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RATIONAL TENSOR REPRESENTATIONS OF Hom(V, V) AND AN EXTENSION OF AN INEQUALITY OF I. SCHUR.

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1. Introduction. Let V be an n-dimensional vector space over the complex numbers equipped with an inner product (x, y), and let (P, μ) be a symmetry class in the mth tensor product of V associated with a permutation group G and a character χ (see below). Then for each $T \in \mathrm{Hom}(V, V)$ the function φ which sends each m-tuple $(v_1, \ldots, v_m)$ of elements of V to the tensor $\mu(Tv_1, \ldots, Tv_m)$ is symmetric with respect to G and χ, and so there is a unique linear map $K(T)$ from P to P such that $\varphi = K(T)\mu$.

It is easily checked that K: $\mathrm{Hom}(V, V) \to \mathrm{Hom}(P, P)$ is a rational representation of the multiplicative semi-group in $\mathrm{Hom}(V, V)$: for any two linear operators S and T on V

$$K(ST) = K(S)K(T).$$

Moreover, if T is normal then, with respect to the inner product induced on P by the inner product on V (see below), $K(T)$ is normal.

In this paper we prove

THEOREM 1. *If S and T are in* $\mathrm{Hom}(V, V)$ *and* rank $T > m$, *then* $K(T) = K(S)$ *if and only if $T = cS$ for some m*th *root of unity, c.*

THEOREM 2. *If $T \in Hom(V, V)$ and* rank $T > m$, *then $K(T)$ is normal if and only if T is normal.*

By considering an $n \times n$ complex matrix as a linear operator on complex n-tuple space, we have

THEOREM 3. *If A is an $n \times n$ complex matrix with* rank $A = m$ *and if $K(A)$ is normal, then A is unitarily similar to the direct sum of a non-singular $m \times m$ upper triangular matrix and the $(n - m) \times (n - m)$ zero matrix.*

We shall show in § 4 how these results can be easily applied to produce the following interesting theorem which was announced recently by R. Kess, H. L. de Vries, and R. Wegmann [**1**].

THEOREM 4. *If A is a non-normal $n \times n$ complex matrix with eigenvalues*

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$\lambda_1, \ldots, \lambda_n$, *if* $D = AA^* - A^*A$, *and if* $\|\ \|$ *denotes the usual Euclidean matrix norm, then*

$$\sum_{i=1}^{n} |\lambda_i|^2 \leqq (\|A\|^4 - \tfrac{1}{2}\|D\|^2)^{\frac{1}{2}} \tag{1}$$

with equality if and only if

$$A = \alpha(vw^* + rwv^*), \tag{2}$$

where α *is a non-zero complex number,* r *is a real number,* $0 \leqq r < 1$, *and where* v *and* w *are orthonormal complex n-tuples.*

It will be seen from our proof of this theorem that inequality (1) is an application of Schur's well known inequality [**2**] to the appropriate transformations.

2. Definitions and notation. Throughout this paper, V will be a finite-dimensional inner product space over the complex numbers **C**, dim $V = n$, G a subgroup of S_m, the symmetric group of degree m, and χ a character of degree 1 on G, i.e., a homomorphism of G into the unit circle. If V is a vector space over **C**, and $\varphi(v_1, \ldots, v_m)$ is an m-multilinear function on the cartesian product $\times_1^m V$ to U, then φ is said to be symmetric with respect to G and χ if

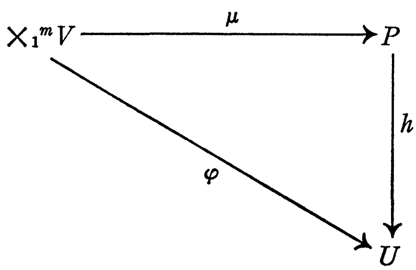
$$\varphi(v_{\sigma(1)}, \ldots, v_{\sigma(m)}) = \chi(\sigma)\varphi(v_1, \ldots, v_m),$$

for any $\sigma \in G$ and for arbitrary $v_i \in V$. By a symmetry class of tensors over V associated with G and χ we shall mean a pair (P, μ), consisting of a vector space P over **C** and an m-multilinear function μ: $\times_1^m V \rightarrow P$, symmetric with respect to G and χ, which is universal for these properties; that is;

(i) $\langle \text{rng } \mu \rangle = P$; i.e., the linear closure of the range of μ is P.

(ii) (Universal Factorization Property) For any vector space U over **C** and any m-multilinear function φ: $\times_1^m V \rightarrow U$, symmetric with respect to G and χ, there exists a linear h: $P \rightarrow U$ such that $\varphi = h\mu$.

(3)



The symmetry class (P, μ) is unique to within canonical isomorphisms, and the linear map h is uniquely determined by φ. The element $\mu(v_1, \ldots, v_m) \in P$ is called decomposable and will sometimes be denoted by $v_1 * \ldots * v_m$. The three most familiar symmetry classes are: (i) the space of m-contravariant tensors, $P = \otimes_1^m V$, $\mu(v_1, \ldots, v_m) = v_1 \otimes \ldots \otimes v_m$, i.e., $G = \{e\}$; (ii) the mth exterior power of V, $P = \wedge^m V$, $\mu(v_1, \ldots, v_m) = v_1 \wedge \ldots \wedge v_m$, i.e., $G = S_m$ and $\chi = \text{sgn} = \epsilon$; (iii) the mth completely symmetric space over V, $P = V^{(m)}$, $\mu(v_1, \ldots, v_m) = v_1 \ldots v_m$, i.e., $G = S_m$ and $\chi \equiv 1$.

Any symmetry class of tensors (P, μ) can be realized as a subspace of $\otimes_1^m V$ by defining

$$\mu(v_1, \ldots, v_m) = \frac{1}{|G|} \sum_{\sigma \in G} \chi(\sigma) v_{\sigma^{-1}(1)} \otimes \ldots \otimes v_{\sigma^{-1}(m)}.$$

In order to describe a basis for an arbitrary symmetry class associated with G and χ, we regard the elements of G as permutations acting on the set of all sequences of length m chosen from the integers $1, \ldots, n$. That is, $\Gamma_n^m = Z_n^{Z_m}$, where $Z_m = \{1, \ldots, m\}$ and for $\sigma \in G$, $\gamma \in \Gamma_n^m$

$$\sigma(\gamma)(t) = \gamma(\sigma^{-1}(t)), \qquad t \in Z_m.$$

Let Δ denote a system of distinct representatives for the orbits in Γ_n^m induced by G, and let $\bar{\Delta}$ denote the set of all those elements $\gamma \in \Delta$ for which the character χ is identically 1 on the stabilizer subgroup $G_\gamma = \{\sigma \in G | \sigma(\gamma) = \gamma\}$. Let $\nu(\gamma) = |G_\gamma|$. It is a routine exercise to verify that if $\{e_1, \ldots, e_n\}$ is a basis of V, then the decomposable elements $e_\gamma^* = e_{\gamma(1)} * \ldots * e_{\gamma(m)}$, $\gamma \in \bar{\Delta}$, form a basis of P. In fact, if $\{e_1, \ldots, e_n\}$ is an orthonormal (hereafter abbreviated o.n.) basis of V, then the $|\bar{\Delta}|$ decomposable elements $(|G|/\nu(\gamma)^{\frac{1}{2}})e_\gamma^*$, $\gamma \in \bar{\Delta}$, form an o.n. basis for P with respect to the induced inner product in $\otimes_1^m V$ defined by

$$(x_1 \otimes \ldots \otimes x_m, y_1 \otimes \ldots \otimes y_m) = \prod_{i=1}^{m} (x_i, y_i).$$

In general, if $x_i = \sum_{j=1}^{n} c_{ij} e_j$, $i = 1, \ldots$, m, then the decomposable element $x_1 * \ldots * x_m$ can be expressed in terms of the basis $\{e_\gamma^*, \gamma \in \bar{\Delta}\}$. Given the group G and character χ, we define the generalized matrix function [**3**], d_χ^G, as a mapping from the set of m-square matrices to **C**, by

$$d_\chi^G(B) = \sum_{\sigma \in G} \chi(\sigma) \prod_{i=1}^{m} b_{i\sigma(i)}. \tag{4}$$

For example, if $G = S_m$ and $\chi = \epsilon$, then $d_\chi^G = \det$; if $G = S_m$ and $\chi \equiv 1$, then $d_\chi^G = \text{per}$. It is a routine calculation to verify that

$$x_1 * \ldots * x_m = \sum_{\gamma \in \bar{\Delta}} \frac{1}{\nu(\gamma)} d_\chi^G(C[1, \ldots, m|\gamma]) e_\gamma^*, \tag{5}$$

where C is the $m \times n$ matrix whose (i, j) entry is c_{ij} and $C[1, \ldots, m|\gamma]$ is the m-square matrix whose (i, j) entry is $c_{i,\gamma(j)}$.

It is an easy task to verify that for arbitrary vectors $x_1, \ldots, x_m, y_1, \ldots, y_m$ in V,

$$(x_1 * \ldots * x_m, y_1 * \ldots * y_m) = \frac{1}{|G|} d_\chi^G([(x_i, y_j)]). \tag{6}$$

If $T \in \mathrm{Hom}(V, V)$, then

$$(7) \qquad \varphi: (v_1, \ldots, v_m) \to Tv_1 * \ldots * Tv_m$$

from $\times_1^m V$ to P is symmetric with respect to G and χ and hence, there is a unique linear map h from P to $U = P$ (see diagram (3)) such that $\varphi = h\mu$. For each linear operator T on V we denote the corresponding linear map h by $K(T)$. Thus for each decomposable element $x_1 * \ldots * x_m$ in P

$$(8) \qquad K(T)x_1 * \ldots * x_m = Tx_1 * \ldots * Tx_m.$$

From (8) we immediately verify for arbitrary S and T in $\mathrm{Hom}(V, V)$ that

$$(9) \qquad K(ST) = K(S)K(T)$$

and

$$(10) \qquad (K(T))^* = K(T^*).$$

If we specialize V to be complex n-tuple space and consider each $n \times n$ complex matrix A to be a linear operator on V, $v \to vA$ for $v \in V$, then with each matrix A we can associate a $|\bar{\Delta}| \times |\bar{\Delta}|$ matrix $K(A)$ defined by (8): if we use the lexicographic ordering in the sequence set $\bar{\Delta}$, and the elements of $\bar{\Delta}$ index the rows and columns of $K(A)$, then the τ, ω entry of the matrix of $K(A)$ relative to the orthonormal basis $\{(|G|/\nu(\gamma))^{\frac{1}{2}} e_\gamma^* | \gamma \in \bar{\Delta}\}$ described above is

$$(11) \qquad (d_\chi{}^G(A[\tau|\omega]))/(\nu(\omega)\nu(\tau))^{\frac{1}{2}}$$

where $B[\tau|\omega]$ means the submatrix of B lying in rows numbered $\tau(1), \ldots, \tau(m)$ and in columns numbered $\omega(1), \ldots, \omega(m)$ [**4**].

Finally, for $v_1, \ldots, v_s$ in V, let $\langle v_1, \ldots, v_s\rangle$ denote the subspace of V spanned by $v_1, \ldots, v_s$.

3. Proofs. In order to prove Theorem 1 we need the following lemma.

LEMMA 1. *If $x_1 * \ldots * x_m = y_1 * \ldots * y_m$, $m < n$, and if $\{x_1, \ldots, x_m\}$ is a linearly independent set, then the sets $\{x_1, \ldots, x_m\}$ and $\{y_1, \ldots, y_m\}$ span the same subspace.*

Proof. Let k be an integer, $1 \leqq k \leqq m$. Since $m < n$, there is a vector $z_k \neq 0$, such that $(y_t, z_k) = 0$ for $t = 1, \ldots, m$. Now let z_i for $i = 1, \ldots, m$, $i \neq k$, be arbitrary vectors in V. Then from (6) we have

$$(12) \qquad (y_1 * \ldots * y_m, z_1 * \ldots * z_m) = \frac{1}{|G|} d_\chi^G([(y_i, z_j)]).$$

Observe that the kth column of the matrix $[(y_i, z_j)]$ is 0 and so the left-hand side of (12) is 0. Thus $(x_1 * \ldots * x_m, z_1 * \ldots * z_m) = 0$ and so $d_\chi{}^G([(x_i, z_j)]) = 0$.

Now choose $z, z_1, \ldots, z_{k-1}, z_{k+1}, \ldots, z_m$ to be biorthogonal to the set $x_k, x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_m$. Then the matrix $[(x_i, z_j)]$ has the following form: its first $k - 1$ columns are those of the $m \times m$ identity matrix, its last $m - k$ columns are all zeros, and its kth column consists of the numbers, in order, $(x_1, z_k), \ldots, (x_m, z_k)$. Thus $0 = d_\chi{}^G([(x_i, z_j)]) = (x_k, z_k) = 0$.

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Hence we have proved that every vector which is perpendicular to the space spanned by $\{y_i, \ldots, y_m\}$ is perpendicular to the space spanned by $\{x_1, \ldots, x_m\}$. Since $\{x_1, \ldots, x_m\}$ is a linearly independent set, it follows that $\{x_1, \ldots, x_m\}$ and $\{y_1, \ldots, y_m\}$ span the same subspace of V.

Proof of Theorem 1. Clearly if $T = cS$ with $c^m = 1$, then $K(T) = K(S)$. Conversely, assume that $K(T) = K(S)$. Let $\{e_1, \ldots, e_n\}$ be a basis for V such that $\{Te_1, \ldots, Te_r\}$ is a basis for $\operatorname{Im} T$ and $\{e_{r+1}, \ldots, e_n\}$ is a basis for the kernel of T.

Let $x_i = Te_i$ and $y_i = Se_i$ for $i = 1, \ldots, n$ and observe that if $\omega \in \Gamma_n^m$, then

$$(13) \qquad \begin{aligned} x_{\omega(1)} * \ldots * x_{\omega(m)} &= Te_{\omega(1)} * \ldots * Te_{\omega(m)} \\ &= K(T)e_{\omega(1)} * \ldots * e_{\omega(m)} \\ &= K(S)e_{\omega(1)} * \ldots * e_{\omega(m)} \\ &= Se_{\omega(1)} * \ldots * Se_{\omega(m)} \\ &= y_{\omega(1)} * \ldots * y_{\omega(m)}. \end{aligned}$$

For $t = 1, \ldots, m + 1$ let ω^t denote the sequence

$$(1, 2, \ldots, t - 1, t + 1, \ldots, m + 1) \in \Gamma_n^m.$$

Since $m < r = \operatorname{rank} T$ it follows that $\{x_1, \ldots, x_{m+1}\}$ is a linearly independent set, and so we can apply Lemma 1 to (13) to conclude that

$$W_t = \langle x_{\omega^t(1)}, \ldots, x_{\omega^t(m)} \rangle = \langle y_{\omega^t(m)}, \ldots, y_{\omega^t(m)} \rangle, \; t = 1, \ldots, m + 1.$$

Now for each k, $1 \leqq k \leqq r$,

$$\bigcap_{\substack{t=1 \\ t \neq k}}^{m+1} W_t = \langle x_k \rangle = \langle y_k \rangle.$$

Thus Te_k and Se_k span the same space for $k = 1, \ldots, r$. Hence $\{Se_1, \ldots, Se_r\}$ is a linearly independent set and

$$x_j = Te_j = c_j Se_j = c_j y_j$$

for $c_j \neq 0$, $j = 1, \ldots, r$. Therefore

$$\begin{aligned} x_{\omega^t}^* &= y_{\omega^t}^* \\ &= \left(\prod_{\substack{j=1 \\ j \neq t}}^{m+1} c_j \right) x_{\omega^t}^* \\ &\neq 0 \end{aligned}$$

and so

$$\prod_{\substack{j=1 \\ j \neq t}}^{m+1} c_j = 1,$$

for $t = 1, \ldots, m + 1$. Thus $c_1 = \ldots = c_{m+1} = c$ with $c^m = 1$. Similarly we can show that $c_{m+1} = \ldots = c_r = c$. Thus $Te_j = cSe_j$ for $j = 1, \ldots, r$. Now let $k \geqq r + 1$. Then, since $x_k = 0$, we have

$$0 = x_1 * \ldots * x_{t-1} * x_{t+1} * \ldots * x_m * x_k$$
$$= y_1 * \ldots * y_{t+1} * y_{t+1} * \ldots * y_m * y_k.$$

Since the latter tensor is zero, the vectors $y_1, \ldots, y_{t-1}, y_{t+1}, \ldots, y_m, y_k$ must be linearly dependent and so y_k belongs to the intersection of the subspaces spanned by $\{y_1, \ldots, y_{t-1}, y_{t+1}, \ldots, y_m\}$, $t = 1, \ldots, m$. But this intersection is the zero vector and so $y_k = 0$ for $k = r + 1, \ldots, n$. Thus $0 = Te_k = cSe_k$ for $k = r + 1, \ldots, n$, and $T = cS$ with $c^m = 1$.

Proof of Theorem 2. It is easily checked from (9) and (10) that if T is normal then so is $K(T)$. Suppose that rank $T > m$ and $K(T)$ is normal. Then $K(TT^*) = K(T^*T)$ and so by Theorem 1, $TT^* = cT^*T$ for some c with $c^m = 1$. But both TT^* and T^*T are positive semi-definite hermitian operators with the same positive trace. Thus $c = 1$ and so T is normal.

In order to prove Theorem 3 we need two lemmas:

LEMMA 2. *If A is an $n \times n$ matrix of the form*

$$\begin{bmatrix} T & L \\ O & C \end{bmatrix}$$

where T is a $p \times p$ upper triangular matrix and C is an $(n - p) \times (n - p)$ upper triangular matrix with zeros along its main diagonal, then for any $\omega \in \Gamma_n{}^m$ for which $\omega(k) > p$ for some k, $1 \leqq k \leqq m$, the matrix $A[\omega|\omega]$ has a zero row and hence $d_\chi{}^G(A[\omega|\omega]) = 0$.

Proof. Assume that A and ω are as in the statement of the lemma and assume that $\omega(k)$ is the largest of the integers $\omega(1), \ldots, \omega(m)$. Then $\omega(k) > p$. Now the entries in row k of $A[\omega|\omega]$ are in succession $a_{\omega(k)\omega(1)}, \ldots, a_{\omega(k)\omega(m)}$. Since A is upper triangular it follows that $a_{\omega(k)\omega(l)} = 0$ when $\omega(k) > \omega(l)$, and since $\omega(k) > p$ it follows that $a_{\omega(k)\omega(l)} = 0$ when $\omega(l) = \omega(k)$. Since $\omega(k) \geqq \omega(l)$ for $l = 1, \ldots, m$, it follows that all the entries of the kth row of $A[\omega|\omega]$ are zero.

LEMMA 3. *If T is a linear operator on V of rank r, then* rank $K(T) = |\bar{\Delta} \cap \Gamma_r{}^m|$.

Proof. Let $\{u_1, \ldots, u_n\}$ be a basis for V for which $\{Tu_1, \ldots, Tu_r\}$ is a basis for Im T, and $\{u_{r+1}, \ldots, u_n\}$ is a basis for the kernel of T. Let $v_i = Tu_i$ $(i = 1, \ldots, r)$. Then since $B = \{K(T)u_\omega{}^* | \omega \in \bar{\Delta} \cap \Gamma_r{}^m\}$ is a subset of a basis, it is a linearly independent set in Im $K(T)$.

On the other hand if for any k, $1 \leqq k \leqq m$, $\omega(k) > r$, then $K(T)u_\omega{}^* = 0$. Thus B is a basis for Im $K(T)$ and so rank $K(T) = |\bar{\Delta} \cap \Gamma_r{}^m|$.

Proof of Theorem 3. If U is a unitary matrix, then $K(U)$ is unitary and so $K(A)$ is normal if and only if $K(U)^*K(A)K(U) = K(U^*AU)$ is normal. Thus we can assume by Schur's triangularization theorem that A is already an upper triangular matrix of the form described in Lemma 2. Thus $K(A)$ is upper triangular and since we assumed that $K(A)$ is also normal, it follows that $K(A)$ is diagonal. By (11) the main diagonal elements of $K(A)$ are

$$\frac{1}{\nu(\omega)} d_\chi^G(A[\omega|\omega]),\ \omega \in \bar{\Delta}.$$

From Lemma 2 if follows that for any $\omega \in \bar{\Delta}$ for which $\omega(k) > p$ for some k, $1 < k < m$, the corresponding main diagonal element of $K(A)$ is zero. Now rank $A = m$ impliesthat $p \leqq m$. If $p < m$, then by the preceding remarks, rank $K(A) \leqq |\bar{\Delta} \cap \Gamma_p{}^m|$. But if $p < m$, then $\bar{\Delta} \cap \Gamma_p{}^m$ is a proper subset of $\bar{\Delta} \cap \Gamma_m{}^m$ (i.e., $\tau = (1, \ldots, m) \in \bar{\Delta} \cap \Gamma_m{}^m$ but is not in $\bar{\Delta} \cap \Gamma_p{}^m$) and so the rank of $K(A)$ would be less than $|\bar{\Delta} \cap \Gamma_m{}^m|$, contradicting Lemma 3. Thus $p = m$ and it follows that C is the zero matrix (otherwise, rank $A > m$).

We can now assume that A has the form

$$\begin{bmatrix} T & L \\ 0 & 0 \end{bmatrix},$$

where T is $m \times m$ upper triangular with the non-zero eigenvalues $\lambda_1, \ldots, \lambda_m$ of A on the main diagonal. The main diagonal element of $K(A)$ in the position corresponding to the sequence τ is $d_\chi{}^G(A[\tau|\tau]) = \lambda_1 \ldots \lambda_m = \det T$. Now, since K is a representation, the main diagonal element of $K(A)K(A)^*$ corresponding to the sequence τ is

$$\begin{aligned} |\det T|^2 &= (K(A)K(A)^*)_{\tau\tau} \\ &= K(AA^*)_{\tau\tau} \\ &= d_\chi{}^G(AA^*[\tau|\tau]) \\ &\geqq \det(AA^*[\tau|\tau]) \qquad (14) \\ &= \det(TT^* + LL^*) \\ &\geqq \det TT^* \qquad (15) \\ &= |\det T|^2. \end{aligned}$$

The inequality (14) is an application of a result of Schur [**17**] and the inequality (15) is an instance of the result which states that if C and D are positive semi-definite hermitian matrices, then

$$\det(C + D) \geqq \det C. \qquad (16)$$

Moreover if C is positive definite, then it is easy to verify that equality holds in (16) if and only if $D = 0$. Thus from the above we conclude that $LL^* = 0$ and hence that $L = 0$.

4. Applications. In order to derive the Kress–de Vries–Wegmann result, we introduce some additional notation.

For an $n \times n$ complex matrix X, $\lambda_1(X), \ldots, \lambda_n(X)$ will denote the eigenvalues of X, $\alpha_1(X), \ldots, \alpha_n(X)$ will denote the singular values of X, $\lambda(X)$ will denote the n-tuple of eigenvalues of X, and $\alpha(X)$ will denote the n-tuple of singular values of X; if f is a symmetric function on the complex numbers, then $f(\lambda(X))$ will denote

$$(f(\lambda_1(X)), \ldots, f(\lambda_n(X)))$$

and $f(\alpha(X))$ will denote

$$(f(\alpha_1(X)), \ldots, f(\alpha_n(X)));$$

$E_r(t_1, \ldots, t_n)$ will denote the rth elementary symmetric function of $t_1, \ldots, t_n$ and $C_m(X)$ will denote the mth compound matrix of X, i.e., $C_m(X)$ is just $K(X)$ with G the full symmetric group of degree m and χ the alternating character on G. The eigenvalues of $C_m(A)$ are just the numbers

$$\lambda_\omega(C_m(A)) = \prod_{i=1}^{m} \lambda_{\omega(i)}(A)$$

as ω runs over $Q_{m,n}$, the set of strictly increasing sequences of length m of integers chosen from $1, \ldots, n$. Thus trace $(C_m(A))$ is just $E_m(\lambda(A))$.

LEMMA 4.

$$(17) \quad ||A||^4 - \tfrac{1}{2}||D||^2 - \left(\sum_{i=1}^{n} |\lambda_i(A)|^2\right)^2 = E_1((\alpha(A^2))^2) - E_1(|\lambda(A^2)|^2) + 2(E_2((\alpha(A))^2) - E_2(|\lambda(A)|^2).$$

Proof. We compute

$$||A||^4 = \left(\sum_{i=1}^{n} \alpha_i^2(A)\right)^2$$
$$= \sum_{i=1}^{n} \alpha_i^4(A) + 2E_2((\alpha(A))^2);$$
$$||D||^2 = \text{trace } ((AA^* - A^*A)^2)$$
$$= \text{trace } ((AA^*)^2 + (A^*A)^2 - AA^*A^*A - A^*AAA^*)$$
$$= 2 \text{ trace } ((AA^*)^2) - 2 \text{ trace } (A^2A^{*2})$$
$$= 2\sum_{i=1}^{n} \lambda_i^2(AA^*) - 2\sum_{i=1}^{n} \alpha_i^2(A^2)$$
$$= 2\sum_{i=1}^{n} \alpha_i^4(A) - 2\sum_{i=1}^{n} \alpha_i^2(A^2);$$
$$\left(\sum_{i=1}^{n} |\lambda_i(A)|^2\right)^2 = \sum_{i=1}^{n} |\lambda_i(A)|^4 + 2E_2(|\lambda(A)|^2)$$
$$= E_1(|\lambda(A^2)|^2) + 2E_2(|\lambda(A)|^2).$$

Thus the left side of (17) is equal to

$$(18) \quad E_1((\alpha(A^2))^2) - E_1(|\lambda(A^2)|^2) + 2[E_2((\alpha(A))^2) - E_2(|\lambda(A)|^2)].$$

We obtain the inequality (1) by rewriting (18) to obtain

$$(19)\quad \left[\|A^2\|^2 - \sum_{i=1}^{n} |\lambda_i(A^2)|^2\right] + 2\left[\|C_2(A)\|^2 - \sum_{\omega \in Q_{2,n}} |\lambda_\omega(C_2(A))|^2\right]$$

and applying Schur's inequality to both A^2 and $C_2(A)$.

Equality holds in (1) if and only if (19) is 0. But from Schur's inequality (19) is 0 if and only if both A^2 and $C_2(A)$ are normal.

Suppose equality holds in (1) and suppose A is not normal. Then by Theorem 2, rank $A \leqq 2$. If rank $A = 2$, then by Theorem 3, A is unitarily similar to a matrix of the form

$$B = \begin{bmatrix} \lambda_1 & a \\ 0 & \lambda_2 \end{bmatrix} \oplus 0_{n-2}$$

where $a \neq 0$ because A is not normal. Since A^2 is normal so is B^2 and so we conclude that $\lambda_2 = -\lambda_1$. Thus A is unitarily similar to a matrix $L \oplus 0_{n-2}$ where

$$L = \begin{bmatrix} \lambda & a \\ 0 & -\lambda \end{bmatrix},$$

with $\lambda \neq 0$. We wish to show that $L \oplus 0_{n-2}$ is unitarily similar to a matrix of the form in (2) and conversely that any matrix of the form in (2) is unitarily similar to a matrix of the form $L \oplus 0_{n-2}$. The converse is just a consequence of Schur's triangularization theorem and the fact that we are assuming that A is not normal.

Now $\{\text{trace}(X), \text{trace}(X^2), \text{trace}(X^*X)\}$ is a complete set of unitary invariants for complex 2×2 matrices X (see [**6**]). Let S denote the matrix in (2). Then using the fact that v and w are orthonormal column vectors it is a routine matter to calculate that $\text{trace}\, S = 0$, $\text{trace}(S^2) = 2r\alpha^2$, and $\text{trace}(S^*S) = |\alpha|^2(1 + r^2)$. Now $\text{trace}\, L = 0$, $\text{trace}\, L^2 = 2\lambda^2$, and $\text{trace}\, L^*L = 2|\lambda|^2 + |a|^2$. Thus the problem of showing that a matrix of the form $L \oplus 0_{n-2}$ is unitarily similar to a matrix of the form S consists of the following: given non-zero complex numbers λ and a, is there a non-zero complex number α and a real number r, $0 < r < 1$, such that

$$r\alpha^2 = \lambda^2,\ 2|\lambda|^2 + |a|^2 = |\alpha|^2(1 + r^2)?$$

But it is a routine calculation to check that the answer to this last question is "yes".

If rank $A = 1$, then $A = \alpha xy^*$ for some complex number $\alpha \neq 0$ and for some complex column n-tuples x and y with $1 = \|x\|^2 = x^*x = y^*y = \|y\|^2$. Since, by assumption, A is not normal it follows that

$$(20)\quad |\alpha|^2 xx^* = AA^* \neq A^*A = |\alpha|^2 yy^*.$$

But A^2 is normal and so

$$|\alpha|^4 (x, y)(y, x) xx^* = A^2(A^2)^* = (A^2)^* A^* = |\alpha|^4 (x, y)(y, x) yy^*.$$

Hence we conclude from (20) that $(x, y) = 0$ and so A has the form (2) with $r = 0$.

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