Corresponding to arbitrary $\epsilon > 0$ there exists a P^n-major function $Q(x)$ and a P^n-minor function $q(x)$ such that the n-convex functions

$$R(x) = Q(x) - F(x), \quad r(x) = F(x) - q(x)$$

satisfy $|R(x)| < \epsilon$, $|r(x)| < \epsilon$, $x \in [a, b]$. The major and minor functions have the property further that $\underline{D}^n Q(x) > -\infty$ and $\bar{D}^n q(x) < +\infty$, $x \in [a, b] - S$, where S is a scattered set, while $Q(x)$ and $q(x)$ satisfy 2.4.5 in S. Thus for each fixed $x \in [a, b] - S$, there exist finite numbers $C_1(x)$ and $C_2(x)$ such that

$$h\theta_n(Q; x, h) > h\, C_1(x)$$
$$h\theta_n(q; x, h) < hC_2(x)$$

for all sufficiently small positive h. But, for $x \in (a, b)$,

$$h\theta_n(R; x, h) = (n/2)\{\gamma_{n-1}(R; x, h) - \gamma_{n-1}(R; x, -h)\}$$

and

$$h\theta_n(r; x, h) = (n/2)\{\gamma_{n-1}(r; x, h) - \gamma_{n-1}(r; x, -h)\},$$

and since $R(x)$ and $r(x)$ are n-convex, it follows that

$$\lim_{h\to 0+} h\theta_n(R; x, h) = (n/2)\{R_{(n-1)}(x+) - R_{(n-1)}(x-)\} \equiv H(n, x),$$

and

$$\lim_{h\to 0+} h\theta_n(r; x, h) = (n/2)\{r_{(n-1)}(x+) - r_{(n-1)}(x-)\} \equiv h(n, x).$$

We have further, for each fixed x, the inequality (Theorem 3.2).

$$\begin{aligned} &\max\{|R_{(n-1)}(x+)|, |R_{(n-1)}(x-)|, |r_{(n-1)}(x+)|, |r_{(n-1)}(x-)|\} \\ &\qquad \leqq \frac{A\epsilon}{\min\{(b-x)^{n-1}(x-a)^{n-1}\}} \end{aligned} \tag{3.2}$$

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where A is a constant independent of ϵ, $R(x)$, $r(x)$, and x. Then since

(3.3) $h\theta_n(q;x,h) + h\theta_n(r;x,h) = h\theta_n(F;x,h) = h\theta_n(Q;x,h) - h\theta_n(R;x,h)$,

we have

(3.4) $hC_2(x) - h\theta_n(r;x,h) > h\theta_n F;x,h) > hC_1(x) - h\theta_n(R;x,h)$,

for all sufficiently small positive h. Similar inequalities hold for negative h and, since ϵ is arbitrary in (3.4), it follows that

$$\lim_{h\to 0} h\theta_n(F;x,h) = 0, \quad x \in [a,b] - S.$$

If $x \in S$, then

$$\limsup_{h\to 0} h\theta_n(F;x,h) \geqq \limsup_{h\to 0} h\theta_n(Q;x,h) - H(n,x)$$
$$\geqq -H(n,x)$$

and

$$\liminf_{h\to 0} h\theta_n(F;x,h) \leqq \liminf_{h\to 0} h\theta_n(q;x,h) + h(n,x)$$
$$< 0 + h(n,x),$$

and the result follows because of (3.2).

Now suppose f is a function defined on $[a,b]$, and let $a < u < c < v < b$. If f is P^n-integrable on $[a,v]$ with respect to some basis, then f is P^n-integrable on $[a,v]$ with respect to the basis

$$A_3 \equiv (c_0, c_2, c_3, \ldots, c_{n-1}, c_n) \equiv (a, c_2, \ldots, c_{n-1}, v),$$

(Theorem 2.5) where, for convenience and without affecting the generality of what we prove, we may assume that $(u, c_2, c_3, \ldots, c_{n-1}, v)$ is a partition of $[u,v]$ into subintervals of equal length, $u < c_2$ and $c_{n/2} < c < c_{(n/2)+1}$. Likewise if f is P^n-integrable on $[u,b]$ with respect to some basis, then it is P^n-integral on $[u,b]$ with respect to the basis

$$A_4 \equiv (c_1, c_2, \ldots, c_{n-1}, c_n') \equiv (u, c_2, c_3, \ldots, c_{n-1}, b).$$

Now if f is P^n-integrable on $[a,v]$ and on $[u,b]$ then f is P^n-integrable on the interval $[u,v]$ with respect to the basis

$$A_5 \equiv (c_1, c_2, \ldots c_{n-1}, c_n) \equiv (u, c_2, \ldots, c_{n-1}, v).$$

Also f is P^n-integrable on the interval $[a,c]$ with respect to the basis

$$A_1 \equiv (c_0, d_1, c_2, d_2, c_3, \ldots, d_{n/2-1}, c_{n/2}, d_{n/2}) \equiv \{a_i\}$$

when $c_0 = a < d_1 < c_2 < d_2 < \ldots < d_{n/2-1} < c_{n/2} < d_{n/2} = c$, and on the interval $[c,b]$ with respect to the basis

$$A_2 = (d_{n/2}, c_{(n/2)+1}, d_{n/2+1}, c_{(n/2)+2}, \ldots, c_{n-1}, d_{n-1}, b) \equiv \{b_i\}$$

where

$$c = d_{n/2} < c_{(n/2)+1} < d_{(n/2)+1} < \ldots < c_{n-1} < d_{n-1} < b.$$

On the other hand if f is P^n-integrable on $[a,b]$ with respect to the basis $(a, c_2, c_3, \ldots, c_{n-1}, b) \equiv (l_1, l_2, \ldots, l_n)$ then it is P^n-integrable on $[a,b]$ with respect to any basis.

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For an arbitary set $A = \{x_0, x_1, \ldots, x_n\}$ of distinct numbers we define a function λ by

$$\lambda(A; x, x_r) \equiv \prod_{i \neq r} \left(\frac{x - x_i}{x_r - x_i}\right).$$

If $F_3(x)$, $F_4(x)$ and $F_5(x)$ denote the associated integrals of f over $[a, v]$, $[u, b]$ and $[u, v]$ with respect to the bases A_3, A_4, and A_5, respectively, then for $x \in [u, v]$, we have (Theorem 2.5)

$$F_3(x) = F_5(x) + \lambda(A_5; x, u)F_3(u), \tag{3.5}$$

and

$$F_4(x) = F_5(x) + \lambda(A_5; x, v)F_4(v). \tag{3.6}$$

Let F be defined on $[a, b]$ as follows:

$$F(x) = \begin{cases} F_3(x) + \lambda(A_3; x, v)\left[F_4(v) + \left(\frac{v-b}{u-b}\right)F_3(u)\right] \\ \qquad \times \left[\frac{(v-a)(u-b)}{(v-u)(a-b)}\right], & x \in [a, v], \\ F_4(x) + \lambda(A_4; x, u)\left[F_3(u) + \left(\frac{u-a}{v-a}\right)F_4(v)\right] \\ \qquad \times \left[\frac{(v-a)(u-b)}{(v-u)(a-b)}\right], & x \in [u, b] \end{cases} \tag{3.7}$$

$$= \begin{cases} F_3(x) + \lambda(A_3; x, v)K_1, \\ F_4(x) + \lambda(A_4; x, u)K_2. \end{cases}$$

We must show that F is well-defined on $[u, v]$. Since f is integrable on $[u, v]$ with respect to the basis A_5 we have from (3.5), (3.6) and (3.7),

$$F(x) = \begin{cases} F_5(x) + \lambda(A_5; x, u)F_3(u) + \lambda(A_3; x, v)K_1 \\ F_5(x) + \lambda(A_5; x, v)F_4(v) + \lambda(A_4; x, u)K_2, & \text{if } x \in [u, v]. \end{cases} \tag{3.8}$$

Since $u - c_i = v - c_{n-i+1}$, $i = 2, 3, \ldots, n - 1$, it is easy to see that

$$\begin{aligned} &\lambda(A_5; x, v)F_4(v) + \lambda(A_4; x, u)K_2 \\ &= g(x)\left\{\frac{(x-u)F_4(v)}{(v-u)} + \frac{(x-b)}{(u-b)}\left[F_3(u) + \left(\frac{u-a}{v-a}\right)F_4(v)\right] \times \left[\frac{(v-a)(u-b)}{(v-u)(a-b)}\right]\right\} \\ &= g(x)\left\{\frac{(x-v)F_3(u)}{(u-v)} + \left(\frac{x-a}{v-a}\right)\left[F_4(v) + \left(\frac{v-b}{u-b}\right)F_3(u)\right] \times \left[\frac{(u-b)(v-a)}{(v-u)(a-b)}\right]\right\} \\ &= \lambda(A_5; x, u)F_3(u) + \lambda(A_3; x, v)K_1, \end{aligned} \tag{3.9}$$

where $g(x) = \dfrac{(x-c_2)(x-c_3)\ldots(x-c_{n-1})}{(v-c_2)(v-c_3)\ldots(v-c_{n-1})}$.

LEMMA 3.1. *If f is integrable on $[a, v]$ and on $[u, b]$ and $u < c < v$, then corresponding to $\epsilon > 0$, there exists a major function $Q_1(x)$ on $[a, c]$ and a major function $Q_2(x)$ on $[c, b]$ such that if*

$$R_1(x) = Q_1(x) - F_1(x), \quad R_2(x) = Q_2(x) - F_2(x),$$

where $F_1(x)$ and $F_2(x)$ denote the associated integrals of f over $[a, c]$ and $[c, b]$, respectively, then

$$(3.10) \quad |R_1(x)| < \epsilon, \quad |R_2(x)| < \epsilon, \quad |R_{1(k)}(c-)| < \epsilon, \quad |R_{2(k)}(c+)| < \epsilon,$$

$k = 1, 2, \ldots, (n - 1)$. *Minor functions $q_1(x)$, $q_2(x)$ exist satisfying similar inequalities.*

Proof. Let $Q_3(x)$, $Q_4(x)$ be major functions on $[a, v]$ and $[u, b]$ respectively. Then

$$Q_1(x) = Q_3(x) - \sum_{i=1}^{n} \lambda(A_1; x, a_i)Q_3(a_i),$$

and

$$Q_2(x) = Q_4(x) - \sum_{i=1}^{n} \lambda(A_2; x, b_i)Q_4(b_i)$$

are major functions of f on $[a, c]$ and $[c, b]$ respectively. Now if

$$R_1(x) = Q_1(x) - F_1(x), \quad x \in [a, c],$$
$$R_2(x) = Q_2(x) - F_2(x), \quad x \in [c, b],$$

we may write

$$R_1(x) = Q_3(x) - F_3(x) - \sum_{i=1}^{n} \lambda(A_1; x, a_i)(Q_3(a_i) - F_3(a_i))$$
$$\equiv R_3(x) - \sum_{i=1}^{n} \lambda(A_1; x, a_i)R_3(a_i), \quad x \in [a, c],$$

and

$$R_2(x) = Q_4(x) - F_4(x) - \sum_{i=1}^{n} \lambda(A_2; x, b_i(Q_4(b_i) - F_4(b_i))$$
$$\equiv R_4(x) - \sum_{i=1}^{n} \lambda(A_2; x, b_i)R_4(b_i), \quad x \in [c, d].$$

Since $R_3(x)$ and $R_4(x)$ are n-convex on $[a, v]$ and $[u, b]$, respectively, then $R_{3(k)}(c)$; $R_{4(k)}(c)$, $1 \leqq k \leqq n - 2$, exist, as do $R_{3(n-1)}(c-)$ and $R_{4(n-1)}(c+)$. It follows that $R_{1(k)}(c-)$ and $R_{2(k)}(c+)$ exist for $1 \leqq k \leqq n - 1$. Moreover by Theorem 3.2 we may choose $R_3(x)$ and $R_4(x)$ so that all the one-sided derivatives of $R_1(x)$ and $R_2(x)$ satisfy the inequalities (3.10).

4. The main result. We are now ready to state and prove our theorem on the additivity of the P^n-integral.

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THEOREM 4.1. *The function f is P^n-integrable on $[a, b]$ if and only if f is P^n-integrable on $[a, v]$ and on $[u, b]$ where $a < u < v < b$. Moreover in the notation of the preceding section we have for $l_s \leqq x < l_{s+1}$, $s = 1, 2, \ldots, n-1$,*

$$(4.1)\qquad F(x) = \begin{cases} F_3(x) + \lambda(A_3; x, v)K_1, & a \leqq x \leqq v \\ F_4(x) + \lambda(A_4; x, u)K_2, & u \leqq x \leqq b, \end{cases}$$

where $F(x)$ denotes the associated integral of f on $[a, b]$ with respect to the basis $(c_0, c_2, \ldots, c_{n-1}, c_n') \equiv (l_1, l_2, \ldots, l_{n-1}, l_n)$.

Proof. The necessity of the condition follows from Theorem 2.5, and verification of (4.1) is a direct result of straightforward calculations. Indeed if $F(x)$ denotes the associated P^n-integral of f over $[a, b]$ then for $l_s \leqq x < l_{s+1}$,

$$(4.2)\qquad F(x) = \begin{cases} F_3(x) + \lambda(A_3; x, v)F(v), & a \leqq x \leqq v \\ F_4(x) + \lambda(A_4; x, u)F(u), & u \leqq x \leqq b \end{cases}$$

Now substituting $x = v$ in both equations of (4.2) and equating we obtain (since $F_3(v) = 0$, $(A_3; v, v) = 1$)

$$F_4(v) = F(v) - \lambda(A_4; v, u)F(u) = F(v) - \left(\frac{v-b}{u-b}\right)F(u).$$

Solving for $F(v)$ and substituting in the first equation of (4.2) gives

$$(4.3)\qquad F(x) = F_3(x) + \left(\frac{u-a}{v-a}\right)\left[F_4(v) + \left(\frac{v-b}{u-b}\right)F(u)\right], \quad a \leqq x \leqq v.$$

Substituting $x = u$ in (4.3) yields

$$F(u) = F_3(u) + \left(\frac{u-a}{v-a}\right)\left[F_4(v) + \left(\frac{v-b}{u-b}\right)F(u)\right],$$

from which we obtain

$$F(u) = \frac{F_3(u) + \left(\dfrac{u-a}{v-a}\right)F_4(v)}{1 - \left(\dfrac{u-a}{v-a}\right)\left(\dfrac{v-b}{u-b}\right)} = K_2.$$

A similar calculation may be made involving K_1.

To prove sufficiency we note first that for $u < c < v$ and $\epsilon > 0$ there exists by Lemma 3.1 a major function $Q_1(x)$ with respect to the basis A_1 on the interval $[a, c]$ and a major function $Q_2(x)$ with respect to the basis A_2 on the interval $[c, b]$ such that the functions $R_1(x) \equiv Q_1(x) - F_1(x)$ and $R_2(x) \equiv Q_2(x) - F_2(x)$ satisfy the following inequalities:

$$|R_1(x)| < \epsilon, \quad |R_2(x)| < \epsilon, \quad |R_{1(k)}(c-)| < \epsilon, \quad |R_{2(k)}(c+)| < \epsilon,$$

$k = 1, 2, \ldots, n-1$. Minor functions $q_1(x)$, $q_2(x)$ satisfying analogous inequalities may be defined similarly, Now define the functions R and r by

$$(4.4)\qquad R(x) = \begin{cases} R_1(x) + \sum_{j=1}^{n/2} \lambda(A_1; x, d_j)\alpha_j \equiv R_1(x) + U(x), & a \leqq x \leqq c \\ R_2(x) + \sum_{j=n/2}^{n-1} \lambda(A_2; x, d_j)\alpha_j \equiv R_2(x) + V(x), & c \leqq x \leqq b \end{cases}$$

$$(4.5)\qquad r(x) = \begin{cases} r_1(x) + \sum_{j=1}^{n/2} \lambda(A_1; x, d_j)\beta_j \equiv r_1(x) + u(x), & a \leqq x \leqq c \\ r_2(x) + \sum_{j=n/2}^{n-1} \lambda(A_2; x, d_j)\beta_j \equiv r_2(x) + v(x), & c \leqq x \leqq b, \end{cases}$$

where the constants α_j and β_j are to be determined so that $R_{(k)}(x)$, $r_{(k)}(x)$ exist at c, $k = 1, 2, \ldots, (n-2)$ and $R(x)$ and $r(x)$ are n-smooth at c. There are thus $(n-1)$ conditions to determine $(n-1)$ constants in each case (of course where the constants exist we will have the relations $\alpha_j = R(d_j)$ and $\beta_j = r(d_j)$).

From this point on we shall restrict our discussion to the function $R(x)$ — analogous statements and proofs hold for $r(x)$.

The first step in showing that the constants α_j with the required properties exist will be to show that the $(n-1)$ conditions mentioned above together with properties of $R_1(x)$ and $R_2(x)$ are equivalent to the condition that $R_{(n-1)}(c)$ exist.

The condition of n-smoothness for $R(x)$ at $x = c$ is given by

$$(4.6)\qquad \frac{1}{h^{n-1}}\left[\frac{R(c+h)+R(c-h)}{2} - \sum_{k=0}^{(n/2)-1} \frac{h^{2k}}{(2k)!} D^{2k}R(c)\right] \to 0, \quad \text{as } h \to 0.$$

Assuming the existence of $R_{(k)}(c)$, $k = 1, 2, \ldots, (n-2)$, the left hand side of (4.6) may be re-written as

$$\begin{aligned} &\frac{1}{2h^{n-1}}\left[R(c+h) - R(c) - \sum_{k=1}^{n-2} \frac{h^k}{k!} R_{(k)}(c+)\right] \\ &\quad - \frac{1}{2(-h)^{n-1}}\left[R(c-h) - R(c) - \sum_{k=1}^{n-2} \frac{(-h)^k}{k!} R_{(k)}(c-)\right] \\ &= \frac{1}{2h^{n-1}}\left[R_2(c+h) - R_2(c) - \sum_{k=1}^{n-2} \frac{h^k}{k!} R_{2(k)}(c+)\right] \\ &\quad + \frac{1}{2h^{n-1}}\left[V(c+h) - V(c) - \sum_{k=1}^{n-2} \frac{h^k}{k!} V_{(k)}(c+)\right] \\ &\quad - \frac{1}{2(-h)^{n-1}}\left[R_1(c-h) - R_1(c) - \sum_{k=1}^{n-2} \frac{(-h)^k}{k!} R_{1(k)}(c-)\right] \\ &\quad - \frac{1}{2(-h)^{n-1}}\left[U(c+h) - U(c) - \sum_{k=1}^{n-2} \frac{(-h)^k}{k!} U_{(k)}(c-)\right]. \end{aligned}$$

Since the one sided derivatives of the polynomials $U(x)$ and $V(x)$ always exist and since by construction $R_{2(n-1)}(c+)$ and $R_{1(n-1)}(c-)$ exist, it follows that if $R_{(k)}(c)$, $k = 1, 2, \ldots, (n-2)$ exist, n-smoothness of $R(x)$ at c is equivalent to the existence and equality of $R_{(n-1)}(c+)$ and $R_{(n-1)}(c-)$.

The next step is to obtain $(n-1)$ equations in the $(n-1)$ unknowns, α_j, by differentiating equation (4.4) $(n-1)$ – times and setting $R_{(k)}(c-) = R_{(k)}(c+)$, $k = 1, 2, \ldots, n-1$. This yields (where for simplicity in notation we write $\lambda_{(k)}(A_1; x, d_j)|_{x=c} \equiv g_{(k)}(d_j)$ and $\lambda_{(k)}(A_2; x, d_j)|_{x=c} \equiv h_{(k)}(d_j)$, $j = 1, 2, \ldots, n-1$, and $k = 1, 2, \ldots, n-1$),

$$\sum_{j=1}^{n/2} \alpha_j g_{(k)}(d_j) - \sum_{j=n/2}^{n-1} \alpha_j h_{(k)}(d_j) = R_{2(k)}(c+) - R_{1(k)}(c-) \equiv \theta_k,$$

$$(g_{(k)}(d_{n/2}) - h_{(k)}(d_{n/2}))\alpha_{n/2} + \sum_{j=1}^{(n/2)-1} \alpha_j g_{(k)}(d_j) - \sum_{j=(n/2)+1}^{n-1} \alpha_j h_{(k)}(d_j) = \theta_k,$$

$k = 1, 2, \ldots, n-1$.

To show that these $(n-1)$ equations have a unique solution we must show that the corresponding determinant is not zero, i.e., the determinant which has as its ith column the following

$$\begin{array}{c} g_{(i)}(d_{n/2}) - h_{(i)}(d_{n/2}) \\ g_{(i)}(d_1) \\ \cdots \\ g_{(i)}(d_{n/2-1}) \\ h_{(i)}(d_{n/2+1}) \\ \cdots \\ h_{(i)}(d_{n-1}), \end{array}$$

for $i = 1, 2, \ldots, n-1$. But this is clearly equivalent to the condition that the polynomials

$$\lambda(A_1; x, c) - \lambda(A_2; x, c), \quad \lambda(A_2; x, d_j), d_j > c, \quad \lambda(A_1; x, d_j), d_j < c, \tag{4.7}$$

be linearly independent on $[a, b]$.

If the polynomials (4.7) were linearly dependent there would exist constants γ_1, $i = 0, 1, 2, \ldots, (n-1)$, such that

$$\gamma_0 \lambda(A_1; x, c) + \sum_{j=1}^{(n/2)-1} \gamma_j \lambda(A_1; x, d_j) = \gamma_0 \lambda(A_2; x, c) + \sum_{j=(n/2)+1}^{n-1} \gamma_j \lambda(A_2; x, d_j), \quad x \in [a, b], \tag{4.8}$$

where each side of the expression is a polynomial of degree $n - 1$ at most. But then (4.8) may be rewritten (in the notation introduced in the previous section) as:

$$(x-a)m(x) \prod_{i=2}^{n/2} (x - c_i) = (x-b)n(x) \prod_{i=(n/2)+1}^{n-1} (x - c_i), \quad x \in [a, b]$$

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where $m(x)$ and $n(x)$ are polynomials of degree at most $(n/2) - 1$. Thus the left hand side has $n/2$ zeros and the right hand side has $n/2$ zeros, all distinct. This would imply that the left hand side which is a polynomial of degree at most $(n - 1)$, has at least n zeros, an impossibility. It follows that the polynomials (4.7) are linearly independent.

This shows that the constants $\alpha_j = R(d_j)$ may be determined so that $R(x)$ is n-smooth at c and so that the unsymmetric derivatives up to order $(n - 2)$ exist and are finite at c. Moreover

$$(4.9) \quad R(d_j) = \sum_{k=1}^{n-1} A_k{}^j(R_{2(k)}(c+) - R_{1(k)}(c-)),$$

and

$$(4.10) \quad r(d_j) = \sum_{k=1}^{n-1} A_k{}^j(r_{2(k)}(c+) - r_{1(k)}(c-)),$$

for each d_j, where the $A_k{}^j$ depends on $\{d_j\}$ but not on f, R_1, R_2, r_1, or r_2.

It is clear therefore that the original choice of $R_1(x)$ and $R_2(x)$, may be made so that

$$(4.11) \quad |R(x)| < \epsilon/2 \quad \text{and} \quad |r(x)| < \epsilon/2, \quad x \in [a, b].$$

Now we claim that $Q(x) = F(x) + R(x)$ is a P^n-major function for $f(x)$ on $[a, b]$ with respect to the basis $(a, c_2, c_3, \ldots, c_{n-1}, b)$.

Both $F(x)$ and $R(x)$ obviously satisfy conditions (2.4.2) of Definition 2.1. That they are continuous and possess derivatives $F_k(x)$, $R_k(x)$, $1 \leqq k \leqq n - 2$, follows from their definitions (cf. equations (3.7) and (4.4)). Moreover since we may write

$$(4.12) \quad Q(x) = \begin{cases} F_1(x) + R_1(x) + \Theta_1(x) = Q_1(x) + \Theta_1(x), & x \in [a, c] \\ F_2(x) + R_2(x) + \Theta_2(x) = Q_2(x) + \Theta_2(x), & x \in [c, b] \end{cases}$$

where $\Theta_1(x)$ and $\Theta_2(x)$ are polynomials of degree at most $(n - 1)$, $Q(x)$ inherits the required n-smoothness property of condition (2.4.1) from $Q_1(x)$ and $Q_2(x)$.

It may be shown also from (4.12) that $Q(x)$ satisfies conditions (2.4.3) and (2.4.4) of Definition 2.1. Since $F(x)$ satisfies condition (2.4.5) at $x = c$ [cf. (3.7) and Theorem 3.3] and $R(x)$, by definition, is n-smooth at $x = c$, it follows that $Q(x)$ satisfies condition (2.4.5) (i).

Similarly $q(x) = F(x) - r(x)$ can be shown to be a P^n-minor function for $f(x)$ on $[a, b]$ with respect to the basis $(a, c_2, c_3, \ldots, c_{n-1}, b)$.

Because of (4.11) we have furthermore that

$$|Q(x) - q(x)| = |R(x) - r(x)| < \epsilon, \quad x \in [a, b],$$

which completes the proof that $f(x)$ is P^n-integrable on $[a, b]$.

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Remark. The lemma corresponding to Skvorcov's Lemma 3 referred to in the introduction of this paper would say that if f is integrable on $[a, c]$ and on $[c, b]$ then functions $R_1(x)$ and $R_2(x)$ (as defined in Lemma 3.1) exist satisfying the inequalities (3.10). If such functions exist for integrable f our methods can be used to prove the result for P^n-integrals corresponding to Theorem 2 [**6**]:

Let $f(x)$ be P^n-integrable over $(a_i; x)$ with associated integral $F_1(x)$ and over $(b_i; x)$ with associated integral $F_2(x)$. Then $f(x)$ is P^n-integrable on $[a, b]$ if and only if there exist constants $\{\theta_j\}$, $j = 1, 2, \ldots, n - 1$ such that the function

$$F(x) = \begin{cases} F_1(x) + \sum_{j=1}^{n/2} \lambda(A_1; x, d_j)\theta_j, & a \leqq x \leqq c, \\ F_2(x) + \sum_{j=n/2}^{n-1} \lambda(A_2; x, d_j)\theta_j, & c \leqq x \leqq b \end{cases}$$

is n-smooth and possesses Peano unsymmetric derivatives up to order $n - 2$ at $x = c$. If such numbers exist then the function $F(x)$ is the associated P^n-integral of $f(x)$ over $(l_1, l_2, \ldots, l_n)$.

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