Canad. Math. Bull. Vol. **60** (4), 2017 pp. 816–829
http://dx.doi.org/10.4153/CMB-2017-004-5

Characterizations of Operator Birkhoff–James Orthogonality

Mohammad Sal Moslehian and Ali Zamani

Abstract. In this paper, we obtain some characterizations of the (strong) Birkhoff–James orthogonality for elements of Hilbert C^*-modules and certain elements of $\mathbb{B}(\mathscr{H})$. Moreover, we obtain a kind of Pythagorean relation for bounded linear operators. In addition, for $T \in \mathbb{B}(\mathscr{H})$ we prove that if the norm attaining set $\mathbb{M}_T$ is a unit sphere of some finite dimensional subspace $\mathscr{H}_0$ of $\mathscr{H}$ and $\|T\|_{\mathscr{H}_0{}^\perp} < \|T\|$, then for every $S \in \mathbb{B}(\mathscr{H})$, T is the strong Birkhoff–James orthogonal to S if and only if there exists a unit vector $\xi \in \mathscr{H}_0$ such that $\|T\|\xi = |T|\xi$ and $S^*T\xi = 0$. Finally, we introduce a new type of approximate orthogonality and investigate this notion in the setting of inner product C^*-modules.

1 Introduction and Preliminaries

Let $\mathbb{B}(\mathscr{H}, \mathscr{K})$ denote the linear space of all bounded linear operators between Hilbert spaces $(\mathscr{H}, [\cdot, \cdot])$ and $(\mathscr{K}, [\cdot, \cdot])$. By I we denote the identity operator. When $\mathscr{H} = \mathscr{K}$, we write $\mathbb{B}(\mathscr{H})$ for $\mathbb{B}(\mathscr{H}, \mathscr{K})$. By $\mathbb{K}(\mathscr{H})$ we denote the algebra of all compact operators on $\mathscr{H}$, and by $\mathcal{C}_1(\mathscr{H})$ the algebra of all trace-class operators on $\mathscr{H}$. Let $\mathbb{S}_{\mathscr{H}} = \{\xi \in \mathscr{H} : \|\xi\| = 1\}$ be the unit sphere of $\mathscr{H}$. For $T \in \mathbb{B}(\mathscr{H})$, let $\mathbb{M}_T$ denote the set of all vectors in $\mathbb{S}_{\mathscr{H}}$ at which T attains norm, *i.e.*, $\mathbb{M}_T = \{\xi \in \mathbb{S}_{\mathscr{H}} : \|T\xi\| = \|T\|\}$. For $T \in \mathbb{B}(\mathscr{H}, \mathscr{K})$, the symbol $m(T) := \inf\{\|T\xi\| : \xi \in \mathbb{S}_{\mathscr{H}}\}$ denotes the minimum modulus of T.

Inner product C^*-modules generalize inner product spaces by allowing inner products to take values in an arbitrary C^*-algebra instead of the C^*-algebra of complex numbers.

In an inner product C^*-module $(V, \langle \cdot, \cdot \rangle)$ over a C^*-algebra $\mathscr{A}$ the following Cauchy–Schwarz inequality holds (see also [1]):

$$\langle x, y\rangle^* \langle x, y\rangle \le \|\langle x, x\rangle\| \langle y, y\rangle \qquad (x, y \in V).$$

Consequently, $\|x\| = \|\langle x, x\rangle\|^{\frac{1}{2}}$ defines a norm on V. If V is complete with respect to this norm, then it is called a *Hilbert $\mathscr{A}$-module*, or a *Hilbert C^*-module over $\mathscr{A}$*. Any C^*-algebra $\mathscr{A}$ can be regarded as a Hilbert C^*-module over itself via $\langle a, b\rangle := a^*b$. For every $x \in V$ the positive square root of $\langle x, x\rangle$ is denoted by $|x|$. In the case of a C^*-algebra, we get the usual notation $|a| = (a^*a)^{\frac{1}{2}}$. By $\mathcal{S}(\mathscr{A})$ we denote the set of all

Received by the editors August 27, 2016; revised January 23, 2017.
Published electronically March 8, 2017.
M. S. Moslehian (the corresponding author) was supported by a grant from Ferdowsi University of Mashhad (No. 2/42627).
AMS subject classification: **46L05**, 46L08, 46B20.
Keywords: Hilbert C^*-module, Birkhoff–James orthogonality, strong Birkhoff–James orthogonality, approximate orthogonality.

states of $\mathscr{A}$, that is, the set of all positive linear functionals of $\mathscr{A}$ whose norm is equal to one.

Furthermore, if $\varphi \in \mathcal{S}(\mathscr{A})$, then $(x, y) \mapsto \varphi(\langle x, y\rangle)$ gives rise to a usual semi-inner product on V, so we have the following useful Cauchy–Schwarz inequality:

$$|\varphi(\langle x, y\rangle)|^2 \leq \varphi(\langle x, x\rangle)\varphi(\langle y, y\rangle) \qquad (x, y \in V).$$

We refer the reader to [11, 17, 20] for more information on the basic theory of C^*-algebras and Hilbert C^*-modules.

A concept of orthogonality in a Hilbert C^*-module can be defined with respect to the C^*- valued inner product in a natural way: two elements x and y of a Hilbert C^*-module V over a C^*-algebra $\mathscr{A}$ are called *orthogonal*, denoted $x \perp y$, if $\langle x, y\rangle = 0$.

In a normed linear space there are several notions of orthogonality, all of which are generalizations of orthogonality in a Hilbert space. One of the most important concepts is that of the Birkhoff–James orthogonality: if x, y are elements of a complex normed linear space $(X, \|\cdot\|)$, then x is orthogonal to y *in the Birkhoff–James sense* [6, 16], in short, $x \perp_B y$, if

$$\|x + \lambda y\| \geq \|x\| \qquad (\lambda \in \mathbb{C}).$$

The central role of Birkhoff–James orthogonality in approximation theory is typified by the fact that $T \in \mathbb{B}(\mathscr{H})$ is a best approximation of $S \in \mathbb{B}(\mathscr{H})$ from a linear subspace M of $\mathbb{B}(\mathscr{H})$ if and only if T is a Birkhoff–James orthogonal projection of S onto M. By the Hahn–Banach theorem, if x, y are two elements of a normed linear space X, then $x \perp_B y$ if and only if there is a norm one linear functional f of X such that $f(x) = \|x\|$ and $f(y) = 0$. If we have additional structures on a normed linear space X, then we obtain other characterizations of Birkhoff–James orthogonality; see [3, 5, 13, 22, 25] and the references therein.

In Section 2, we present some characterizations of Birkhoff–James orthogonality for elements of a Hilbert $\mathbb{K}(\mathscr{H})$-module and elements of $\mathbb{B}(\mathscr{H})$. Next, we will give some applications. In particular, for $T, S \in \mathbb{B}(\mathscr{H})$ with $m(S) > 0$, we prove that there exists a unique $\gamma \in \mathbb{C}$ such that

$$\left\|(T + \gamma S) + \lambda S\right\|^2 \geq \left\|T + \gamma S\right\|^2 + |\lambda|^2 m^2(S) \qquad (\lambda \in \mathbb{C}).$$

As a natural generalization of the notion of Birkhoff–James orthogonality, the concept of strong Birkhoff–James orthogonality, which involves modular structure of a Hilbert C^*-module was introduced in [2]. When x and y are elements of a Hilbert $\mathscr{A}$-module V, x is orthogonal to y in the *strong Birkhoff–James sense*, in short, $x \perp_B^s y$ if

$$\|x + ya\| \geq \|x\| \qquad (a \in \mathscr{A});$$

i.e., the distance from x to $\overline{y\mathscr{A}}$, the $\mathscr{A}$-submodule of V generated by y, is exactly $\|x\|$. This orthogonality is "between" $\perp$ and $\perp_B$, *i.e.*,

$$x \perp y \Longrightarrow x \perp_B^s y \Longrightarrow x \perp_B y, \qquad (x, y \in V),$$

while the converses do not hold in general (see [2]). It was shown in [2] that the following relation between the strong and the classical Birkhoff–James orthogonality is valid:

$$x \perp_B^s y \Leftrightarrow x \perp_B y\langle y, x\rangle \qquad (x, y \in V).$$

https://doi.org/10.4153/CMB-2017-004-5 Published online by Cambridge University Press

In particular, by [3, Proposition 3.1], if $\langle x, y\rangle \geq 0$, then

$$x \perp_B^s y \Leftrightarrow x \perp_B y \qquad (x, y \in V). \tag{1.1}$$

If V is a full Hilbert $\mathscr{A}$-module, then the only case where the orthogonalities $\perp_B^s$ and $\perp_B$ coincide is when $\mathscr{A}$ is isomorphic to $\mathbb{C}$ (see [3, Theorem 3.5]), while orthogonalities $\perp_B^s$ and $\perp$ coincide only when $\mathscr{A}$ or $\mathbb{K}(V)$ is isomorphic to $\mathbb{C}$ (see [3, Theorems 4.7, 4.8]). Further, by [3, Lemma 4.2], we have

$$x \perp_B \left(\|x\|^2 y - y\langle x, x\rangle \right) \qquad (x, y \in V), \tag{1.2}$$

$$x \perp_B^s \left(\|x\|^2 x - x\langle x, x\rangle \right) \qquad (x \in V). \tag{1.3}$$

In Section 2, we obtain a characterization of strong Birkhoff–James orthogonality for elements of a C^*-algebra. We also present some characterizations of strong Birkhoff–James orthogonality for certain elements of $\mathbb{B}(\mathscr{H})$. In particular, for $T \in \mathbb{B}(\mathscr{H})$ we prove that if $\mathbb{S}_{\mathscr{H}_0} = \mathbb{M}_T$, where $\mathscr{H}_0$ is a finite dimensional subspace of $\mathscr{H}$ and $\|T\|_{\mathscr{H}_0^\perp} < \|T\|$, then for every $S \in \mathbb{B}(\mathscr{H})$, $T \perp_B^s S$ if and only if there exists a unit vector $\xi \in \mathscr{H}_0$ such that $\|T\|\xi = |T|\xi$ and $S^*T\xi = 0$.

For given $\varepsilon \geq 0$, elements x, y in an inner product $\mathscr{A}$-module V are said to be *approximately orthogonal* or *ε-orthogonal*, in short, $x \perp^\varepsilon y$ if $\|\langle x, y\rangle\| \leq \varepsilon\|x\|\|y\|$. For $\varepsilon \geq 1$, it is clear that every pair of vectors is ε-orthogonal, so the interesting case is when $\varepsilon \in [0, 1)$.

In an arbitrary normed space X, Chmieliński [7, 8] introduced the approximate Birkhoff–James orthogonality $x \perp_B^\varepsilon y$ by

$$\|x + \lambda y\|^2 \geq \|x\|^2 - 2\varepsilon|\lambda|\|x\|\|y\| \qquad (\lambda \in \mathbb{C}).$$

Inspired by the above approximate Birkhoff–James orthogonality, we propose a new type of approximate orthogonality in inner product C^*-modules.

Definition 1.1 For given $\varepsilon \in [0, 1)$, elements x, y of an inner product $\mathscr{A}$-module V are said to be *approximate strongly Birkhoff–James orthogonal*, denoted by $x \perp_{B^\varepsilon}^s y$, if

$$\|x + ya\|^2 \geq \|x\|^2 - 2\varepsilon\|a\|\|x\|\|y\| \qquad (a \in \mathscr{A}).$$

In Section 3, we investigate this notion of approximate orthogonality in inner product C^*-modules. In particular, we show that

$$x \perp^\varepsilon y \Longrightarrow x \perp_{B^\varepsilon}^s y \Longrightarrow x \perp_B^\varepsilon y, \qquad (x, y \in V),$$

while the converses do not hold in general.

As a result, we show that if $T\colon V \to W$ is a linear mapping between inner product $\mathscr{A}$-modules such that $x \perp_B y \Rightarrow Tx \perp_{B^\varepsilon}^s Ty$ for all $x, y \in V$, then

$$(1 - 16\varepsilon)\|T\|\|x\| \leq \|Tx\| \leq \|T\|\|x\| \qquad (x \in V).$$

Some other related topics can be found in [14, 15, 23, 24].

2 Operator (Strong) Birkhoff–James Orthogonality

The characterization of the (strong) Birkhoff–James orthogonality for elements of a Hilbert C^*-module by means of the states of the underlying C^*-algebra is known. For

https://doi.org/10.4153/CMB-2017-004-5 Published online by Cambridge University Press

elements x, y of a Hilbert $\mathscr{A}$-module V, the following results were obtained in [2, 5]:

(2.1) $x \perp_B y \Longleftrightarrow \big(\exists \varphi \in \mathcal{S}(\mathscr{A}) : \varphi(\langle x, x\rangle) = \|x\|^2 \text{ and } \varphi(\langle x, y\rangle) = 0\big)$

(2.2) $x \perp_B^s y \Longleftrightarrow \big(\exists \varphi \in \mathcal{S}(\mathscr{A}) : \varphi(\langle x, x\rangle) = \|x\|^2 \text{ and } \varphi(\langle x, y\rangle a) = 0 \ \forall a \in \mathscr{A}\big)$.

In the following result we establish a characterization of Birkhoff–James orthogonality for elements of a Hilbert $\mathbb{K}(\mathscr{H})$-module.

Theorem 2.1 *Let V be a Hilbert $\mathbb{K}(\mathscr{H})$-module and $x, y \in V$. Then the following statements are equivalent:*

(i) $x \perp_B y$.

(ii) *There exists a positive operator $P \in \mathcal{C}_1(\mathscr{H})$ of trace one such that*

$$\|x + \lambda y\|^2 \geq \|x\|^2 + |\lambda|^2 \mathrm{tr}(P|y|^2) \qquad (\lambda \in \mathbb{C}).$$

Proof Let $x \perp_B y$. By (2.1), there exists a state φ over $\mathbb{K}(\mathscr{H})$ such that $\varphi(\langle x, x\rangle) = \|x\|^2$ and $\varphi(\langle x, y\rangle) = 0$. For every $\lambda \in \mathbb{C}$, we therefore have

$$\begin{aligned} \|x + \lambda y\|^2 &\geq \varphi\big(\langle x + \lambda y, x + \lambda y\rangle\big) \\ &= \varphi(\langle x, x\rangle) + \lambda\varphi(\langle x, y\rangle) + \overline{\lambda\varphi(\langle x, y\rangle)} + |\lambda|^2\varphi(\langle y, y\rangle) \\ &= \|x\|^2 + |\lambda|^2\varphi(|y|^2). \end{aligned}$$

Thus,

$$\|x + \lambda y\|^2 \geq \|x\|^2 + |\lambda|^2\varphi(|y|^2) \qquad (\lambda \in \mathbb{C}).$$

Now, by [20, Theorem 4.2.1], there exists a positive operator $P \in \mathcal{C}_1(\mathscr{H})$ of trace one such that $\varphi(T) = \mathrm{tr}(PT)$, $T \in \mathbb{K}(\mathscr{H})$. Thus, we have

$$\|x + \lambda y\|^2 \geq \|x\|^2 + |\lambda|^2\varphi(|y|^2) = \|x\|^2 + |\lambda|^2 \mathrm{tr}(P|y|^2) \qquad (\lambda \in \mathbb{C}).$$

Conversely, if (ii) holds, then, since $|\lambda|^2\mathrm{tr}(P|y|^2) \geq 0$ for all $\lambda \in \mathbb{C}$, we get

$$\|x + \lambda y\| \geq \sqrt{\|x\|^2 + |\lambda|^2\mathrm{tr}(P|y|^2)} \geq \|x\| \qquad (\lambda \in \mathbb{C}).$$

Hence, $x \perp_B y$. ∎

Remark 2.2 Let V be a Hilbert $\mathbb{K}(\mathscr{H})$-module and $x, y \in V$. Using the same argument as in the proof of Theorem 2.1 and (2.2) we obtain $x \perp_B^s y$ if and only if there exists a positive operator $P \in \mathcal{C}_1(\mathscr{H})$ of trace one such that

$$\|x + ya\|^2 \geq \|x\|^2 + \mathrm{tr}(P|ya|^2) \qquad (a \in \mathscr{A}).$$

In the following result we establish a characterization of strong Birkhoff–James orthogonality for elements of a C^*-algebra.

Theorem 2.3 *Let $\mathscr{A}$ be a C^*-algebra, and $a, b \in \mathscr{A}$. Then the following statements are equivalent:*

(i) $a \perp_B^s b$.

(ii) *There exist a Hilbert space $\mathscr{H}$, a representation $\pi\colon \mathscr{A} \to \mathbb{B}(\mathscr{H})$, and a unit vector $\xi \in \mathscr{H}$ such that*

$$\|a + bc\|^2 \geq \|a\|^2 + \|\pi(bc)\xi\|^2 \qquad (c \in \mathscr{A}).$$

Proof Suppose that $a \perp_B^s b$. By (2.2) applied to $V = \mathscr{A}$ and using the same argument as in the proof of Theorem 2.1, there exists a state φ of $\mathscr{A}$ such that $\|a + bd\|^2 \geq \|a\|^2 + \varphi(|bd|^2)$ for all $d \in \mathscr{A}$. Now, by [11, Proposition 2.4.4] there exist a Hilbert space $\mathscr{H}$, a representation $\pi\colon \mathscr{A} \to \mathbb{B}(\mathscr{H})$, and a unit vector $\xi \in \mathscr{H}$ such that for any $c \in \mathscr{A}$ we have $\varphi(c) = [\pi(c)\xi, \xi]$. Hence,

$$\begin{aligned}\|a + bc\|^2 &\geq \|a\|^2 + \varphi(|bc|^2) = \|a\|^2 + \big[\pi(|bc|^2)\xi, \xi\big]\\ &= \|a\|^2 + \big[\pi(bc)\xi, \pi(bc)\xi\big] = \|a\|^2 + \|\pi(bc)\xi\|^2,\end{aligned}$$

for all $c \in \mathscr{A}$.

The converse is obvious. ∎

Corollary 2.4 *Let $\mathscr{A}$ be a unital C^*-algebra with the unit e. For every self-adjoint noninvertible $a \in \mathscr{A}$, there exist a Hilbert space $\mathscr{H}$, a representation $\pi\colon \mathscr{A} \to \mathbb{B}(\mathscr{H})$ and a unit vector $\xi \in \mathscr{H}$ such that*

$$\|e + ab\|^2 \geq 1 + \|\pi(ab)\xi\|^2 \qquad (b \in \mathscr{A}).$$

Proof Since a is noninvertible, a^2 is noninvertible as well. Therefore there is a state φ of $\mathscr{A}$ such that $\varphi(a^2) = 0$. We have $\varphi(ee^*) = \|e\|^2 = 1$ and

$$|\varphi(eab)| \leq \sqrt{\varphi(eaa^*e^*)\varphi(b^*b)} = \sqrt{\varphi(a^2)\varphi(b^*b)} = 0 \qquad (b \in \mathscr{A}).$$

Thus, by (2.2) we get $e \perp_B^s a$. Hence, by Theorem 2.3, there exist a Hilbert space $\mathscr{H}$, a representation $\pi\colon \mathscr{A} \to \mathbb{B}(\mathscr{H})$, and a unit vector $\xi \in \mathscr{H}$ such that $\|e + ab\|^2 \geq 1 + \|\pi(ab)\xi\|^2$ for all $b \in \mathscr{A}$. ∎

Now, we are going to obtain some characterizations of (strong) Birkhoff–James orthogonality in the Hilbert C^*-module $\mathbb{B}(\mathscr{H})$. Let $T, S \in \mathbb{B}(\mathscr{H})$. It was proved in [4, Theorem 1.1 and Remark 3.1] and [2, Proposition 2.8] that $T \perp_B S$ (resp. $T \perp_B^s S$) if and only if there is a sequence of unit vectors $(\xi_n) \subset \mathscr{H}$ such that

$$\lim_{n\to\infty} \|T\xi_n\| = \|T\| \quad \text{and} \quad \lim_{n\to\infty} [T\xi_n, S\xi_n] = 0 \quad (\text{resp. } \lim_{n\to\infty} S^*T\xi_n = 0). \tag{2.3}$$

When $\mathscr{H}$ is finite dimensional, it holds that $T \perp_B S$ (resp. $T \perp_B^s S$) if and only if there is a unit vector $\xi \in \mathscr{H}$ such that

$$\|T\xi\| = \|T\| \quad \text{and} \quad [T\xi, S\xi] = 0 \quad (\text{resp. } S^*T\xi = 0). \tag{2.4}$$

The following results are immediate consequences of the above characterizations.

Corollary 2.5 *Let $T \in \mathbb{B}(\mathscr{H})$ be an isometry and $S \in \mathbb{B}(\mathscr{H})$ be an invertible positive operator. Then $T \not\perp_B TS$.*

Corollary 2.6 *Let $S \in \mathbb{B}(\mathscr{H})$. Then the following statements are equivalent:*

(i) *S is non-invertible.*
(ii) *$T \perp_B S$ for every unitary operator $T \in \mathbb{B}(\mathscr{H})$.*

Proof By [10, Proposition 3.3], $S \in \mathbb{B}(\mathscr{H})$ is not invertible if and only if

$$0 \in \Big\{\lambda \in \mathbb{C} : \exists (\xi_n) \subset \mathscr{H}, \|\xi_n\| = 1, \lim_{n\to\infty} [T^*S\xi_n, \xi_n] = \lambda\Big\},$$

for every unitary operator T. Hence, by using (2.3), the statements are equivalent. ∎

https://doi.org/10.4153/CMB-2017-004-5 Published online by Cambridge University Press

Corollary 2.7 *Let $T, S \in \mathbb{B}(\mathscr{H})$. Then the following statements hold:*

(i) *If $\dim \mathscr{H} < \infty$, then $T \perp_B S$ if and only if there is a unit vector $\xi \in \mathscr{H}$ such that $\|T\|\xi = |T|\xi$ and $[T\xi, S\xi] = 0$.*

(ii) *If $\dim \mathscr{H} = \infty$, then $T \perp_B S$ if and only if there is a sequence of unit vectors $(\xi_n) \subset \mathscr{H}$ such that $\lim_{n\to\infty}\big(\|T\|\xi_n - |T|\xi_n\big) = 0$ and $\lim_{n\to\infty}[T\xi_n, S\xi_n] = 0$.*

(iii) *If $\dim \mathscr{H} < \infty$, then $T \perp_B^s S$ if and only if there is a unit vector $\xi \in \mathscr{H}$ such that $\|T\|\xi = |T|\xi$ and $S^*T\xi = 0$.*

(iv) *If $\dim \mathscr{H} = \infty$, then $T \perp_B^s S$ if and only if there is a sequence of unit vectors $(\xi_n) \subset \mathscr{H}$ such that $\lim_{n\to\infty}\big(\|T\|\xi_n - |T|\xi_n\big) = 0$ and $\lim_{n\to\infty} S^*T\xi_n = 0$.*

Proof (i) Let $T \perp_B S$. Take the same vector ξ as in (2.4). So, we have

$$\|T\xi\|^2 = [T\xi, T\xi] = [|T|^2\xi, \xi] \le \||T|\|^2\|\xi\|^2 \le \|T\|^2\|\xi\|^2 = \|T\xi\|^2.$$

This forces $|T|^2\xi = \|T\|^2\xi$ and thus $|T|\xi = \|T\|\xi$, as asserted.

The converse is trivial.

Using (2.3) and (2.4), we can similarly prove statements (ii)–(iv). ∎

Theorem 2.8 *Let $S \in \mathbb{B}(\mathscr{H})$. Let $\mathscr{H}_0 \ne \{0\}$ be a closed subspace of $\mathscr{H}$ and let P be the orthogonal projection onto $\mathscr{H}_0$. Then the following statements hold:*

(i) *If $\dim \mathscr{H} < \infty$, then $P \perp_B S$ if and only if there is a unit vector $\xi \in \mathscr{H}_0$ such that $[S\xi, \xi] = 0$.*

(ii) *If $\dim \mathscr{H} = \infty$, then $P \perp_B S$ if and only if there is a sequence of unit vectors $(\xi_n) \subset \mathscr{H}_0$ such that $\lim_{n\to\infty}[S\xi_n, \xi_n] = 0$.*

Proof (i) Let $P \perp_B S$. By (2.4), there is a unit vector $\zeta \in \mathscr{H}$ such that $\|P\zeta\| = \|P\| = 1$ and $[P\zeta, S\zeta] = 0$. We have $\zeta = \xi + \eta$, where $\xi \in \mathscr{H}_0$ and $\eta \in \mathscr{H}_0^{\perp}$. Since $\|\xi\| = \|P(\xi + \eta)\| = \|P\zeta\| = 1$ and $\|\xi\|^2 + \|\eta\|^2 = 1$, so we get $\eta = 0$. Hence, $[S\xi, \xi] = [S(\xi + \eta), \xi] = [S(\xi + \eta), P(\xi + \eta)] = \overline{[P\zeta, S\zeta]} = 0$.

The converse is trivial.

(ii) Let $P \perp_B S$. Take the vector sequence (ζ_n) of $\mathscr{H}$ as in (2.3). We have $\zeta_n = \mu_n + \eta_n$, where $\mu_n \in \mathscr{H}_0$ and $\eta_n \in \mathscr{H}_0^{\perp}$. Since

$$\lim_{n\to\infty}\|\mu_n\| = \lim_{n\to\infty}\|P(\mu_n + \eta_n)\| = \lim_{n\to\infty}\|P\zeta_n\| = 1 \quad \text{and} \quad \|\mu_n\|^2 + \|\eta_n\|^2 = 1,$$

we get $\lim_{n\to\infty}\|\eta_n\| = 0$. We can assume that $\|\mu_n\| \ge \frac{1}{2}$ for every $n \in \mathbb{N}$. Let us put $\xi_n = \frac{\mu_n}{\|\mu_n\|}$. We have

$$\begin{aligned}
\big|[S\xi_n, \xi_n]\big| &= \frac{1}{\|\mu_n\|^2}\big|[S\mu_n, \mu_n]\big| \\
&= \frac{1}{\|\mu_n\|^2}\big|[S\zeta_n, P\zeta_n] + [S\mu_n, \mu_n] - [S\zeta_n, P\zeta_n]\big| \\
&\le \frac{1}{\|\mu_n\|^2}\big|[S\zeta_n, P\zeta_n]\big| + \frac{1}{\|\mu_n\|^2}\big|[S\mu_n, \mu_n] - [S(\mu_n + \eta_n), \mu_n]\big| \\
&\le \frac{1}{\|\mu_n\|^2}\big|[S\zeta_n, P\zeta_n]\big| + \frac{1}{\|\mu_n\|^2}\big|[S\eta_n, \mu_n]\big|
\end{aligned}$$

$$\leq \frac{1}{\|\mu_n\|^2}\big|[S\zeta_n, P\zeta_n]\big| + \frac{1}{\|\mu_n\|}\|S\|\|\eta_n\|$$
$$\leq 4\big|[S\zeta_n, P\zeta_n]\big| + 2\|S\|\|\eta_n\|,$$

whence

$$\big|[S\xi_n, \xi_n]\big| \leq 4\big|[S\zeta_n, P\zeta_n]\big| + 2\|S\|\|\eta_n\|.$$

Since $\lim_{n\to\infty}[P\zeta_n, S\zeta_n] = 0$ and $\lim_{n\to\infty}\|\eta_n\| = 0$, from the above equality we get $\lim_{n\to\infty}[S\xi_n, \xi_n] = 0$.

The converse is trivial. ■

Theorem 2.9 *Let $T, S \in \mathbb{B}(\mathscr{H})$. Then the following statements are equivalent:*

(i) $T \perp_B S$;

(ii) $\|T + \lambda S\|^2 \geq \|T\|^2 + |\lambda|^2 m^2(S)$ $(\lambda \in \mathbb{C})$, *where $m(S)$ is the minimum modulus of S.*

Proof (i)⇒(ii) Let $T \perp_B S$ and $\dim \mathscr{H} = \infty$. By (2.3), there exists a sequence of unit vectors $(\xi_n) \subset \mathscr{H}$ such that $\lim_{n\to\infty}\|T\xi_n\| = \|T\|$ and $\lim_{n\to\infty}[T\xi_n, S\xi_n] = 0$. We have

$$\|T + \lambda S\|^2 \geq \|(T + \lambda S)\xi_n\|^2 = \|T\xi_n\|^2 + \overline{\lambda}[T\xi_n, S\xi_n] + \lambda[S\xi_n, T\xi_n] + |\lambda|^2\|S\xi_n\|^2,$$

for all $\lambda \in \mathbb{C}$ and $n \in \mathbb{N}$. Thus,

$$\|T + \lambda S\|^2 \geq \|T\|^2 + |\lambda|^2 \limsup_{n\to\infty}\|S\xi_n\|^2 \geq \|T\|^2 + |\lambda|^2 m^2(S) \qquad (\lambda \in \mathbb{C}).$$

When $\dim \mathscr{H} < \infty$, by using (2.4), we can similarly prove the statement (ii).

(ii)⇒(i) This implication is trivial. ■

Remark 2.10 Notice that for $S \in \mathbb{B}(\mathscr{H})$ it is straightforward to show that $m(S) > 0$ if and only if S is bounded below, or equivalently, S is left invertible. So in the implication (i)⇒(ii) of Theorem 2.9, if S is left invertible, then $m(S) > 0$.

It is well known that Pythagoras' equality does not hold in $\mathbb{B}(\mathscr{H})$. The following result is a kind of Pythagorean inequality for bounded linear operators.

Corollary 2.11 *Let $T, S \in \mathbb{B}(\mathscr{H})$ with $m(S) > 0$. Then there exists a unique $\gamma \in \mathbb{C}$, such that*

$$\big\|(T + \gamma S) + \lambda S\big\|^2 \geq \big\|T + \gamma S\big\|^2 + |\lambda|^2 m^2(S) \qquad (\lambda \in \mathbb{C}).$$

Proof The function $\lambda \mapsto \|T + \lambda S\|$ attains its minimum at, say, γ (there may be of course many such points) and hence $T + \gamma S \perp_B S$. So, by Theorem 2.9, we have

$$\big\|(T + \gamma S) + \lambda S\big\|^2 \geq \big\|T + \gamma S\big\|^2 + |\lambda|^2 m^2(S) \qquad (\lambda \in \mathbb{C}).$$

Now, suppose that ξ is another point satisfying the inequality

$$\big\|(T + \xi S) + \lambda S\big\|^2 \geq \big\|T + \xi S\big\|^2 + |\lambda|^2 m^2(S) \qquad (\lambda \in \mathbb{C}).$$

Choose $\lambda = \gamma - \xi$ to get

$$\begin{aligned}\|T+\gamma S\|^2 &= \|(T+\xi S)+(\gamma-\xi)S\|^2 \geq \|T+\xi S\|^2 + |\gamma-\xi|^2 m^2(S)\\ &\geq \|T+\gamma S\|^2 + |\gamma-\xi|^2 m^2(S).\end{aligned}$$

Hence $0 \geq |\gamma-\xi|^2 m^2(S)$. Since $m^2(S) > 0$, we get $|\gamma-\xi|^2 = 0$, or equivalently, $\gamma = \xi$. This shows that γ is unique. ■

Let $T \in \mathbb{B}(\mathscr{H})$. For every $S \in \mathbb{B}(\mathscr{H})$, it is easy to see that if there exists a unit vector $\xi \in \mathscr{H}$ such that $\|T\|\xi = |T|\xi$ and $S^*T\xi = 0$; then $T \perp_B^s S$. The question is under which conditions the converse is true. When the Hilbert space is finite dimensional, it follows from Corollary 2.7(iii) that there exists a unit vector $\xi \in \mathscr{H}$ such that $\|T\|\xi = |T|\xi$ and $S^*T\xi = 0$.

The following example shows that the finite dimensionality in statement (iii) of Corollary 2.7 is essential.

Example 2.12 Consider operators $T, S\colon \ell^2 \to \ell^2$ defined by

$$T(\xi_1,\xi_2,\xi_3,\dots) = \Big(\frac{1}{2}\xi_1, \frac{2}{3}\xi_2, \frac{3}{4}\xi_3,\dots\Big) \quad \text{and} \quad S(\xi_1,\xi_2,\xi_3,\dots) = (\xi_1,0,0,\dots).$$

One can easily observe that $T \perp_B S$ and $T^*S(\xi_1,\xi_2,\xi_3,\dots) = \frac{1}{2}\xi_1^{\,2} \geq 0$. So, by (1.1), we get $T \perp_B^s S$. But there does not exist $\xi \in \ell^2$ such that $\|T\|\xi = |T|\xi$.

We now settle the problem for any infinite dimensional Hilbert space. The proof of Theorem 2.13 is a modification of one given by Paul et al. [21, Theorem 3.1].

Theorem 2.13 *Let* $\dim \mathscr{H} = \infty$ *and* $T \in \mathbb{B}(\mathscr{H})$. *If* $\mathbb{S}_{\mathscr{H}_0} = \mathbb{M}_T$, *where* $\mathscr{H}_0$ *is a finite dimensional subspace of* $\mathscr{H}$ *and* $\|T\|_{\mathscr{H}_0^\perp} = \sup\{\|T\xi\| : \xi \in \mathscr{H}_0^{\perp}, \|\xi\| = 1\} < \|T\|$, *then for every* $S \in \mathbb{B}(\mathscr{H})$, *the following statements are equivalent:*

(i) $T \perp_B^s S$.
(ii) *There exists a unit vector* $\xi \in \mathscr{H}_0$ *such that* $\|T\xi\| = \|T\|$ *and* $S^*T\xi = 0$.
(iii) *There exists a unit vector* $\xi \in \mathscr{H}_0$ *such that* $\|T\|\xi = |T|\xi$ *and* $S^*T\xi = 0$.

Proof (i)⇒(ii) Suppose (i) holds. By (2.3), there exists a sequence of unit vectors $\{\zeta_n\}$ in $\mathscr{H}$ such that

$$\lim_{n\to\infty} \|T\zeta_n\| = \|T\| \quad \text{and} \quad \lim_{n\to\infty} S^*T\zeta_n = 0. \tag{2.5}$$

For each $n \in \mathbb{N}$, we have $\zeta_n = \xi_n + \eta_n$, where $\xi_n \in \mathscr{H}_0$ and $\eta_n \in \mathscr{H}_0^{\perp}$.

Since $\mathscr{H}_0$ is a finite dimensional subspace and $\|\xi_n\| \leq 1$, $\{\xi_n\}$ has a convergent subsequence converging to some element of $\mathscr{H}_0$. Without loss of generality we assume that $\lim_{n\to\infty} \xi_n = \xi$. Since $\mathbb{S}_{\mathscr{H}_0} = \mathbb{M}_T$,

$$\lim_{n\to\infty} \|T\xi_n\| = \|T\xi\| = \|T\|\|\xi\| \tag{2.6}$$

and

$$\lim_{n\to\infty} \|\eta_n\|^2 = \lim_{n\to\infty} (\|\zeta_n\|^2 - \|\xi_n\|^2) = 1 - \|\xi\|^2. \tag{2.7}$$

https://doi.org/10.4153/CMB-2017-004-5 Published online by Cambridge University Press

Now for each non-zero element $\xi_n \in \mathscr{H}_0$, by hypothesis $\frac{\xi_n}{\|\xi_n\|} \in \mathbb{S}_{\mathscr{H}_0} = \mathbb{M}_T$, and so $\|T\xi_n\| = \|T\|\|\xi_n\|$. Thus,

$$\|T^*T\|\|\xi_n\|^2 = \|T\|^2\|\xi_n\|^2 = \|T\xi_n\|^2 = [T^*T\xi_n, \xi_n] \le \|T^*T\xi_n\|\|\xi_n\| \le \|T^*T\|.$$

Hence, $[T^*T\xi_n, \xi_n] = \|T^*T\xi_n\|\|\xi_n\|$. By the equality case of Cauchy–Schwarz inequality $T^*T\xi_n = \lambda_n\xi_n$ for some $\lambda_n \in \mathbb{C}$, and therefore

$$[T^*T\xi_n, \eta_n] = [T^*T\eta_n, \xi_n] = 0. \tag{2.8}$$

By (2.5), (2.6), and (2.8) we have

$$\begin{aligned}\|T\|^2 &= \lim_{n\to\infty}\|T\zeta_n\|^2 = \lim_{n\to\infty}[T^*T\zeta_n, \zeta_n]\\ &= \lim_{n\to\infty}\big([T^*T\xi_n, \xi_n] + [T^*T\xi_n, \eta_n] + [T^*T\eta_n, \xi_n] + [T^*T\eta_n, \eta_n]\big)\\ &= \lim_{n\to\infty}\|T\xi_n\|^2 + \lim_{n\to\infty}\|T\eta_n\|^2 = \|T\|^2\|\xi\|^2 + \lim_{n\to\infty}\|T\eta_n\|^2,\end{aligned}$$

whence by (2.7) we reach

$$\lim_{n\to\infty}\|T\eta_n\|^2 = \|T\|^2(1 - \|\xi\|^2) = \|T\|^2\lim_{n\to\infty}\|\eta_n\|^2. \tag{2.9}$$

By the hypothesis $\|T\|_{\mathscr{H}_0^\perp} < \|T\|$, and so by (2.9) there does not exist any non-zero subsequence of $\{\|\eta_n\|\}$. So we conclude that $\eta_n = 0$ for all $n \in \mathbb{N}$. Hence, (2.5) and (2.7) imply

$$\|\xi\| = 1, \quad \|T\xi\| = \|T\|, \quad \text{and} \quad S^*T\xi = 0.$$

(ii)⇒(iii) This implication follows from the proof of Corollary 2.7.
(iii)⇒(i) This implication is trivial. ∎

Corollary 2.14 *Let* $\dim\mathscr{H} = \infty$ *and* $T \in \mathbb{B}(\mathscr{H})$. *If* $\mathbb{S}_{\mathscr{H}_0} = \mathbb{M}_T$, *where* $\mathscr{H}_0$ *is a finite dimensional subspace of* $\mathscr{H}$ *and* $\|T\|_{\mathscr{H}_0^\perp} < \|T\|$, *then there exists a unit vector* $\xi \in \mathscr{H}_0$ *such that* $\|T\|\xi = |T|\xi$ *and* $\|T\|^2T^*T\xi = (T^*T)^2\xi$.

Proof By (1.3), $T \perp_B^s (\|T\|^2T - TT^*T)$. So, by Theorem 2.13, there exists a unit vector $\xi \in \mathscr{H}_0$ such that $\|T\|\xi = |T|\xi$ and $(\|T\|^2T - TT^*T)^*T\xi = 0$. Thus, $\|T\|^2T^*T\xi = (T^*T)^2\xi$. ∎

Corollary 2.15 *Let* $\dim\mathscr{H} = \infty$ *and let* $T \in \mathbb{B}(\mathscr{H})$ *be a nonzero positive operator. If* $\mathbb{S}_{\mathscr{H}_0} = \mathbb{M}_T$, *where* $\mathscr{H}_0$ *is a finite dimensional subspace of* $\mathscr{H}$ *and* $\|T\|_{\mathscr{H}_0^\perp} < \|T\|$, *then for every* $S \in \mathbb{B}(\mathscr{H})$ *the following statements are equivalent:*

(i) $T \perp_B^s S$.
(ii) *There exists a unit vector* $\xi \in \mathscr{H}_0$ *such that* $T\xi = \|T\|\xi$ *and* $S^*\xi = 0$.

Proof Obviously, (ii)⇒(i).

Suppose (i) holds. By Theorem 2.13, there exists a unit vector $\xi \in \mathscr{H}_0$ such that $\|T\xi\| = \|T\|$ and $S^*T\xi = 0$. Since $T \ge 0$, $\|T\xi\| = \|T\| \Leftrightarrow T\xi = \|T\|\xi$. Therefore, $S^*T\xi = 0 \Leftrightarrow S^*\xi = 0$, as $T \neq 0$. ∎

3 An Approximate Strong Birkhoff–James Orthogonality

Recall that in an inner product $\mathscr{A}$-module V and for $\varepsilon \in [0,1)$, we say x, y are *approximate strongly Birkhoff–James orthogonal*, in short $x \perp_{B^\varepsilon}^s y$, if

$$\|x + ya\|^2 \geq \|x\|^2 - 2\varepsilon\|a\|\|x\|\|y\| \qquad (a \in \mathscr{A}).$$

The following proposition states some basic properties of the relation $\perp_{B^\varepsilon}^s$.

Proposition 3.1 *Let $\varepsilon \in [0, \frac{1}{2})$ and let V be an inner product $\mathscr{A}$-module. Then the following statements hold for every $x, y \in V$:*

(i) $x \perp_{B^\varepsilon}^s x \Leftrightarrow x = 0$.
(ii) $x \perp_{B^\varepsilon}^s y \Rightarrow \alpha x \perp_{B^\varepsilon}^s \beta y$ *for all* $\alpha, \beta \in \mathbb{C}$.
(iii) $x \perp^\varepsilon y \Rightarrow x \perp_{B^\varepsilon}^s y$.
(iv) $x \perp_{B^\varepsilon}^s y \Rightarrow x \perp_B^\varepsilon y$.
(v) $x \perp_{B^\varepsilon}^s y \Leftrightarrow x \perp_B^\varepsilon ya$ *for all* $a \in \mathscr{A}$.

Proof (i) Let $x \perp_{B^\varepsilon}^s x$. Also, suppose that $(e_i)_{i\in I}$ is an approximate unit for $\mathscr{A}$. We have

$$\|x - xe_i\|^2 \geq \|x\|^2 - 2\varepsilon\| - e_i\|\|x\|\|x\| \quad (i \in I).$$

Since $\lim_i \|x - xe_i\| = 0$ and $\|e_i\| = 1$, we get $(1 - 2\varepsilon)\|x\|^2 \leq 0$. Thus, $x = 0$.

The converse is obvious.

(ii) Let $x \perp_{B^\varepsilon}^s y$ and let $\alpha, \beta \in \mathbb{C}$. Excluding the obvious case $\alpha = 0$, we have

$$\begin{aligned}\|\alpha x + \beta ya\|^2 = |\alpha|^2\Big\|x + y\frac{\beta}{\alpha}a\Big\|^2 &\geq |\alpha|^2\Big(\|x\|^2 - 2\varepsilon\|a\|\|x\|\Big\|\frac{\beta}{\alpha}y\Big\|\Big)\\ &= \|\alpha x\|^2 - 2\varepsilon\|a\|\|\alpha x\|\|\beta y\|.\end{aligned}$$

Hence, $\alpha x \perp_{B^\varepsilon}^s \beta y$.

(iii) Let $x \perp^\varepsilon y$. For any $a \in \mathscr{A}$, we have

$$\begin{aligned}\|x + ya\|^2 &= \|\langle x + ya, x + ya\rangle\| = \|\langle x, x\rangle + \langle ya, ya\rangle + \langle x, ya\rangle + \langle ya, x\rangle\|\\ &\geq \|\langle x, x\rangle + \langle ya, ya\rangle\| - \|\langle x, ya\rangle + \langle ya, x\rangle\|\\ &\geq \|\langle x, x\rangle\| - \|\langle x, ya\rangle + \langle ya, x\rangle\|\\ &\geq \|x\|^2 - \|\langle x, ya\rangle\| - \|\langle ya, x\rangle\| \geq \|x\|^2 - 2\|a\|\|\langle x, y\rangle\|\\ &\geq \|x\|^2 - 2\varepsilon\|a\|\|x\|\|y\|.\end{aligned}$$

Thus, $\|x + ya\|^2 \geq \|x\|^2 - 2\varepsilon\|a\|\|x\|\|y\|$, or equivalently, $x \perp_{B^\varepsilon}^s y$.

(iv) Let $x \perp_{B^\varepsilon}^s y$. Hence, for any $\lambda \in \mathbb{C}$ and an approximate unit $(e_i)_{i\in I}$ for $\mathscr{A}$, we have

$$\begin{aligned}\big(\|x + \lambda y\| + |\lambda|\|ye_i - y\|\big)^2 \geq \|x + \lambda ye_i\|^2 &\geq \|x\|^2 - 2\varepsilon\|\lambda e_i\|\|x\|\|y\|\\ &\geq \|x\|^2 - 2\varepsilon|\lambda|\|x\|\|y\|.\end{aligned}$$

Since $\lim_i \|ye_i - y\| = 0$, whence we get $\|x + \lambda y\|^2 \geq \|x\|^2 - 2\varepsilon|\lambda|\|x\|\|y\|$, or equivalently, $x \perp^\varepsilon y$.

https://doi.org/10.4153/CMB-2017-004-5 Published online by Cambridge University Press

(v) Let $x \perp_{B^\varepsilon}^s y$ and let $(e_i)_{i\in I}$ be an approximate unit for $\mathscr{A}$. We have

$$\begin{aligned}\big(\|x+\lambda ya\| + \|\lambda yae_i - \lambda ya\|\big)^2 \geq \|x+\lambda yae_i\|^2 &\geq \|x\|^2 - 2\varepsilon\|\lambda ae_i\|\|x\|\|y\| \\ &\geq \|x\|^2 - 2\varepsilon|\lambda|\|a\|\|x\|\|y\|\end{aligned}$$

for all $a \in \mathscr{A}$ and all $\lambda \in \mathbb{C}$. Since $\lim_i \|yae_i - ya\| = 0$, we obtain from the above inequality

$$\|x+\lambda ya\|^2 \geq \|x\|^2 - 2\varepsilon|\lambda|\|a\|\|x\|\|y\|,$$

for all $a \in \mathscr{A}$ and all $\lambda \in \mathbb{C}$. Thus, $x \perp_B^\varepsilon ya$ for all $a \in \mathscr{A}$.

The converse is trivial. ■

Proposition 3.1 shows that in an arbitrary inner product C^*-module the relation $\perp^\varepsilon$ is weaker than the relation $\perp_{B^\varepsilon}^s$ and this relation is weaker than the relation $\perp_B^\varepsilon$, but the converses are not true in general (see the example below).

Example 3.2 Suppose that $\varepsilon \in [0, \frac{1}{2})$. Consider $\mathbb{M}_2(\mathbb{C})$, regarded as an inner product $\mathbb{M}_2(\mathbb{C})$-module. Let $I = \left[\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right]$, $A = \left[\begin{smallmatrix}-1 & 0\\ 0 & 1\end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right]$. Then

$$\begin{aligned}\|I+\lambda A\|^2 = \left\|\begin{bmatrix}1-\lambda & 0\\ 0 & 1+\lambda\end{bmatrix}\right\|^2 &= \big(\max\{|1-\lambda|, |1+\lambda|\}\big)^2 \\ \geq 1 \geq 1 - 2\varepsilon|\lambda| &= \|I\|^2 - 2\varepsilon|\lambda|\|I\|\|A\|\end{aligned}$$

for all $\lambda \in \mathbb{C}$. Hence $I \perp_B^\varepsilon A$, but not $I \perp_{B^\varepsilon}^s A$, since

$$\|I + A(-A)\|^2 = 0 < 1 - 2\varepsilon = \|I\|^2 - 2\varepsilon\|-A\|\|I\|\|A\|.$$

On the other hand, for any $C = \left[\begin{smallmatrix}c_1 & c_2\\ c_3 & c_4\end{smallmatrix}\right]$, we have

$$\begin{aligned}\|I+BC\|^2 &= \left\|\begin{bmatrix}1+c_1 & c_2\\ 0 & 1\end{bmatrix}\right\| \\ &= \left[\frac{1}{2}\big(|1+c_1|^2 + |c_2|^2 + 1\big) + \frac{1}{2}\sqrt{(|1+c_1|^2 + |c_2|^2 + 1)^2 - 4|1+c_1|^2}\right]^{\frac{1}{2}} \\ &\geq 1 \geq 1 - 2\varepsilon\|C\|\|B\| = \|I\|^2 - 2\varepsilon\|C\|\|I\|\|B\|.\end{aligned}$$

Therefore, $I \perp_{B^\varepsilon}^s B$. But not $I \perp^\varepsilon B$ since

$$\|\langle I, B\rangle\| = \|B\| = 1 > \varepsilon = \varepsilon\|I\|\|B\|.$$

By combining Proposition 3.1(iv) and [19, Theorem 3.5], we obtain the following result (see also [9, 12, 18]).

Corollary 3.3 *Let V, W be inner product $\mathscr{A}$-modules, $\varepsilon \in [0, \frac{1}{2})$ and let $T\colon V \to W$ be a linear mapping satisfying $x \perp_B y \Rightarrow Tx \perp_{B^\varepsilon}^s Ty$. Then*

$$(1-16\varepsilon)\|T\|\|x\| \leq \|Tx\| \leq \|T\|\|x\| \qquad (x \in V).$$

Proposition 3.4 *Let $\varepsilon \in [0,1)$. Let x, y be elements in an inner product $\mathscr{A}$-module V such that $\langle x, x\rangle \perp_{B^\varepsilon}^s \langle x, y\rangle$; then $x \perp_{B^\varepsilon}^s y$.*

Proof We assume that $x \neq 0$. Since $\langle x, x\rangle \perp^s_{B^\varepsilon} \langle x, y\rangle$, therefore for every $a \in \mathscr{A}$, we have

$$\|\langle x, x\rangle + \langle x, y\rangle a\|^2 \geq \|\langle x, x\rangle\|^2 - 2\varepsilon\|a\|\|\langle x, x\rangle\|\|\langle x, y\rangle\|$$

or equivalently,

$$\|\langle x, x + ya\rangle\|^2 \geq \|x\|^4 - 2\varepsilon\|a\|\|x\|^2\|\langle x, y\rangle\|.$$

Hence, we get

$$\|x\|^2\|x + ya\|^2 \geq \|x\|^4 - 2\varepsilon\|a\|\|x\|^3\|y\| \qquad (a \in \mathscr{A}).$$

Since $\|x\|^2 \neq 0$, we obtain from the above inequality

$$\|x + ya\|^2 \geq \|x\|^2 - 2\varepsilon\|a\|\|x\|\|y\| \qquad (a \in \mathscr{A}).$$

Thus, $x \perp^s_{B^\varepsilon} y$. ∎

Proposition 3.5 *Let x, y be two elements in an inner product $\mathscr{A}$-module V and let $\varepsilon \in [0, 1)$. If there exists a state φ on $\mathscr{A}$ such that $\varphi(\langle x, x\rangle) = \|x\|^2$ and $|\varphi(\langle x, y\rangle a)| \leq \varepsilon\|a\|\|x\|\|y\|$ for all $a \in \mathscr{A}$, then $x \perp^s_{B^\varepsilon} y$.*

Proof We assume that $x \neq 0$. Let $a \in \mathscr{A}$. By the Cauchy–Schwarz inequality, we have

$$\begin{aligned}\|x\|^2 &= \varphi(\langle x, x\rangle) = |\varphi(\langle x, x + ya\rangle) - \varphi(\langle x, ya\rangle)|\\ &\leq |\varphi(\langle x, x + ya\rangle)| + |\varphi(\langle x, ya\rangle)|\\ &\leq \sqrt{\varphi(\langle x, x\rangle)\varphi(\langle x + ya, x + ya\rangle)} + \varepsilon\|a\|\|x\|\|y\|\\ &\leq \|x\|\|x + ya\| + \varepsilon\|a\|\|x\|\|y\|.\end{aligned}$$

Thus, $\|x\|^2 \leq \|x\|\|x + ya\| + \varepsilon\|a\|\|x\|\|y\|$, *i.e*, $\|x + ya\| \geq \|x\| - \varepsilon\|a\|\|y\|$. We consider two cases.

Case 1: If $\|x\| - \varepsilon\|a\|\|y\| \geq 0$, then we get

$$\begin{aligned}\|x + ya\|^2 &\geq \big(\|x\| - \varepsilon\|a\|\|y\|\big)^2 = \|x\|^2 - 2\varepsilon\|a\|\|x\|\|y\| + \varepsilon^2\|a\|^2\|y\|^2\\ &\geq \|x\|^2 - 2\varepsilon\|a\|\|x\|\|y\|.\end{aligned}$$

Case 2: If $\|x\| - \varepsilon\|a\|\|y\| < 0$, then we reach

$$\begin{aligned}\|x + ya\|^2 &\geq 0 > \|x\|\big(\|x\| - \varepsilon\|a\|\|y\|\big) \geq \|x\|\big(\|x\| - \varepsilon\|a\|\|y\|\big) - \varepsilon\|a\|\|x\|\|y\|\\ &= \|x\|^2 - 2\varepsilon\|a\|\|x\|\|y\|.\end{aligned}$$

Hence, $x \perp^s_{B^\varepsilon} y$. ∎

Proposition 3.6 *Let x, y be two elements in an inner product $\mathscr{A}$-module V and let $\varepsilon \in [0, \frac{1}{2})$. If $x \perp^s_{B^\varepsilon} y$ then there exists a state φ on $\mathscr{A}$ such that*

$$|\varphi(\langle x, y\rangle a)| \leq \sqrt{2\varepsilon}\|a\|\|x\|\|y\| \qquad (a \in \mathscr{A}).$$

https://doi.org/10.4153/CMB-2017-004-5 Published online by Cambridge University Press

Proof Suppose that $x \perp^s_{B^\varepsilon} y$. Because of the homogeneity of relation $\perp^s_{B^\varepsilon}$, we can assume, without loss of generality, that $\|x\| = \|y\| = 1$. Then for arbitrary $a \in \mathscr{A}$, we have

$$\|x + ya\|^2 \geq 1 - 2\varepsilon\|a\|\|y\|.$$

Since $\| - \langle y, x\rangle\| \leq \|y\|\|x\| = 1$, for $a = -\langle y, x\rangle \in \mathscr{A}$ we get

$$\|x - y\langle y, x\rangle\|^2 \geq 1 - 2\varepsilon.$$

On the other hand, by [20, Theorem 3.3.6], there is $\varphi \in \mathcal{S}(\mathscr{A})$ such that

$$\varphi\Big(\big\langle x - y\langle y, x\rangle, x - y\langle y, x\rangle\big\rangle\Big) = \|x - y\langle y, x\rangle\|^2.$$

Also, we have

$$\begin{aligned}
&\varphi\Big(\big\langle x - y\langle y, x\rangle, x - y\langle y, x\rangle\big\rangle\Big)\\
&\quad = \varphi(\langle x, x\rangle) - 2\varphi(\langle x, y\rangle\langle y, x\rangle) + \varphi(\langle x, y\rangle\langle y, y\rangle\langle y, x\rangle)\\
&\quad \leq \|x\|^2 - 2\varphi(\langle x, y\rangle\langle y, x\rangle) + \varphi(\langle x, y\rangle\|y\|^2\langle y, x\rangle)\\
&\quad = 1 - \varphi(\langle x, y\rangle\langle y, x\rangle),
\end{aligned}$$

so, we get

$$1 - \varphi(\langle x, y\rangle\langle y, x\rangle) \geq \varphi\Big(\big\langle x - y\langle y, x\rangle, x - y\langle y, x\rangle\big\rangle\Big) = \|x - y\langle y, x\rangle\|^2 \geq 1 - 2\varepsilon.$$

Therefore, $\varphi(\langle x, y\rangle\langle y, x\rangle) \leq 2\varepsilon$. Now, by the Cauchy–Schwarz inequality, we reach

$$|\varphi(\langle x, ya\rangle)| \leq \sqrt{\varphi(\langle x, y\rangle\langle y, x\rangle)\varphi(a^*a)} \leq \sqrt{2\varepsilon}\|a\| \qquad (a \in \mathscr{A}). \qquad \blacksquare$$

Acknowledgments The authors would like to sincerely thank the referee for several useful suggestions.

References

[1] J. M. Aldaz, S. Barza, M. Fujii, and M. S. Moslehian, *Advances in operator Cauchy–Schwarz inequalities and their reverses*. Ann. Funct. Anal. **6**(2015), no. 3, 275–295. http://dx.doi.org/10.15352/afa/06-3-20

[2] Lj. Arambašić and R. Rajić, *A strong version of the Birkhoff–James orthogonality in Hilbert C^*-modules*. Ann. Funct. Anal. **5**(2014), no. 1, 109–120. http://dx.doi.org/10.15352/afa/1391614575

[3] ———, *On three concepts of orthogonality in Hilbert C^*-modules*. Linear Multilinear Algebra **63**(2015), no. 7, 1485–1500. http://dx.doi.org/10.1080/03081087.2014.947983

[4] R. Bhatia and P. Šemrl, *Orthogonality of matrices and some distance problems*. Linear Algebra Appl. **287**(1999), no. 1–3, 77–85. http://dx.doi.org/10.1016/S0024-3795(98)10134-9

[5] T. Bhattacharyya and P. Grover, *Characterization of Birkhoff-James orthogonality*. J. Math. Anal. Appl. **407**(2013), no. 2, 350–358. http://dx.doi.org/10.1016/j.jmaa.2013.05.022

[6] G. Birkhoff, *Orthogonality in linear metric spaces*. Duke Math. J. **1**(1935), no. 2, 169–172. http://dx.doi.org/10.1215/S0012-7094-35-00115-6

[7] J. Chmieliński, *On an ε-Birkhoff orthogonality*. J. Inequal. Pure Appl. Math. **6**(2005), no. 3, Art. 79.

[8] ———, *Linear mappings approximately preserving orthogonality*. J. Math. Anal. Appl. **304**(2005), 158–169. http://dx.doi.org/10.1016/j.jmaa.2004.09.011

[9] ———, *Orthogonality equation with two unknown functions*. Aequationes Math. **90**(2016), no. 1, 11–23. http://dx.doi.org/10.1007/s00010-015-0359-x

[10] C. Diogo, *Algebraic properties of the set of operators with 0 in the closure of the numerical range*. Oper. Matrices **9**(2015), no. 1, 83–93. http://dx.doi.org/10.7153/oam-09-04

[11] J. Dixmier, *C^*-Algebras*. North-Holland, Amsterdam, 1981.

[12] M. Frank, A. S. Mishchenko, and A. A. Pavlov, *Orthogonality-preserving, C^*-conformal and conformal module mappings on Hilbert C^*-modules*. J. Funct. Anal. **260**(2011), no. 2, 327–339. http://dx.doi.org/10.1016/j.jfa.2010.10.009
[13] P. Ghosh, D. Sain and K. Paul, *Orthogonality of bounded linear operators*. Linear Algebra Appl. **500**(2016), 43–51. http://dx.doi.org/10.1016/j.laa.2016.03.009
[14] P. Grover, *Orthogonality of matrices in the Ky Fan k-norms*. Linear Multilinear Algebra **65**(2017), no. 3, 496–509. http://dx.doi.org/10.1080/03081087.2016.1193118
[15] D. Ilišević and A. Turnšek, *Approximately orthogonality preserving mappings on C^*-modules*. J. Math. Anal. Appl. **341**(2008), no. ,1 298–308. http://dx.doi.org/10.1016/j.jmaa.2007.10.028
[16] R. C. James, *Orthogonality in normed linear spaces*. Duke Math. J. **12**(1945), 291–302. http://dx.doi.org/10.1215/S0012-7094-45-01223-3
[17] E. C. Lance, *Hilbert C^*-modules. A toolkit for operator algebraists*. London Mathematical Society Lecture Note Series, 210, Cambridge University Press, Cambridge, 1995. http://dx.doi.org/10.1017/CBO9780511526206
[18] C.-W. Leung, C.-K. Ng, and N.-C. Wong, *Linear orthogonality preservers of Hilbert C^*-modules*. J. Operator Theory **71**(2014), no. 2, 571–584. http://dx.doi.org/10.7900/jot.2012jul12.1966
[19] B. Mojškerc and A. Turnšek, *Mappings approximately preserving orthogonality in normed spaces*. Nonlinear Anal. **73**(2010), no. 12, 3821–3831. http://dx.doi.org/10.1016/j.na.2010.08.007
[20] G. J. Murphy, *C^*-Algebras and operator theory*. Academic Press, Boston, MA, 1990.
[21] K. Paul, D. Sain, and P. Ghosh, *Birkhoff–James orthogonality and smoothness of bounded linear operators*. Linear Algebra Appl. **506**(2016), 551–563. http://dx.doi.org/10.1016/j.laa.2016.06.024
[22] D. Sain, K. Paul, and S. Hait, *Operator norm attainment and Birkhoff–James orthogonality*. Linear Algebra Appl. **476**(2015), 85–97. http://dx.doi.org/10.1016/j.laa.2015.03.002
[23] P. Wójcik, *Norm-parallelism in classical M-ideals*. Indag. Math., to appear. http://dx.doi.org/10.1016/j.indag.2016.07.001
[24] A. Zamani and M. S. Moslehian, *Exact and approximate operator parallelism*. Canad. Math. Bull. **58**(2015), no. 1, 207–224. http://dx.doi.org/10.4153/CMB-2014-029-4
[25] A. Zamani, M. S. Moslehian, and M. Frank, *Angle preserving mappings*. Z. Anal. Anwend. **34**(2015), no. 4, 485–500. http://dx.doi.org/10.4171/ZAA/1551

(M. S. Moslehian, A. Zamani) *Department of Pure Mathematics, Ferdowsi University of Mashhad, P.O. Box 1159, Mashhad 91775, Iran*
e-mail: moslehian@um.ac.ir zamani.ali85@yahoo.com

(A. Zamani) *Department of Mathematics, Farhangian University, Iran*