TRIANGULAR DISSECTIONS OF N-GONS

J. W. Moon and L. Moser

Let f(n) denote the number of dissections of a regular n-gon into n-2 triangles by n-3 non-intersecting diagonals. It is known that

$$f(n) = \frac{1}{n-1} \binom{2(n-2)}{n-2}$$

and that

(1) $f(n) = f(2)f(n-1) + f(3)f(n-2) + \ldots + f(n-1)f(2)$

for $n = 3, 4, \ldots,$ where $f(2) = 1$ by definition. (For pertinent references on this and related problems see, e.g., Motzkin [2].) The object of this note is to obtain a simple expression for g(n), the number of such dissections remaining when those which differ only by a rotation, reflection, or both are not considered as being different. For convenience we shall let $g(2) = 1$ and $f(k) = 0$ when k is not an integer. We shall prove the following:

THEOREM.

$$g(n) = f(n)/2n + f(n/3 + 1)/3 + 3f(n/2 + 1)/4, \quad \text{if } n \equiv 0(2);$$

$$g(n) = f(n)/2n + f(n/3 + 1)/3 + f(\tfrac{1}{2}(n+1))/2, \quad \text{if } n \equiv 1(2).$$

Our proof makes use of the following combinatorial lemma, a proof of which may be found in Burnside [1].

LEMMA. If G is a finite group of transformations operating on a finite set of objects and if two objects are equivalent when one is transformed into the other by a transformation of G, then the number of inequivalent objects is

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(2) $$F = \frac{1}{h} \Sigma N(t) ,$$

where h is the order of G , N(t) the number of objects left invariant by transformation t of G , and the sum is over all elements of G .

In the present problem the objects are the triangular dissections of a regular n-gon and the transformation group is the dihedral group of order 2n whose elements, $I, R, R^2, \ldots, R^{n-1}, T, TR, \ldots, TR^{n-1}$, are generated by the rotation $R = (1\ 2\ \ldots\ n)$ and the reflection $T = (1)(2, n)(3, n-1) \ldots ([n+2)/2], [(n+3)/2])$, where the square brackets denote the greatest integer function.

Before applying the lemma we make two preliminary observations. In the first place, if a triangular dissection of a regular n-gon remains invariant when subjected to a rotation of $2\pi d/n$ radians, where $0 < d \leq n/2$, then $d = n/2$ or $n/3$. For if the centre of the n-gon lies on a diagonal of the dissection then this diagonal joins diametrically opposite vertices of the n-gon and the only value of d in the above range which would permit the dissection to remain invariant would be $d = n/2$. The other alternative is that the centre lies in the interior of some triangle of the dissection. Labelling the vertices of the n-gon in counterclockwise order, let the vertices of this triangle be P_1, P_j, and P_k, where $1 < j < k < n+1$. For the dissection to remain invariant in this case it must be that $j - 1 = k - j = n + 1 - k = d$ which implies that $d = n/3$.

Appealing to the definition of a triangular dissection it is not difficult to verify the following observation also. The only axis of symmetry a triangular dissection of a regular n-gon can have when n is even is one through diametrically opposite vertices. In this case if the axis of symmetry does not coincide with a diagonal of the dissection then there is precisely one diagonal which crosses the axis; this diagonal is perpendicular to the axis and both of the vertices which it joins are connected by diagonals to the two vertices through which the axis passes. If n is odd and the axis of symmetry passes through P_1, then diagonals join P_1 to $P_{(n+1)/2}$ and to $P_{(n+3)/2}$.

We now complete the proof of the theorem for the case that $n \equiv 0\,(6)$. Clearly $N(I) = f(n)$. From the first observation above it follows that $N(R^{n/3}) = N(R^{2n/3}) = \frac{n}{3} f(n/3 + 1)$, since having chosen the vertices there labelled P_1, $P_{n/3+1}$, and $P_{2n/3+1}$ in one of $n/3$ ways the remaining diagonals may be selected in one of $f(n/3+1)$ ways so that the resulting dissection remains invariant under a rotation of $\pm 2\pi/3$ radians. Similarly we find that $N(R^{n/2}) = \frac{n}{2} f(n/2 + 1)$. For $m = 0, 1, \ldots, n/2-1$, it may be seen that

$$N(TR^{2m}) = f(n/2+1) + f(2)f(n/2) + f(3)f(n/2-1) + \ldots + f(n/2)f(2)$$

$$= 2f(n/2+1),$$

upon classifying the various possibilities consistent with the second observation above according to the length of the diagonal that is perpendicular to the axis of symmetry and using (1). The earlier remarks imply that the remaining values of $N(t)$ are zero. Substituting these values into (2) completes the proof of the theorem for this case. The remaining cases may be treated in a similar manner.

Motzkin [2] has listed the values of $g(n)$ up to $n = 13$. His numbers agree with those given by our theorem, except that it gives $g(12) = 733$ instead of 783, as given by him.

In closing we remark that with slight changes in formulation the hypothesis that the n-gons are regular can be dropped, a fact that tacitly has been used.

REFERENCES

1. W. Burnside, Theory of Groups of Finite Order, 2nd ed., Cambridge University Press, Cambridge, 1911. p. 191.

2. Th. Motzkin, Relations between hypersurface cross ratios, and a combinatorial formula for partitions of a polygon, for permanent preponderance, and for non-associative products, Bull. Amer. Math. Soc., 54 (1948) 352-360.

University of Alberta