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ON A GENERALIZED DIVISOR PROBLEM II

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Abstract. We investigate the $\Omega_\pm$-result of $\Delta_a(x)$ and its number of sign-changes in an interval $[1, T]$, denoted by $X_a(T)$, for $-1 \le a < -1/2$. We can prove that $T \ll_a X_a(T)$ which is the best possible in order of magnitude.

§1. Introduction

Let $-1 < a < 0$ and define

$$\Delta_a(x) = \sum_{n \le x} \sigma_a(n) - \zeta(1-a)x - \frac{\zeta(1+a)}{1+a} x^{1+a} + \frac{1}{2}\zeta(-a) \tag{1.1}$$

where $\sigma_a(n) = \sum_{d|n} d^a$. The case $a = -1$ is defined by taking the right-hand limit. Here, we do not half the last term in the sum when x is an integer, in order to match the definition of some authors and to help simplifying later calculations. As was discussed in [4], the behaviour of $\Delta_a(x)$ for $-1 \le a < -1/2$ is different from the case $-1/2 < a \le 0$ and $a = -1/2$ appears as a critical point. Furthermore, we find in [5] that the limiting distribution for the case $-1 \le a < -1/2$ is symmetric while the case $a = 0$ is not. This further supports the change in nature. Therefore, we want to explore more properties of $\Delta_a(x)$ in these two ranges in order to realize their differences.

In [4], we investigated the oscillatory nature of $\Delta_a(x)$ for $-1/2 \le a < 0$. In this paper, we continue our study for the other case by considering the extreme values and the number of sign-changes of $\Delta_a(x)$. Certainly, large extreme values show a great amplitude of fluctuation and plenty of sign-changes tell us that it is very oscillatory.

Through the mean square formula (see [4, Section 1]), we expect that $\Delta_a(x) \ll x^\epsilon$ when $-1 \le a < -1/2$. The theorem below gives a result in the opposite direction.

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THEOREM 1. *We have, for $-1 < a \le -1/2$,*

$$\Delta_a(x) = \Omega_\pm\left(\exp\left((1+o(1))\frac{1}{1-|a|}\left(\frac{|a|}{2}\right)^{1-|a|}\frac{(\log x)^{1-|a|}}{\log\log x}\right)\right)$$

and $\Delta_{-1}(x) = \Omega_\pm(\log\log x)$.

This result, which seems to be the sharpest to date, was obtained by Pétermann [9] for $-1 \le a < -25/38$. We extend the range of a up to $-1/2$ by his argument together with a simple idea, which is the use of an averaged result of $G_{|a|}(x)$ (in Lemma 2.3) instead of a bound derived by the method of exponent pairs.

Concerning with the sign-changes of $\Delta_a(x)$, Pétermann studied this problem as well and he obtained in [6] that

$$X_a(T) \ge \frac{8}{3}\left(1 - \frac{\zeta(2|a|)}{4\zeta(2+2|a|)}\right)T + o(T) \tag{1.2}$$

where $X_a(T)$ denotes the number of sign-changes of $\Delta_a(x)$ in $[1,T]$. (Here, a sign-change of a function f at x_0 means $f(x_0-)f(x_0+) < 0$.) It should be remarked that (i) $X_a(T) \ll T$ and (ii) the main term in (1.2) is positive only when $a < -0.6236622010\ldots$. It is apparent that $\Delta_a(x)$ decreases by an amount of $-\zeta(1-a) + o(1)$ (as $n \to \infty$) when x varies over $[n, n+1)$ where n is an integer. The sign-changes counted in $X_a(T)$ may be due to the fact that plenty of $\Delta_a(n)$ ($n \in \mathbb{N}$) just lie above the x-axis. This leads to the consideration of sign-changes at integral points. In [2] and [8], problem of this type has been studied for the Euler Phi function. Let us denote by $N_a(T)$ the number of sign-changes on integers (i.e. $\Delta_a(n)\Delta_a(n+1) < 0$ with $n \in \mathbb{N}$). Clearly, the determination of $N_a(T)$ is harder and it was shown in [8] that $N_{-1}(T) \gg T^{0.71468244}$.

Our next result can extend the range of a in (1.2) to $-1/2$ and this shows the consistency in oscillatory behaviour of $\Delta_a(x)$ for $a \in [-1, -1/2)$. (Note that the case $a = -1/2$ is not included.) Moreover, it yields a lower bound for $N_{-1}(T)$. Let us say that a real-valued function $f(x)$ has a sign-change behind an integer n if $f(n)f(n+r) < 0$ and

$$f(n+1) = f(n+2) = \cdots = f(n+r-1) = 0$$

for some natural number r (independent of n). Then we have

THEOREM 2. *Let* $N_{a,0}(T) = \mathrm{Card}\{n \in [1,T] : \Delta_a(x)$ *has a sign-change behind* $n\}$. *Then,* $N_{a,0}(T) \gg_a T$ *for* $-1 \le a < -1/2$ *and all sufficiently large* T.

An immediate consequence is $X_a(T) \gg T$ for $-1 \le a < -1/2$ by looking at the graph of $\Delta_a(x)$. Another consequence is an improvement of the lower bound for $N_{-1}(T)$, which is best possible in order of magnitude.

COROLLARY. *We have* $N_{-1}(T) \gg T$.

§2. Proof of Theorem 1

To prove Theorem 1, we need some lemmas. Lemma 2.1 is our basic tool. By using it, we obtain Lemmas 2.2 and 2.3 which rely on the arguments in [7] and [9].

LEMMA 2.1. *For* $-1 \le a \le -1/2$, *let* $\psi(u) = u - [u] - 1/2$ *where* $[u]$ *is the integral part of* u,

$$\Delta_a(t) = -\sum_{n \le \sqrt{t}} n^a \psi\left(\frac{t}{n}\right) - t^a \sum_{n \le \sqrt{t}} n^{|a|} \psi\left(\frac{t}{n}\right) + O(t^{a/2}).$$

For the case $-1 < a \le -1/2$, it was proved in Chowla [1, Lemma 15] but, in fact, the argument applies to the case $a = -1$ as well.

Define $G_a(x) = \sum_{n \le \sqrt{x}} n^a \psi(x/n)$. Then one can find the following result in [7] or [9]. We include a proof here as it helps us to prove the next lemma.

LEMMA 2.2. *Let* A *be a squarefree integer and* B *be an integer with* $|B| \le A - 1$. *For* $-1 \le a \le -1/2$, *we have*

$$\frac{1}{X} \sum_{m \le X} G_a(Am + B) = \sum_{n \le \sqrt{AX+B}} (A,n) n^{a-1} \psi\left(\frac{B}{(A,n)}\right) + O(A(AX)^{a/2})$$

where (A,n) *is the greatest common divisor of* A *and* n. *In particular,*

$$\frac{1}{X} \sum_{m \le X} G_a(Am) = -\frac{1}{2}\zeta(1-a) \prod_{p|A} (1 + p^a - p^{a-1}) + O(A(AX)^{a/2}),$$

$$\frac{1}{X} \sum_{m \le X} G_a(Am-1) = \frac{1}{2}\zeta(1-a) \prod_{p|A} (1 + p^a - p^{a-1}) - \zeta(1-a) + O(A(AX)^{a/2}).$$

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Proof. Let $n^* = n/(A,n)$. Then it is not difficult to see that

$$\sum_{u\le m\le v} \psi\left(\frac{Am+B}{n}\right) = \frac{v-u}{n^*}\sum_{m=0}^{n^*-1}\psi\left(\frac{m}{n^*}+\frac{B}{n}\right)+O(n^*) \tag{2.1}$$
$$= \frac{v-u}{n^*}\psi\left(\frac{B}{(A,n)}\right)+O(n^*).$$

(See [7, Lemma 1] for details.) From the definition of $G_a(x)$ and (2.1),

$$\begin{aligned}
&\frac{1}{X}\sum_{m\le X} G_a(Am+B) \\
&= \frac{1}{X}\sum_{n\le\sqrt{AX+B}} n^a\left((X-\max(1,\frac{n^2-B}{A}))\frac{1}{n^*}\psi\left(\frac{B}{(A,n)}\right)+O(n^*)\right) \\
&= \sum_{n\le\sqrt{AX+B}}(A,n)n^{a-1}\psi\left(\frac{B}{(A,n)}\right) \\
&\quad - X^{-1}\sum_{n\le\sqrt{AX+B}}(A,n)n^{a-1}\max\left(1,\frac{n^2-B}{A}\right)\psi\left(\frac{B}{(A,n)}\right) \\
&\quad + O\left(X^{-1}\sum_{n\le\sqrt{AX+B}} n^{a+1}\right).
\end{aligned} \tag{2.2}$$

The O-term is obviously $\ll A(AX)^{a/2}$ and the second sum in (2.2) is

$$\begin{aligned}
&\ll (AX)^{-1}\sum_{n\le\sqrt{AX}}(A,n)n^{a+1}+AX^{-1} \\
&\ll (AX)^{-1}\sum_{d|A} d^{a+2}\sum_{n\ll\sqrt{AX}/d} n^{a+1}+AX^{-1} \\
&\ll (AX)^{a/2}\sigma_0(A)+AX^{-1}\ll A(AX)^{a/2}.
\end{aligned} \tag{2.3}$$

This yields the first part of Lemma 1.

When $B=0$, we have $\psi(0)=-1/2$ and the first sum in (2.2) is equal

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to

$$\begin{aligned}
&-\frac{1}{2}\sum_{n\le\sqrt{AX}} n^{a-1}(A,n)\\
&= -\frac{1}{2}\sum_{d|A} d^a \sum_{\substack{n=1\\(n,A/d)=1}}^{\infty} n^{a-1} + O(A(AX)^{a/2})\\
&= -\frac{1}{2}\zeta(1-a)\prod_{p|A}(1+p^a-p^{a-1}) + O(A(AX)^{a/2}).
\end{aligned}$$

The case $B=-1$ follows by similar argument with $\psi(-1/(A,n)) = 1/2 - 1/(A,n)$.

Lemma 2.3. *Let $-1 \le a \le -1/2$. For $B = 0$ or -1, we have*

$$\frac{1}{X}\sum_{m\le X}(Am+B)^a G_{|a|}(Am+B) \ll (AX)^{a/2}(A+(AX)^{\epsilon})$$

where A is a squarefree integer.

Proof. Consider the case $B=0$, we have, from (2.1), $\sum_{u\le m\le v}\psi(\frac{Am}{n}) \ll (v-u)/n^* + n^*$. This yields

$$\sum_{n^2/A\le m\le X} m^a\psi\left(\frac{Am}{n}\right) \ll \frac{X^{a+1}}{n^*} + \left(\frac{n^2}{A}\right)^a n^*.$$

Then,

$$\begin{aligned}
&X^{-1}\sum_{m\le X}(Am)^a G_{|a|}(Am)\\
&= A^a X^{-1}\sum_{n\le\sqrt{AX}} n^{|a|}\sum_{n^2/A\le m\le X} m^a\psi\left(\frac{Am}{n}\right)\\
&\ll A^a X^{-1}\sum_{n\le\sqrt{AX}} n^{|a|}\left(\frac{X^{a+1}}{n^*} + \left(\frac{n^2}{A}\right)^a n^*\right)\\
&\ll (AX)^a\sum_{n\le\sqrt{AX}} n^{|a|-1}(A,n) + X^{-1}\sum_{n\le\sqrt{AX}} n^{a+1}\\
&\ll \sigma_0(A)(AX)^{a/2} + A(AX)^{a/2}
\end{aligned}$$

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by using the argument in (2.3).

To prove the case $B = -1$, it suffices to check that $G_{|a|}(Am) - G_{|a|}(Am-1) \ll (Am)^{|a|/2+\epsilon}$. This follows from the observation that if n does not divide Am,

$$\psi\left(\frac{Am}{n}\right) - \psi\left(\frac{Am-1}{n}\right) = \frac{1}{n},$$

and $\ll 1$ otherwise.

Proof of Theorem 1. Taking $A = \prod_{2<p\le y} p$ where y is chosen such that $A \asymp X^{|a|/(2+a)}$, by Lemmas 2.2 and 2.3, we have immediately that

$$X^{-1} \sum_{m\le X} \Delta_a(Am) = \frac{1}{2}\zeta(1-a)\prod_{p|A}(1+p^a-p^{a-1}) + O(A(AX)^{a/2}),$$

and

$$X^{-1} \sum_{m\le X} \Delta_a(Am-1)$$
$$= -\frac{1}{2}\zeta(1-a)\prod_{p|A}(1+p^a-p^{a-1}) + \zeta(1-a) + O(A(AX)^{a/2}).$$

The value of $\prod_{2<p\le y}(1+p^a-p^{a-1})$ is equal to

$$\exp\left(\sum_{2<p\le y} p^a + O(\sum_{2<p\le y} p^{2a})\right) \tag{2.4}$$
$$= \begin{cases} \exp((1+o(1))y^{1+a}/((1+a)\log y), & \text{if } -1 < a \le -1/2 \\ \exp(\log\log y + O(1)) & \text{if } a = -1, \end{cases}$$

by the Prime Number Theorem. Observing that $\sup_{1\le u\le AX} \Delta_a(u) \ge X^{-1} \sum_{m\le X} \Delta_a(Am)$ (and $\sup_{1\le u\le AX}(-\Delta_a(u)) \ge -X^{-1}\sum_{m\le X} \Delta_a(Am-1)$), our result follows after replacing AX by x. Noting that $x \asymp X^{2/(2+a)} \asymp e^{2y/|a|}$, we have $y = (|a|\log x)/2 + O(1)$.

§3. Proof of Theorem 2 and Corollary

Our approach is to show that there are many integers at which $\Delta_a(x)$ takes negative values. From the definition, we see that the graph of $\Delta_a(x)$ is essentially a straight line of negative slope on each interval $[n, n+1)$. If the number of integers n satisfying $\Delta_a(n) > 0$ is small, then the absolute value of the integral of $\Delta_a(x)$ should be large since the positive area cannot give much cancellation to the negative. We find that it is not the case and hence Theorem 2 can be proved. (This method can be applied to the case of the Euler Phi function as well.)

To complete the first task, we consider the distribution functions. Let $P_{a,X}(u) = X^{-1}\operatorname{Card}\{1 \le n \le X : \Delta_a(n) \le u\}$ and $D_{a,X}(u) = X^{-1}\mu\{t \in [1,X] : \Delta_a(t) \le u\}$ where Card means the cardinality and μ is the Lebesgue measure. We have

$$D_{a,X}(u - \zeta(1-a)) \le P_{a,X}(u) \le D_{a,X}(u) + O(X^{-1}). \tag{3.1}$$

(This can be seen as follows: by (1.1), for any $t \in [n, n+1)$,

$$\Delta_a(n) - \Delta_a(t) = (t-n)(\zeta(1-a) + \zeta(1+a)\xi^a)$$

for some $\xi \in (n,t)$. As $\zeta(1+a) < 0$ for $-1 < a < 0$, $\Delta_a(t) \le \Delta_a(n)$ for all sufficiently large n. This yields the right side. Also, it follows that $\Delta_a(n) \le \Delta_a(t)+\zeta(1-a)$ and hence the left side of (3.1).) From [5, Theorem 3], we see that $D_a(u) = \lim_{X\to\infty} D_{a,X}(u)$ is a symmetric (i.e. $1 - D_a(u) = D_a(-u)$) probability distribution function. Moreover, we can prove

LEMMA 3.1. *For all real u, $0 < D_a(u) < 1$.*

Proof. As a distribution function is increasing, it suffices to show $D_a(-u) > 0$ for all sufficiently large u. Let u be any large number, and define y by the equations $\log u = y^{1+a}/\log y$ if $-1 < a < -1/2$ or $u = \log y$ if $a = -1$. Write $A = \prod_{2\le p\le y} p$, then $\sigma_a(A) = \prod_{p|A}(1 + p^a) \gg u$ (see (2.5)). Since $\sigma_a(Am) \ge \sigma_a(A)$ for any integer m, we get

$$\Delta_a(Am) - \Delta_a(Am-1) = \sigma_a(Am) + O(1) \gg u.$$

This implies $|\Delta_a(Am)| \gg u$ or $|\Delta_a(Am-1)| \gg u$; hence

$$1 - P_{a,X}(u) + P_{a,X}(-u) = X^{-1}\operatorname{Card}\{1 \le n \le X : |\Delta_a(n)| \ge u\} \gg A^{-1}.$$

Using (3.1) and taking $X \to \infty$, we deduce that

$$1 - D_a(u - \zeta(1-a)) + D_a(-u) \gg A^{-1} > 0.$$

Replacing u by $u + \zeta(1-a)$ and observing that $D_a(-u) \ge D_a(-u-\zeta(1-a))$ (since $\zeta(1-a) > 0$), we conclude with the symmetry of $D_a(u)$ that $2D_a(-u) = 1 - D_a(u) + D_a(-u) > 0$. Our proof is then complete.

The next lemma is to show that the integral of $\Delta_a(x)$ is small on average.

Lemma 3.2. *Let* $-1 \le a < -1/2$ *and* $1 \ll h \ll \sqrt{T}$. *Define* $E_a(t) = \int_0^t \Delta_a(v)\,dv$. *We have*

$$\int_T^{2T} (E_a(t+h) - E_a(t))^2\,dt \ll Th^{3+2a} \min((1-|a|)^{-1}, \log h).$$

Proof. From [3], we can establish a (truncated) Voronoi-type formula for $E_a(t)$. This is obtained by taking $\delta = 1+a$, $\rho = 1$, $\phi(s) = \psi(s) = \pi^{-s}\zeta(s)\zeta(s-a)$ and $\Delta(s) = \Gamma(s/2)\Gamma((s-a)/2)$ there. Then we see that $a_n = b_n = \sigma_a(n)$, $\lambda_n = \mu_n = \pi n$, $A = 1$, $h = 4$, $k_0(\rho) = -3/4$, $e_0(\rho) = 1/(2\sqrt{2\pi})$ and $\theta_\rho = 3/4 + a/2$ as $\rho = 1$. Noting that $E_a(t) = \pi^{-1}E_{A,1}(\pi t)$, we apply [3, Theorem 1] with $X = 2T$, $Z = 4T$ and observe that the second sum is $\ll T^{-1/2}\sum_{n<4T}\sigma_a(n)n^{-(3/4+a/2)}\min(1, |t-n|^{-2}) \ll 1$. We get for $t \in [T, 2T]$,

$$E_a(t) = \frac{1}{2\sqrt{2}\pi^2} t^{3/4+a/2} \sum_{n\le 4T} \frac{\sigma_a(n)}{n^{5/4+a/2}} w_T(n) \cos\left(4\pi\sqrt{nt} - \frac{3\pi}{4}\right) + O(1)$$

where $w_T(u) = 1$ for $1 \le u \le 2T$ and $w_T(u) = 2 - u/(2T)$ for $2T \le u \le 4T$. Then Lemma 3.2 is complete with the argument in [4, Theorem 1].

Proof of Theorem 2. Let $\epsilon > 0$ be a small fixed number. By Lemma 3.1, $D_a(-\epsilon - \zeta(1-a))$ is a positive constant. Hence, for all sufficiently large T, we have from (3.1) that $P_{a,T}(-\epsilon) \ge D_{a,T}(-\epsilon - \zeta(1-a)) \ge \kappa$ for some positive constant κ depending on a and ϵ. Let H be a large number which will be chosen later. Cutting the interval $[T, 2T]$ into subintervals of length H, there are at least $\kappa T/H - O(1)$ subintervals, each of which contains an integer n such that $\Delta_a(n) < -\epsilon < 0$. We can then form a class $\mathcal{C}$ from these subintervals such that (i) the cardinality of $\mathcal{C} \ge \kappa T/(4H)$, (ii) any two intervals in $\mathcal{C}$ is separated by a distance not less than $2H$, and (iii) for any $I \in \mathcal{C}$, $\Delta_a(n) < 0$ for some $n \in I$. This can be done by picking one from every three consecutive subintervals.

Now, we single out the interval $I \in \mathcal{C}$ which has the following property: there is $n_I \in I$ such that $\Delta_a(m) \le 0$ for all $n_I \le m \le n_I + 2H$. Let M be the number of such intervals. When $\Delta_a(m) \le 0$, $\int_m^{m+1} \Delta_a(u)\,du \le -\zeta(1-a)/2 + O(m^a)$. Then, we have for all real $t \in [n_I, n_I + H]$,

$$|E_a(t+H) - E_a(t)| = \left|\int_t^{t+H} \Delta_a(u)\,du\right| \gg_a H.$$

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Hence $\int_{n_I}^{n_I+H}(E_a(t+H)-E_a(t))^2\,dt \gg H^3$ and this yields $MH^3 \ll \int_T^{2T}(E_a(t + H) - E_a(t))^2\,dt$. By Lemma 3.2, $M \ll TH^{2a}\log H$. We select a large constant H so that $\kappa T/(4H) - O(TH^{2a}\log H) \gg_a T$. Therefore there are $\gg_a T$ subintervals, in which there is an integer n satisfying $\Delta_a(n) < 0$ but $\Delta_a(m) > 0$ for some integer m in $[n+1, n+H]$. This completes the proof.

At last, we prove the corollary. It follows from the fact that the limiting distribution $P_{-1}(u) = \lim_{X\to\infty} P_{-1,X}(u)$ is a continuous function (see the last section of [8]). Hence the number of $n \in [1,T]$ such that $\Delta_{-1}(n)$ equals zero is $o(T)$. This means that $N_{-1}(T) = N_{-1,0}(T) + o(T)$ and the result follows.

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