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REGULAR RINGS ARE VERY REGULAR

BY
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The following problem arose in a conversation with Abraham Zaks: "Suppose R is an associative ring with identity such that every finitely generated left ideal is generated by idempotents. Is R von-Neumann regular?" In the literature the "*s*" in "idempotents" is missing, and is replaced by "an idempotent". The answer is, "Yes!"

THEOREM. *Let R be an associative ring with unit for which every finitely generated left ideal is generated by idempotents. Then R is regular.*

Proof. Let $a \in R$. We wish to show $Ra = Re$, for some idempotent e. We know $Ra = \sum_{i=1}^{n} Re_i$, $e_1, \ldots, e_n$ idempotents. Let us assume n is as small as possible. Consider $Re_1 + Re_2$. We have that $Re_1 + Re_2 = Re_1 \oplus Re_2(1-e_1)$. The submodule $Re_2(1-e_1)$ of Re_2 is pure in Re_2. To see this suppose $\sum_{i=1}^{n} r_i x_i = re_2(1-e_1)$, with $x_i \in Re_2$, $i = 1, \ldots, m$. Then $y_i = x_i(1-e_1) \in Re_2(1-e_1)$ for each $i = 1 \cdots m$, and $\sum r_i y_i = re_2(1-e_1)$ so $Re_2(1-e_1)$ is pure in Re_2. Since Re_2 is a direct summand, Re_2 is pure in R. This gives $Re_2(1-e_1)$ pure in R. Next we have that $0 \to Re_2(1-e_1) \to R \to R/Re_2(1-e_1) \to 0$ is exact and since $Re_2(1-e_1)$ is pure, $R/Re_2(1-e_1)$ is flat. Now $R/Re_2(1-e_1)$ is finitely presented, so is projective. Therefore the sequence splits and $Re_2(1-e_1) = Rf$ for some $f = f^2$. Now form $g = (f - e_1 f)$. We have $ge_1 = (f - e_1 f)e_1 = (1-e_1)fe_1 = 0$ since $f = f(1-e_1)$. Clearly, $e_1 g = 0$ and $fg = f$ so that $Re_1 + Re_2 = Re_1 \oplus Rg$. Let $e_1' = e_1 + g$. Then $Re_1' + Re_3 + \cdots + Re_n = Ra$ and this contradicts the minimality of n, unless $n = 1$. I.e., $Ra = Re_1$ and R is regular. Q.E.D.

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