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A GENERALIZATION OF HILBERT'S THEOREM 94

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In this paper we shall prove the following theorem conjectured by Miyake in [3] (see also Jaulent [2]).

THEOREM. *Let k be a finite algebraic number field and K be an unramified abelian extension of k, then all ideals belonging to at least $[K: k]$ ideal classes of k become principal in K.*

Since the capitulation homomorphism is equivalently translated to a group-transfer of the galois group (see Miyake [3]), it is enough to prove the following group-theoretical verison:

THEOREM (The group-theoretical version). *Let H be a finite group and N be a normal subgroup of H containing the commutator subgroup H^c of H. Then $[H: N]$ divides the order of the kernel of the group-transfer $V_{H\to N}: H^{ab} \to N^{ab}$.*

Hilbert's theorem 94 and the principal ideal theorem immediately follow from our theorem.

§ 1. Notations and two lemmas

For a group H, we denote the commutator group of H by H^c, and the augmentation ideal of the integral group algebra $\mathbf{Z}[H]$ by I_H. Put also

$$H^{ab} = H/H^c\,,$$
$$\mathrm{Tr}_H = \sum_{g\in H} g \in \mathbf{Z}[H]\,,$$

and

$$A_H = \mathbf{Z}[H]/(\mathrm{Tr}_H)\,.$$

For a $\mathbf{Z}[H]$-module M, we denote the $\mathbf{Z}[H]$-submodule consisting of all the H-invariant elements of M by M^H and the Pontrjagin dual of M by

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$M^\wedge$. The $\mathbf{Z}[H]$-module generated by $v_1, \cdots, v_m \in M$ is denoted by $\langle v_1, \cdots, v_m \rangle$. We denote the cardinality of a finite set S by $^\sharp S$.

In this section we shall prove the following two lemmas:

LEMMA 1. *Let G be a finite abelian group and M be a monogenerated $\mathbf{Z}[G]$-module of finite order. Then the order of $H^{-1}(G, M)$ divides the order of $H^0(G, M)$.*

Proof. For a natural number r, we define a standard perfect pairing on the group algebra over the quotient ring $\mathbf{Z}/r\mathbf{Z}$,

$$\mathbf{Z}/r\mathbf{Z}[G] \times \mathbf{Z}/r\mathbf{Z}[G] \longrightarrow \mathbf{Q}/\mathbf{Z}$$

by $(g, h) = 1/r \cdot \delta_{g,h}$ for $g, h \in G$. Then for $v, w, w' \in \mathbf{Z}/r\mathbf{Z}[G]$, we can see

$$(uw, w') = (w, \operatorname{inv}(u) \cdot w'),$$

where $\operatorname{inv}: \mathbf{Z}[G] \cong \mathbf{Z}[G]$ is the inverted isomorphism given by $\operatorname{inv}(g) = g^{-1}$ for $g \in G$. Since $\mathbf{Z}/r\mathbf{Z}[G]$ is self-dual by this pairing, we have an injective homomorphism $i: M \hookrightarrow \oplus^m \mathbf{Z}/r\mathbf{Z}[G]$, by taking the dual of a $\mathbf{Z}/r\mathbf{Z}[G]$-presentation of rank m of $M^\wedge$ for some natural numbers r and m; here $\oplus^m \mathbf{Z}/r\mathbf{Z}[G]$ is a direct sum of m-copies of the algebra $\mathbf{Z}/r\mathbf{Z}[G]$. We define a perfect pairing

$$\overset{m}{\oplus} \mathbf{Z}/r\mathbf{Z}[G] \times \overset{m}{\oplus} \mathbf{Z}/r\mathbf{Z}[G] \longrightarrow \mathbf{Q}/\mathbf{Z}$$

by

$$(w, w') = \sum_{i=1}^{m} (w_i, w'_i),$$

where

$$w = (w_1, \cdots, w_m), \qquad w' = (w'_1, \cdots, w'_m) \in \overset{m}{\oplus} \mathbf{Z}/r\mathbf{Z}[G].$$

Take a generator $v = (v_1, \cdots, v_m) \in \oplus^m \mathbf{Z}/r\mathbf{Z}[G]$ of M. Then for $w = (w_1, \cdots, w_m) \in \oplus^m \mathbf{Z}/r\mathbf{Z}[G]$ and $a \in \mathbf{Z}[G]$,

$$\begin{aligned}
&(av, w) = 0 && (\forall a \in \mathbf{Z}[G]) \\
\Longleftrightarrow\ &((av_1, \cdots, av_m), (w_1, \cdots, w_m)) = 0 && (\forall a \in \mathbf{Z}[G]) \\
\Longleftrightarrow\ &\sum_{i=1}^{m} (av_i, w_i) = 0 && (\forall a \in \mathbf{Z}[G]) \\
\Longleftrightarrow\ &(a, \sum_{i=1}^{m} \operatorname{inv}(v_i) \cdot w_i) = 0 && (\forall a \in \mathbf{Z}[G]) \\
\Longleftrightarrow\ &\sum_{i=1}^{m} \operatorname{inv}(v_i) \cdot w_i = 0 .
\end{aligned}$$

Hence the orthogonal $M^{\perp}$ of M is given by

$$M^{\perp} = \operatorname{Ker}(\operatorname{inv}(v)\cdot : \bigoplus^{m} \mathbf{Z}/r\mathbf{Z}[G] \longrightarrow \mathbf{Z}/r\mathbf{Z}[G]),$$

where $\operatorname{inv}(v)\cdot$ is the homomorphism defined by

$$\operatorname{inv}(v)\cdot w = \sum_{i=1}^{m} \operatorname{inv}(v_i)\cdot w_i$$

for $w = (w_1, \cdots, w_m) \in \bigoplus^m \mathbf{Z}/r\mathbf{Z}[G]$. Then we have

$$M^{\wedge} \cong \operatorname{Im} \operatorname{inv}(v)\cdot ,$$

and

$$(M^G)^{\wedge} \cong \operatorname{Im} \operatorname{inv}(v)\cdot / I_G \operatorname{Im} \operatorname{inv}(v)\cdot .$$

Since we have $\operatorname{inv}(I_G) = I_G$, the isomorphism $\operatorname{inv}: \mathbf{Z}[G] \cong \mathbf{Z}[G]$ induces an isomorphism

$$(M^G)^{\wedge} \cong \operatorname{Im} v\cdot / I_G \operatorname{Im} v\cdot ,$$

where $v\cdot : \bigoplus^m \mathbf{Z}/r\mathbf{Z}[G] \rightarrow \mathbf{Z}/r\mathbf{Z}[G]$ is the homomorphism given by

$$v\cdot w = \sum_{i=1}^{m} v_i \cdot w_i$$

for $w = (w_1, \cdots, w_m) \in \bigoplus^m \mathbf{Z}/r\mathbf{Z}[G]$.

Put

$$q = {}^{\#}\operatorname{Im} v\cdot / I_G \operatorname{Im} v\cdot .$$

Then we have

$$\begin{aligned} q &= {}^{\#}\operatorname{Im} v\cdot / I_G \operatorname{Im} v\cdot \\ &= {}^{\#}\operatorname{Im} \operatorname{inv}(v)\cdot / I_G \operatorname{Im} \operatorname{inv}(v)\cdot \\ &= {}^{\#}(M^G)^{\wedge} \\ &= {}^{\#}M^G . \end{aligned}$$

Now there exist two matrices $U \in \mathrm{M}(m, \mathbf{Z})$ and $J \in \mathrm{M}(m, I_G)$ such that

$$vU = vJ \quad \text{and} \quad \det U = q ,$$

because $\operatorname{Im} v\cdot = \langle v_1, \cdots, v_m \rangle = \mathbf{Z}v_1 + \cdots + \mathbf{Z}v_m + I_G \operatorname{Im} v\cdot$, and $I_G \operatorname{Im} v\cdot = I_G v_1 + \cdots + I_G v_m$. Therefore we have

$$\det(U - J)v = 0 \qquad \text{in } \bigoplus_{i=1}^{m} \mathbf{Z}/r\mathbf{Z}[G] .$$

This implies

$$q \cdot M/I_G M = 0\,,$$

because $\det(U - J) \equiv \det U \equiv q \bmod I_G$. Since $M = \mathbf{Z}[G]v = \mathbf{Z}v + I_G M$, the order of $M/I_G M$ divides $q = {}^{\#}M^G$. Furthermore we have

$${}^{\#}M/\mathrm{Ker}\,(\mathrm{Tr}_G\colon M \longrightarrow M) = {}^{\#}\mathrm{Tr}_G\, M\,,$$

because ${}^{\#}M$ is finite. Therefore

$$\begin{aligned} {}^{\#}H^0(G, M) &= q/{}^{\#}\mathrm{Tr}_G\, M \\ &= {}^{\#}\mathrm{Ker}\,(\mathrm{Tr}_G\colon M \longrightarrow M)/I_G M \cdot q/{}^{\#}M/I_G M \\ &= {}^{\#}H^{-1}(G, M) \cdot q/{}^{\#}M/I_G M \end{aligned}$$

is divisible by ${}^{\#}H^{-1}(G, M)$.

LEMMA 2. *Let G be a finite abelian group, and put $n = {}^{\#}G$ and $A_G = \mathbf{Z}[G]/(\mathrm{Tr}_G)$. Then for any m-generated $\mathbf{Z}[G]$-submodule Y of $\bigoplus^{m-1} A_G \otimes_{\mathbf{Z}} \mathbf{Q}$, the order of $Y/I_G Y$ divides n^{m-1}.*

Proof. Let $\{y_1, \cdots, y_m\}$ be a set of generators of Y. For each maximal ideal $\mathfrak{m}$ of $A_G \otimes_{\mathbf{Z}} \mathbf{Q}$, take an element $c_{\mathfrak{m}} \in A_G \backslash \mathfrak{m}$ which belongs to all the other maximal ideals of $A_G \otimes_{\mathbf{Z}} \mathbf{Q}$. Then $c_{\mathfrak{m}}$ becomes 0 at every maximal ideal except $\mathfrak{m}$. If, for some $\mathfrak{m}$,

$$(\langle y_1, \cdots, y_{m-1} \rangle \otimes_{\mathbf{Z}} \mathbf{Q})_{\mathfrak{m}} \neq (Y \otimes_{\mathbf{Z}} \mathbf{Q})_{\mathfrak{m}}\,,$$

the $(A_G \otimes_{\mathbf{Z}} \mathbf{Q})_{\mathfrak{m}}$-dimension of the space in the left hand is less than $m-1$. If we take an omissible $(A_G \otimes_{\mathbf{Z}} \mathbf{Q})_{\mathfrak{m}}$-generator and put $i = i(\mathfrak{m})$, then we have

$$(\langle y_1, \cdots, y_{i-1}, y_i + c_{\mathfrak{m}} y_m, y_{i+1}, \cdots, y_{m-1} \rangle \otimes_{\mathbf{Z}} \mathbf{Q})_{\mathfrak{m}} = (Y \otimes_{\mathbf{Z}} \mathbf{Q})_{\mathfrak{m}}\,,$$

and we may change the generator y_i to $y_i + c_{\mathfrak{m}} y_m$. Thus we may assume

$$(\langle y_1, \cdots, y_{m-1} \rangle \otimes_{\mathbf{Z}} \mathbf{Q})_{\mathfrak{m}} = (Y \otimes_{\mathbf{Z}} \mathbf{Q})_{\mathfrak{m}}$$

for every $\mathfrak{m}$, namely

$$\langle y_1, \cdots, y_{m-1} \rangle \otimes_{\mathbf{Z}} \mathbf{Q} = Y \otimes_{\mathbf{Z}} \mathbf{Q}\,.$$

Let $\pi\colon \bigoplus^{m-1} A_G \otimes_{\mathbf{Z}} \mathbf{Q} \to Y \otimes_{\mathbf{Z}} \mathbf{Q}$ be the $\mathbf{Z}[G]$-homomorphism which maps the standard i-th generator $\bar{e}_i = (0, \cdots, 0, 1, 0, \cdots, 0)$ to y_i for every $i = 1, \cdots, m-1$. Take an element $y \in \bigoplus^{m-1} A_G \otimes_{\mathbf{Z}} \mathbf{Q}$ such that $\pi(y) = y_m$, and put

$$Y' = \langle \bar{e}_1, \cdots, \bar{e}_{m-1}, y \rangle \subseteq \bigoplus^{m-1} A_G \otimes_{\mathbf{Z}} \mathbf{Q}\,.$$

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Then $\pi(Y') = Y$ shows that the order $^{\#}Y/I_G Y$ divides the order $^{\#}Y'/I_G Y'$. Now taking Y' in place of Y, we may further assume that

$$y_i = \bar{e}_i$$

is the standard i-th generator of $\bigoplus^{m-1} A_G$ for each $i = 1, \cdots, m-1$, and the last element

$$y_m = y$$

is an arbitrary element of $\bigoplus^{m-1} A_G \otimes_{\mathbf{Z}} \mathbf{Q}$. Now we may naturally identify $A_G \otimes_{\mathbf{Z}} \mathbf{Q}$ with the direct summand $I_G \otimes_{\mathbf{Z}} \mathbf{Q}$ of $\mathbf{Q}[G]$; its unit element is

$$e = 1 - 1/n \cdot \mathrm{Tr}_G = \sum_{g \in G} - 1/n \cdot (g - 1)\,.$$

Let

$$\mathrm{pr}: \bigoplus^{m-1} A_G \otimes_{\mathbf{Z}} \mathbf{Q} \longrightarrow \bigoplus^{m-1} A_G \otimes_{\mathbf{Z}} \mathbf{Q} / \bigoplus^{m-1} I_G = \bigoplus^{m-1} I_G \otimes_{\mathbf{Z}} \mathbf{Q}/\mathbf{Z}$$

be the natural projection. In a direct forward way, it is easy to see that

$$(\bigoplus^{m-1} I_G \otimes_{\mathbf{Z}} \mathbf{Q}/\mathbf{Z})^G = \langle \mathrm{pr}\,(\bar{e}_1), \cdots, \mathrm{pr}\,(\bar{e}_{m-1}) \rangle \cong \bigoplus^{m-1} \mathbf{Z}/n\mathbf{Z}\,.$$

In particular $I_G \langle \mathrm{pr}\,(\bar{e}_1), \cdots, \mathrm{pr}\,(\bar{e}_{m-1}) \rangle = 0$. Let M be the $\mathbf{Z}[G]$-submodule of $\bigoplus^{m-1} I_G \otimes_{\mathbf{Z}} \mathbf{Q}/\mathbf{Z}$ generated by the single element $\mathrm{pr}\,(y)$. Then we have

$$\begin{aligned}
^{\#}Y/I_G Y &= {}^{\#}\mathrm{pr}\,(Y)/I_G\, \mathrm{pr}\,(Y) \\
&= {}^{\#}(M + \langle \mathrm{pr}\,(\bar{e}_1), \cdots, \mathrm{pr}\,(\bar{e}_{m-1}) \rangle)/I_G M \\
&= {}^{\#}(M + (\bigoplus^{m-1} I_G \otimes_{\mathbf{Z}} \mathbf{Q}/\mathbf{Z})^G)/I_G M \\
&= {}^{\#}M/I_G M \cdot {}^{\#}(M + (\bigoplus^{m-1} I_G \otimes_{\mathbf{Z}} \mathbf{Q}/\mathbf{Z})^G)/M \\
&= {}^{\#}M/I_G M \cdot {}^{\#}(\bigoplus^{m-1} I_G \otimes_{\mathbf{Z}} \mathbf{Q}/\mathbf{Z})^G / {}^{\#}M \cap (\bigoplus^{m-1} I_G \otimes_{\mathbf{Z}} \mathbf{Q}/\mathbf{Z})^G \\
&= n^{m-1} \cdot {}^{\#}H^{-1}(G, M)/{}^{\#}H^0(G, M)\,.
\end{aligned}$$

Since M is a monogenerated $\mathbf{Z}[G]$-module of finite order, Lemma 1 implies Lemma 2.

§ 2. Proof of the theorem

2.1. Put $G = H/N$. We may assume that G is an abelian p-group, for some rational prime number p. Put $n = {}^{\#}G$.

Let $(f_{g,h})$ be a 2-cocycle in the cohomology class of the group extension

$$1 \longrightarrow N^{ab} \longrightarrow H/N^c \longrightarrow G \longrightarrow 1 .$$

Let $\{x_g \mid g \in G\backslash\{1\}\}$ be a set of symbols parametrized by $G\backslash\{1\}$, and W be the $\mathbf{Z}[G]$-module

$$N^{ab} \oplus \Big(\bigoplus_{g \in G\backslash\{1\}} \mathbf{Z}\cdot x_g \Big)$$

with group action

$$g\cdot x_h = x_{g\cdot h} - x_g + f_{g,h} \qquad (g, h \in G) .$$

Then we have an exact sequence

$$0 \longrightarrow N^{ab} \longrightarrow W \longrightarrow I_G \longrightarrow 0$$

by assigning $g - 1 \in I_G$ to x_g for $g \in G\backslash\{1\}$; furthermore we also have $W/I_G W \cong H^{ab}$; and the trace homomorphism Tr_G: $W/I_G W \to N^{ab}$ coincides with the group-transfer $V_{H\to N}$: $H^{ab} \to N^{ab}$ (see Artin-Tate [1] and Miyake [3], § 3, for example). Therefore it is enough to show ${}^{\#}H^{-1}(G, W) \geq n$.

Let

$$H^{ab} = W/I_G W \cong \bigoplus_{i=1}^{m} \mathbf{Z}/q_i\mathbf{Z}$$

and take a $\mathbf{Z}[G]$-homomorphism φ: $\oplus^m \mathbf{Z}[G] \to W$ which maps the i-th generator $e_i = (0, \cdots, 0, 1, 0, \cdots, 0)$ of $\oplus^m \mathbf{Z}[G]$ to a representative of the i-th generator $h_i = (0, \cdots, 0, 1, 0, \cdots, 0)$ of $\oplus_{i=1}^m \mathbf{Z}/q_i\mathbf{Z}$. Then we have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ker}\,\mathrm{nat}\circ\varphi & \longrightarrow & \bigoplus^{m} \mathbf{Z}[G] & \xrightarrow{\mathrm{nat}\circ\varphi} & I_G & & \\
 & & \downarrow & & \downarrow{\scriptstyle\varphi} & & \| & & \\
0 & \longrightarrow & N^{ab} & \longrightarrow & W & \xrightarrow{\mathrm{nat}} & I_G & \longrightarrow & 0
\end{array}$$

with exact rows. Moreover Nakayama's lemma shows that the localization of φ at (p) is surjective. Namely the cokernel of φ is a $\mathbf{Z}[G]$-module of finite order s prime to p. Hence there exists an element $u_i \in \mathrm{Ker}\,\varphi$ such that $u_i \equiv s\cdot q_i\cdot e_i \bmod \oplus^m I_G$ for each $i = 1, \cdots, m$. Put $U = \langle u_1, \cdots, u_m\rangle$, and denote the p-primary part of a finite $\mathbf{Z}[G]$-module A by A_p in general. Then identifying by the isomorphism $(\oplus^m \mathbf{Z}[G]/(U + \oplus^m I_G))_p \cong (W/I_G W)_p$ induced by φ, we have

$$H^{-1}(G, \overset{m}{\oplus} \mathbf{Z}[G]/U) = \operatorname{Ker}(\operatorname{Tr}_G : \overset{m}{\oplus} \mathbf{Z}[G]/(U + \overset{m}{\oplus} I_G) \longrightarrow \operatorname{Ker} \operatorname{nat} \circ \varphi/U)_p$$
$$\subseteq \operatorname{Ker}(\operatorname{Tr}_G : W/I_G W \longrightarrow N^{ab})_p .$$

Therefore it is enough to show $^{\#}H^{-1}(G, \oplus^m \mathbf{Z}[G]/U) \geq n = {}^{\#}G$. Put $\tau = \operatorname{nat} \circ \varphi$, and $t_i = s \cdot q_i$ for each i.

2.2. The $\mathbf{Z}[G]$-homomorphism $\tau : \oplus^m \mathbf{Z}[G] \to I_G$ has a finite cokernel. Therefore $I_G \operatorname{Im} \tau$ is also of finite index in I_G. Since

$$0 \longrightarrow \operatorname{Ker} \tau \cap \overset{m}{\oplus} I_G \longrightarrow \overset{m}{\oplus} I_G \longrightarrow I_G \operatorname{Im} \tau \longrightarrow 0$$

is exact and $I_G \otimes_{\mathbf{Z}} \mathbf{Q} = A_G \otimes_{\mathbf{Z}} \mathbf{Q}$ is a finite direct sum of finite field extensions of $\mathbf{Q}$, we have

$$(2.2.1) \qquad (\operatorname{Ker} \tau \cap \overset{m}{\oplus} I_G) \otimes_{\mathbf{Z}} \mathbf{Q} \cong \overset{m-1}{\oplus} I_G \otimes_{\mathbf{Z}} \mathbf{Q} = \overset{m-1}{\oplus} A_G \otimes_{\mathbf{Z}} \mathbf{Q} .$$

In particular Lemma 2 *holds for* $\operatorname{Ker} \tau \cap \oplus^m I_G$ *in place of* $\oplus^{m-1} A_G \otimes_{\mathbf{Z}} \mathbf{Q}$.

We are now in the following situation.

(2.2.2) *We may assume that there exist a natural number* t_i *and an element* u_i *of* $\operatorname{Ker} \tau$ *such that* $u_i \equiv t_i \cdot e_i \bmod \oplus^m I_G$ *for each* $i = 1, \cdots, m$, *where* e_i *is the standard i-th generator of* $\oplus^m \mathbf{Z}[G]$. *Put* $U = \langle u_1, \cdots, u_m \rangle$ *and* $W_0 = \oplus^m \mathbf{Z}[G]/U$.

Now it is enough to prove the following:

LEMMA 3. *Under the situation* (2.2.2), *the order* n *of* G *divides the order of* $H^{-1}(G, W_0)$.

Proof. Since we have

$$\begin{aligned} H^{-1}(G, W_0) &\cong H^0(G, U) \\ &\cong H^0(G, nU) \\ &\cong H^{-1}(G, \overset{m}{\oplus} \mathbf{Z}[G]/nU) , \end{aligned}$$

we may take nU instead of U. In particular, we may assume that n divides t_i for every i. Put $d_i = t_i/n$.

The fact $\operatorname{Tr}_G \equiv n \bmod I_G$ shows that $\operatorname{Ker} \operatorname{Tr}_G \cap W_0/I_G W_0 \subseteq {}_n(W_0/I_G W_0)$, where ${}_nA$ means the submodule consisting of all the elements of A of order dividing n. By the assumption $n | t_i$, ${}_n(W_0/I_G W_0)$ is isomorphic to $\oplus^m \mathbf{Z}/n\mathbf{Z}$ and generated by the elements $d_i \cdot e_i$; $i = 1, \cdots, m$. Put $y_i = d_i \cdot \operatorname{Tr}_G \cdot e_i - u_i$ for each $i = 1, \cdots, m$, and let Y be the $\mathbf{Z}[G]$-module generated by all the y_i. Then we have

$$Y = \langle y_1, \cdots, y_m \rangle \subseteq \overset{m}{\oplus} I_G \cap \operatorname{Ker} \tau ,$$

and $I_G Y = I_G U$. By the choice of u_i, we also have

$$\begin{aligned} U/U \cap \overset{m}{\oplus} I_G &\cong U + \overset{m}{\oplus} I_G / \overset{m}{\oplus} I_G \\ &\cong \overset{m}{\oplus} \mathbf{Z} \cong U/I_G U . \end{aligned}$$

Therefore $U \cap \oplus^m I_G$ must coincide with $I_G U = I_G Y$, because $I_G U \subseteq U \cap \oplus^m I_G$. By the following identification

$$\begin{aligned} (\operatorname{Ker} \tau \cap (U + \overset{m}{\oplus} I_G))/U &\cong \operatorname{Ker} \tau \cap \overset{m}{\oplus} I_G / U \cap \overset{m}{\oplus} I_G \cap \operatorname{Ker} \tau \\ &= (\operatorname{Ker} \tau \cap \overset{m}{\oplus} I_G)/I_G Y , \end{aligned}$$

we have the commutative diagram

$$\begin{array}{ccccc} {}_n(W_0/I_G W_0) & \xrightarrow{\operatorname{Tr}_G} & (\operatorname{Ker} \tau \cap (U + \oplus^m I_G))/U & \hookrightarrow & \operatorname{Ker} \tau / U \\ \Vert & & \uparrow \cong & & \\ & & \operatorname{Ker} \tau \cap \oplus^m I_G / I_G Y & & \\ \Vert & & \uparrow & & \\ \oplus^m \mathbf{Z}/n\mathbf{Z} & \xrightarrow{\eta} & Y/I_G Y , & & \end{array}$$

where η is the $\mathbf{Z}[G]$-homomorphism which maps the standard i-th generator $(0, \cdots, 0, 1, 0, \cdots, 0)$ of $\oplus^m \mathbf{Z}/n\mathbf{Z}$ to $y_i \bmod I_G Y$. Then we have

$$\operatorname{Ker} (\operatorname{Tr}_G \colon W_0/I_G W_0 \longrightarrow \operatorname{Ker} \tau / U) = \operatorname{Ker} \eta .$$

Since Y is a m-generated submodule of $\operatorname{Ker} \tau \cap \oplus^m I_G$, (2.2.1) shows that the order ${}^\# Y/I_G Y$ divides n^{m-1}. Since we have

$$\begin{aligned} {}^\# H^{-1}(G, W_0) &= {}^\# \operatorname{Ker} \eta \\ &= n^m / {}^\# (Y/I_G Y) , \end{aligned}$$

the order of $H^{-1}(G, W_0)$ is certainly divided by n. Q.E.D.

Thus our theorem is also proved.

Remark. In the above proof, it is easy to see that there exists a finite group H such that ${}^\# \operatorname{Ker} V_{H \to N} = [H \colon N]$, if each q_i is divisible by n.

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