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CONJUGACY OF ELEMENTS IN A NORMAL RING

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Let $(R, *)$ be a ring R with an involution $*$, i.e., $*$ is a map $R \to R$ such that for all $a, b \in R$

$$(a+b)^* = a^* + b^*$$
$$(ab)^* = b^* a^*$$
$$a^{**} = a.$$

The *trace* and *norm* of an element a in $(R, *)$ are respectively

$$T(a) = a + a^*, \qquad N(a) = aa^*.$$

$(R, *)$ is said to be a *normal ring* if for all $a \in R$

$$N(a) = N(a^*)$$

or equivalently,

$$aa^* = a^*a.$$

It is well-known that two real quaternionic elements a and b have the same trace and norm if and only if they are conjugates, i.e., there exists a non-zero quaternion x such that $xa = bx$. This result is now extended to a normal ring $(R, *)$, R being not commutative and having no zero divisors.

As usual, we write $[x, y] = xy - yx$ for all $x, y \in R$. The symbol Z denotes the center of R. Clearly, $x \in Z$ implies $x^* \in Z$.

Following Dyson [1], a ring $(R, *)$ is said to have the *scalar product property* (and is henceforth abbreviated as a *SPP-ring*) if for all $a, b \in R$

$$[a^*, b^*] = [a, b]$$

or equivalently,

$$T(ab) = T(ba).$$

LEMMA 1. (i) *A normal ring $(R, *)$ is a SPP-ring.*
(ii) *A 2-torsionfree SSP-ring $(R, *)$ is a normal ring.*

Proof. (i) For all $a, b \in R$

$$\begin{aligned} T(ab) &= N(a+b^*) - N(a) - N(b^*) \\ &= N(a^*+b) - N(a^*) - N(b) \\ &= T(ba). \end{aligned}$$

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(ii) For all $a \in R$, $2aa^* = T(aa^*) = T(a^*a) = 2a^*a$. Hence, $aa^* = a^*a$.

A SPP-ring which is not 2-torsionfree need not be normal. We have the following

Example 1. Let F be a field of char 2 and R be the F-algebra of matrices of the form:

$$\begin{bmatrix} x & y & z \\ 0 & x & w \\ 0 & 0 & x \end{bmatrix}, \qquad x, y, z, w \in F.$$

The map which sends

$$\begin{bmatrix} x & y & z \\ 0 & x & w \\ 0 & 0 & x \end{bmatrix} \text{ to } \begin{bmatrix} x & w & z \\ 0 & x & y \\ 0 & 0 & x \end{bmatrix}$$

is an involution * on R. It is easy to verify that $(R, ^*)$ is a SPP-ring. It is not normal because for

$$a = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$aa^* \neq a^*a$.

Lemma 2. *Let $(R, ^*)$ be a normal ring which is not commutative. Then for all $a, b \in R$*

$$T(a) = T(b), \qquad N(a) = N(b)$$

imply $xa = bx$ for some $x \in R$, $x \neq 0$.

Proof. First assume $b \neq a^*$. Since $T(a) = T(b)$, $X = b - a^* = a - b^* \neq 0$ and we have $xa = (b - a^*)a = ba - a^*a = ba - bb^* = b(a - b^*) = bx$.

Next assume $b = a^*$ and $a \notin Z$. Then $x = [a, y] \neq 0$ for some $y \in R$, $y \neq 0$. Whence, $xa = [a, y]a = [a, ya] = [a^*, a^*y^*] = a^*[a^*, y^*] = a^*[a, y] = bx$.

Lastly, assume $b = a^*$ and $a \in Z$. Since R is not commutative, there exists non-zero elements y, z in R such that $x = [y, z] \neq 0$. Hence, $xa = [y, z]a = [y, za] = [y^*, a^*z^*] = a^*[y^*, z^*] = a^*[y, z] = bx$.

The converse to the above is not true in general.

Example 2. Let F be a field of char $\neq 2$ and R be the ring of 2×2 matrices over F. The map * defined by

$$\begin{bmatrix} t & u \\ v & w \end{bmatrix}^* = \begin{bmatrix} w & -u \\ -v & t \end{bmatrix}, \qquad t, u, v, w \in F$$

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is an involution on R. It is easily verified that $(R, *)$ is a normal ring. For

$$a=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \qquad b=x=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix},$$

$xa=x=bx$ but clearly a and b have distinct traces and norms.

The converse, however, is true if R has no zero divisors.

THEOREM. *Let $(R, *)$ be a normal ring which is not commutative and has no zero divisors. Then for all $a, b \in R$,*

$$T(a)=T(b), \qquad N(a)=N(b)$$

if and only if

$$xa=bx \quad \textit{for some} \quad x \in R, \qquad x \neq 0.$$

Proof. Assume $xa=bx$ for sime $x \in R$, $x \neq 0$. Then $xT(a)x^*=T(xax^*)=T(bxx^*)=T(bx^*x)=T(xbx^*)=xT(b)x^*$ and $xaa^*x^*=bxx^*b^*=bx^*xb^*=bx^*(bx^*)^*=(bx^*)^*bx^*=xb^*bx^*=xbb^*x^*$. Hence,

$$T(a)=T(b) \quad \text{and} \quad aa^*=bb^*.$$

The converse is Lemma 2.

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