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ON FINITE G-LOCALLY PRIMITIVE GRAPHS AND THE WEISS CONJECTURE

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A graph Γ is said to be a G-locally primitive graph, for $G \leqslant \operatorname{Aut}\Gamma$, if for every vertex α, the stabiliser G_α induces a primitive permutation group on $\Gamma(\alpha)$ the set of vertices adjacent to α. In 1978 Richard Weiss conjectured that there exists a function $f : \mathbb{N} \longrightarrow \mathbb{N}$ such that for any finite connected vertex-transitive G-locally primitive graph of valency d and a vertex α of the graph, $|G_\alpha| \leqslant f(d)$. The purpose of this paper is to prove that, in the case $\operatorname{Soc}(G) = Sz(q)$, the conjecture is true.

1. INTRODUCTION

In 1978 Weiss [**16**] conjectured that, for a finite connected vertex-transitive locally primitive graph Γ, the number of automorphisms fixing a given vertex is bounded above by some function of the valency of Γ.

Let G be a transitive permutation group on a set Ω. We say G is *quasiprimitive* on Ω if each of nontrivial normal subgroups of G is transitive on Ω.

For a graph Γ, denote by $\Gamma(\alpha)$ the set of vertices adjacent to a vertex α. We say that a graph Γ is *locally primitive* if for every vertex α of Γ, the stabiliser in $\operatorname{Aut}\Gamma$ of α induces a primitive permutation group on $\Gamma(\alpha)$. Let G be a subgroup of $\operatorname{Aut}\Gamma$, the automorphism group of Γ. Then Γ is said to be a *G-locally primitive graph*, if for every vertex α of Γ the stabiliser G_α in G of α induces a primitive permutation group $G_\alpha^{\Gamma(\alpha)}$ on $\Gamma(\alpha)$. The *socle* $\operatorname{Soc}(G)$ of a group G is the product of its minimal normal subgroups, and a finite group is said to be an *almost simple* if its socle is a nonabelian simple group. Also by $G \rtimes H$ we denote the semidirect product of G by H.

Recently Conder, Li and Praeger [**1**] proved that the Weiss conjecture is true for finite non-bipartite graphs provided that is true for such graphs which admit almost simple locally primitive groups of automorphisms; especially they proved [**1**, Proposition 3.2] the conjecture where $\operatorname{Soc}(G) = PSL(2,q)$. In this paper we prove that if G is an almost simple group with $\operatorname{Soc}(G) = Sz(q)$, then the graph Γ is non-bipartite and the conjecture is true. The question of extending this result to the remaining two families of groups of Lie type of Lie rank one will be the subject of a future paper.

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THEOREM 1.1. (Main Theorem.) *If Γ is a finite G-locally primitive graph of valency d for an almost simple group G with $\mathrm{Soc}(G) = Sz(q)$, a Suzuki simple group, then the graph Γ is non-bipartite, and $|G_\alpha|$ is bounded by a function of d.*

We note also that, in the special case of 2-arc transitive graphs, the Weiss conjecture has been proved as a culmination of a long sequence of papers by several authors: namely, Gardiner [**2, 3, 4**], Weiss [**13, 14, 15**] and Trofimov [**9, 10, 11, 12**].

2. PRELIMINARIES

First we define a core-free subgroup of a group and we bring some previous results.

DEFINITION 2.1: A subgroup H of a group G is said to be *core-free* in G if $\bigcap_{g \in G} H^g = \{1\}$.

DEFINITION 2.2: Let H be a core-free subgroup of G and $g \in G$ be an element such that $\langle H, g\rangle = G$ and $g^2 \in H$. Then a *coset graph* $\Gamma = \Gamma(G, H, g)$ is defined by $V\Gamma = \{Hx \mid x \in G\}$ with $\{Hx, Hy\}$ an edge if and only if $xy^{-1} \in HgH$.

A coset graph has the following properties:

LEMMA 2.3. [7] *Let $\Gamma = \Gamma(G, H, g)$ be a coset graph defined as above. Then G, acting by right multiplication, is a subgroup of* $\mathrm{Aut}\,\Gamma$ *and $G_\alpha = H$, where α is the vertex H of Γ. Moreover, Γ is a connected G-arc transitive graph of valency $|H : H \cap H^g|$. Conversely if G acts faithfully and arc-transitively on a finite connected graph Γ then $\Gamma \cong \Gamma(G, H, g)$ for some core-free subgroup H of G and some $g \in G$ such that $\langle H, g\rangle = G$ and $g^2 \in H$.*

DEFINITION 2.4: A permutation group G on Ω is said to be *regular* if it is transitive and $G_\alpha = \{1\}$ for every $\alpha \in \Omega$.

LEMMA 2.5. ([**5**, Lemma 2.1].) *Let $\Gamma = (V, E)$ be a connected graph, and suppose that $G \leqslant \mathrm{Aut}\,\Gamma$ is transitive on $V\Gamma$. Suppose that N is a normal subgroup of G which is regular on $V\Gamma$. Then $G = N \rtimes G_\alpha$ and G_α is faithful on $\Gamma(\alpha)$.*

In[**14**] Weiss proved his conjecture for a special class of local actions, namely the case where $G_\alpha^{\Gamma(\alpha)}$ is primitive with a regular normal elementary Abelian subgroup. Such primitive groups are called *affine primitive groups.*

THEOREM 2.6. (Weiss [**1**, Theorem 3.1].) *If $G_\alpha^{\Gamma(\alpha)}$ is an affine primitive group and $d = |\Gamma(\alpha)|$ then $|G_\alpha|$ is bounded by a function of d.*

For a non-negative integer i, by $\Gamma_i(\alpha)$ and $G_i(\alpha)$, we mean the set of vertices of Γ which are at distance at most i from α, and the kernel of G_α on $\Gamma_i(\alpha)$ respectively.

PROPOSITION 2.7. ([**1**, Lemma 1.6].) *Suppose that G is quasiprimitive on the vertex set of a finite connected graph Γ and that $G_\alpha^{\Gamma(\alpha)}$ is primitive, for $\alpha \in V\Gamma$. Let N be a nontrivial normal subgroup of G. If $N_i(\alpha) = \{1\}$ for some integer i, then $G_{i+1}(\alpha) = \{1\}$.*

3. Proof of the Main Theorem

The Suzuki groups $Sz(q)$ form an infinite family of simple groups of Lie type, and were defined in [**8**] as subgroups of the groups $SL_4(q)$. Let $T \leqslant Sz(q)$. Then by [**8**] T is conjugate to one of the groups in case(i), or T is conjugate to a subgroup of one of the groups in (ii)–(v).

(i) $Sz(2^l)$ for some divisor l of m where $q = 2^m$,
(ii) B_0; a dihedral group of order $2(q-1)$,
(iii) $A_i (i = 1, 2)$; a cyclic group of order $q \pm r + 1$, where $r^2 = 2q$
(iv) $B_i (i = 1, 2)$; the normaliser $N_G(A_i)$ of order $4(q \pm r + 1)$ or
(v) F; a Frobenius group of order $q^2(q-1)$.

Furthermore the group F in case (v) is a semidirect product of two groups, a cyclic group of order $q-1$ and a normal subgroup of order q^2.

Proof of the Main Theorem: By Praeger [**6**, Section 3], we may assume that every nontrivial normal subgroup of G has at most two orbits on vertices. By assumption $S = Sz(q)$ is the unique minimal normal subgroup of G and $G/S \leqslant \mathrm{Out}(S)$ is cyclic of odd order. Then S has one or two orbits on vertices. If S has two orbits then G has a normal subgroup of index 2 containing S, but this is not the case since $|G : S|$ is odd. Hence S is transitive on vertices and S has no subgroup of index 2. If the graph Γ be a bipartite graph then G would have a subgroup of index 2, and this is not the case. Thus Γ is a non-bipartite graph.

Set $S_\alpha := G_\alpha \cap S$ and consider G_α / S_α. By this fact that G/S is a subgroup of $\mathrm{Out}(S)$ which is a cyclic group, we conclude that G/S and hence G_α/S_α are soluble groups. If S_α be also soluble, then G_α is soluble, and by Theorem 2.6, $|G_\alpha|$ is bounded by a function of d as required.

Thus we may assume that S_α is insoluble, and so lies in case (i) above, that is, $S_\alpha = Sz(2^l)$ for some l. Let $S_1(\alpha)$ be the kernel of S_α on $\Gamma(\alpha)$, that is, the normal subgroup of S_α which fixes every vertex in $\Gamma(\alpha)$. Now, $S_\alpha \lhd G_\alpha$ and since G_α is primitive on $\Gamma(\alpha)$, we conclude that S_α acts either trivially or transitively on $\Gamma(\alpha)$.

If S_α acts trivially on $\Gamma(\alpha)$, then $G_\alpha/G_1(\alpha)$ is soluble. In this case by Theorem 2.6 $|G_\alpha|$ is bounded by a function of d. Thus we may assume that $S_1(\alpha)$ is a proper normal subgroup of the simple group S_α. Hence $S_1(\alpha) = G_1(\alpha) \cap S = \{1\}$, so $G_1(\alpha) \leqslant C_{G_\alpha}(S_\alpha) \leqslant Z_{m/l}$. By Proposition 2.7, $G_2(\alpha) = \{1\}$ and $G_1(\alpha)$ acts faithfully on $\Gamma_2(\alpha) \setminus \Gamma(\alpha)$. Therefore $G_1(\alpha)$ is isomorphic to a subgroup of $\prod_{\beta \in \Gamma(\alpha)} G_1(\alpha)^{\Gamma(\beta)}$. The groups $G_1(\alpha)^{\Gamma(\beta)}$ $(\beta \in \Gamma(\alpha))$ are permuted transitively by conjugation by G_α, and so are isomorphic. Since $G_1(\alpha)$ is cyclic, it follows that $G_1(\alpha)$ is isomorphic to $G_1(\alpha)^{\Gamma(\beta)} \leqslant S_{d-1}$, and hence $|G_\alpha|$ is bounded by a function of d as required. $\square$

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