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EQUALITY OF DECOMPOSABLE SYMMETRIZED TENSORS

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Let V be an n-dimensional vector space over the field F. Let $\otimes^m V$ be the mth tensor power of V. If $\sigma \in S_m$, the symmetric group, there exists a linear operator $P(\sigma^{-1})$ on $\otimes^m V$ such that

$$P(\sigma^{-1})\, x_1 \otimes \ldots \otimes x_m = x_{\sigma(1)} \otimes \ldots \otimes x_{\sigma(m)},$$

for all $x_1, \ldots, x_m \in V$. (Here, $x_1 \otimes \ldots \otimes x_m$ denotes the decomposable tensor product of the indicated vectors.) If c is any function of S_m taking its values in F, we define

(1) $$\theta = \sum_{\sigma \in S_m} c(\sigma) P(\sigma).$$

The linear operator θ on $\otimes^m V$ is called a *symmetrizer*. Symmetrizers provide the vehicle for connecting the irreducible representations of S_m with those of the full linear group [**1**]. In the form

(2) $$\frac{\lambda(\mathrm{id})}{o(G)} \sum_{\sigma \in G} \lambda(\sigma) P(\sigma),$$

where G is a subgroup of S_m and λ is an irreducible F-character of G, symmetrizers have proved useful in the discovery of inequalities for certain matrix functions (e.g. [**3**]). In this latter connection, the following questions arise very naturally: Let

(3) $$x_1 * \ldots * x_m = \theta\, x_1 \otimes \ldots \otimes x_m.$$

For which vectors $x_1, \ldots, x_m \in V$, is it the case that $x_1 * \ldots * x_m = 0$? Moreover, when can it happen that $x_1 * \ldots * x_m = y_1 * \ldots * y_m \neq 0$? (Naturally, such information is very important to the study of these *decomposable symmetrized tensors* (3). Surpringly, the answers are not known in general.)

1. *Example.* If $G = S_m$, and λ is the alternating character in (2), the range of θ is the space of skew symmetric tensors. In this case, $x_1 * \ldots * x_m$ is commonly written $x_1 \wedge \ldots \wedge x_m$. It is a classical result that $x_1 \wedge \ldots \wedge x_m \neq 0$ if (and only if) $x_1, \ldots, x_m$ are linearly independent. Moreover, if $x_1 \wedge \ldots \wedge x_m = y_1 \wedge \ldots \wedge y_m \neq 0$, then $\langle x_1, \ldots, x_m \rangle = \langle y_1, \ldots, y_m \rangle$, i.e., the space spanned by $x_1, \ldots, x_m$ is the same as the space spanned by $y_1, \ldots, y_m$.

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Recently, Marcus and Gordon [**3**, Lemma 1] extended the above result as follows: Let F be the field of complex numbers. Let θ be defined by (2), where λ is a linear character on G ($\lambda(\mathrm{id}) = 1$). *If* $x_1 * \ldots * x_m = y_1 * \ldots * y_m$, $m < n$, *and if* $\{x_1, \ldots, x_m\}$ *is a linearly independent set, then* $\langle x_1, \ldots, x_m \rangle = \langle y_1, \ldots, y_m \rangle$.

In his book [**2**, p. 136], Marcus reproves the result in the more general case that F is an arbitrary field of characteristic 0. He also makes clear that if $m \leqq n$ *and* $x_1 * \ldots * x_m = 0$, *then* $x_1, \ldots, x_m$ *are linearly dependent.*

In this note, we extend the classical skew symmetric theorem still further.

2. THEOREM. *Let F be an arbitrary field. Let $c : S_m \to F$ be an arbitrary function. Let θ ve defined as in* (1). *If* $x_1 * \ldots * x_m = y_1 * \ldots * y_m \neq 0$, *then* $\langle x_1, \ldots, x_m \rangle = \langle y_1, \ldots, y_m \rangle$. *Moreover, if c is not identically zero, and if* $x_1, \ldots, x_m$ *are linearly independent, then* $x_1 * \ldots * x_m \neq 0$.

Proof. We will make use of the fact that the dual space of the space of m-linear functionals on V is a model for $\otimes^m V$, in which

$$x_1 \otimes \ldots \otimes x_m(\phi) = \phi(x_1, \ldots, x_m).$$

Suppose first that $x_1 * \ldots * x_m = y_1 * \ldots * y_m \neq 0$. Let $W = \langle x_1, \ldots, x_m \rangle$. Since $x_1 * \ldots * x_m \neq 0$, there exists an m-linear $\phi : W \times \ldots \times W \to F$ such that $x_1 * \ldots * x_m(\phi) \neq 0$. Since every m-linear ϕ is a linear combination of products of linear functionals, there exist $f_1, \ldots, f_m$ in the dual space of W such that

$$x_1 * \ldots * x_m \left(\prod_{t=1}^{m} f_t \right) \neq 0.$$

Now, if $y_i \notin W$, we may extend each f_t to $\langle W, y_i \rangle$ by defining $f_t(y_i) = 0$, $1 \leqq t \leqq m$. Then

$$\begin{aligned} 0 &\neq x_1 * \ldots * x_m \left(\prod_{t=1}^{m} f_t \right) \\ &= y_1 * \ldots * y_m \left(\prod_{t=1}^{m} f_t \right) \\ &= \sum_{\sigma \in S_m} c(\sigma^{-1}) \prod_{t=1}^{m} f_t(y_{\sigma(t)}) \\ &= 0, \end{aligned}$$

since for each σ there is a t such that $\sigma(t) = i$, and $f_t(y_i) = 0$. This contradiction proves that $\langle y_1, \ldots, y_m \rangle \subset \langle x_1, \ldots, x_m \rangle$. Clearly the proof is symmetric.

Suppose, now, that $x_1, \ldots, x_m$ are linearly independent. Then

$$\{x_{\sigma(1)} \otimes \ldots \otimes x_{\sigma(m)} : \sigma \in S_m\}$$

is a linearly independent set. Thus,

$$x_1 * \ldots * x_m = \sum_{\sigma \in S_m} c(\sigma^{-1}) x_{\sigma(1)} \otimes \ldots \otimes x_{\sigma(m)} = 0$$

if and only if $c(\sigma) = 0$ for all $\sigma \in S_m$.

References

1. H. Boerner, *Representations of groups* (American Elsevier, 1970).

2. M. Marcus, *Finite dimensional multilinear algebra*, Part I (Marcel Dekker, 1973).

3. M. Marcus and W. R. Gordon, *Rational tensor representations of* Hom (V, V) *and an extension of an inequality of I. Schur*, Can. J. Math. *24* (1972), 686–695.

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